COMPARISON OF PROOF SIZES IN FREGE SYSTEMS AND SUBSTITUTION FREGE SYSTEMS

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***Abstract**: It is known that the minimal number of the steps in a proof of a tautology in a Frege system can be exponentially larger than in a substitution Frege system, but it is an open problem whether Frege systems can polynomially simulate substitution Frege systems by sizes. Many people conjecture that the answer is no. We prove that the answer is **yes**. As a bridge between substitution Frege systems and Frege systems we consider the Frege systems, augmented with restricted substitution (single renaming) rule. We prove that Frege systems with single renaming rule polynomially simulate by size Frege systems with substitution rule without any restrictions, and Frege systems without substitution rule polynomially simulate Frege systems with single renaming rule both by steps and by size.*

***Keywords**: Frege system, proof complexity, depth-restricted substitution rule, k-bounded substitution rule, polynomial simulation, exponential speed-up.*

***ACM Classification Keywords**: F.4.1 Mathematical Logic and Formal Languages, Mathematical Logic, Proof theory*

1. Introduction

It is well known that the investigations of the propositional proof complexity are very important due to their relation to the main problem of the complexity theory: P $\stackrel{?}{=}$ NP.

One of the most fundamental problems of the proof complexity theory is to find an efficient proof system for propositional calculus. There is a wide spread understanding that polynomial time computability is the correct mathematical model of feasible computation. According to the opinion, a truly "effective" system must have a polynomial size, p(n) proof for every tautology of size n. In [Cook, Reckhow, 1979] Cook and Reckhow named such a system, a *super system*. They showed that if there exists a super system, then NP = coNP.

It is well known that many systems are not super. This question about Frege system, the most natural calculi for propositional logic, is still open. It is interesting how efficient can be Frege systems augmented with new, not sound rules, in particular, Frege systems with different modifications of substitution rules.

In the field of proof complexity the relation of Frege systems ($\mathcal{F}$-systems) to Frege systems with substitution ($S\mathcal{F}$-systems) has been discussed in [Cejtin, Chubaryan, 1975], [Krajicek, 1989], [Buss, 1995]. It has been proved that there exist tautologies which have n line substitution Frege proofs, but which require Frege proofs of 2^{cn} lines for some constant c. It has also been proved, that substitution Frege proofs of these tautologies can be transformed into Frege proofs only with quadratic increase of size [Buss, 1995], [Chubaryan, 2000]. It is an open problem whether Frege systems can polynomially simulate substitution Frege systems [Buss, 1995].

In [Chubaryan, 2000] a special construction of substitution Frege proofs is described. By their transformation into Frege proofs the maximum (exponential) increase in the number of lines is obtained, although increase in size is at most polynomial (it seems, the latter size increase is maximal). The paper [Chubaryan, 2000] reveals also some important properties of substitution Frege proofs, which can be simulated by Frege systems.

In [Chubaryan et al. 2008], [Chubaryan, Nalbandyan, 2009] the substitution rules with two different restrictions are introduced:

a) if for any constant k≥ 1 we allow substitution instead of occurrences of no more than k different variables at a time (k – bounded substitution)

b) if for any constant d ≥ 0 we allow substitution of formulas, depth of which is no more than d (d – depth restricted substitution).

For every type restriction in [Chubaryan et al. 2008] and [Chubaryan, Nalbandyan, 2009] it is proved that:

1) the minimal numbers of steps (the minimal sizes) of the proofs of tautology in any two restricted substitution Frege systems are polynomially related

2) the minimal sizes of the proofs of tautology in without restrictions substitution Frege system and in restricted substitution Frege system are also polynomially related

3) the minimal number of steps of a tautology in restricted substitution Frege system can be exponentially larger than in the system with substitution rule without restrictions**.**

Here it is proved that:

4) the minimal number of steps of a tautology in Frege system without substitution rule can be exponentially larger than in Frege system with restricted substitution rule

The question about the increase of sizes by transformation in the case 4) was also open.

Here we consider the substitution rule with double restriction: 1 – bounded (single) and 0 – depth (renaming).

We prove that Frege systems with such double restricted substitution rule and Frege systems without substitution rule are polynomially equivalent both by steps and by size.

2. Preliminary

We shall use generally accepted concepts of Frege system and Frege system with substitution.

A Frege system $\mathcal{F}$ uses a denumerable set of propositional variables, a finite, complete set of propositional connectives; $\mathcal{F}$ has a finite set of inference rules defined by a *figure* of the form $\frac{A_1 A_2 \dots A_m}{B}$ (the rules of inference with zero hypotheses are the axioms schemes); $\mathcal{F}$ must be sound and complete, i.e. for each rule of inference $\frac{A_1 A_2 \dots A_m}{B}$ every truth-value assignment, satisfying A_1, A_2, ..., A_m, also satisfies B, and $\mathcal{F}$ must prove every tautology.

A substitution Frege system S$\mathcal{F}$ consists of a Frege system $\mathcal{F}$ augmanted with the substitution rule with inferences of the form $\frac{A}{A\sigma}$ for any substitution σ = $\begin{pmatrix} \varphi_{i_1} & \varphi_{i_2} \dots \varphi_{i_s} \\ p_{i_1} & p_{i_2} \dots p_{i_s} \end{pmatrix}$, s≥ 1, consisting of a mapping from propositional variables to propositional formulas, and $A\sigma$ denotes the result of applying the substitution to formula A, which replaces each variable in A with its image under σ. This definition of substitution rule allows to use the simultaneous substitution of multiple formulas for multiple variables of A without any restrictions. The substitution rule is not sound.

If the depths of formulas φ_{i_j} (1 ≤ j ≤ s) are restricted by some fixed d (d ≥ 0), then we have *d-restricted substitution rule* and we denote the corresponding system by $S^d\mathcal{F}$. 0-restricted substitution rule is named renaming rule.

If for any constant k≥ 1 we allow substitution instead of occurrences of no more than k different variables at a time, then we have *k – bounded substitution rule*. The k – bounded substitution Frege system $S_k\mathcal{F}$ consists of a Frege system augmented with the k – bounded substitution rule.

We use also the well-known notions of proof, proof complexities and p – simulation given in [1]. The proof in any system Φ (Φ-proof) is a finite sequence of such formulas, each being an axiom of Φ, or is inferred from earlier formulas by one of the rules of Φ.

The total number of symbols, appearing in a formula φ, we call size of φ and denote by | φ |.

We define ℓ – complexity to be the size of a proof (= the total number of symbols) and *t* – complexity to be its length (= the total number of lines).

The minimal ℓ – complexity (*t* – complexity) of a formula φ in a proof system Φ we denote by $\mathrm{l}_{\varphi}^{\Phi}(t_{\varphi}^{\Phi})$.

Let Φ_1 and Φ_2 be two different proof systems.

Definition 1. The system Φ_2 p-ℓ-simulates Φ_1 ($\Phi_1 \prec_\ell \Phi_2$), if there exists a polynomial p( ) such that for each formula φ, provable both in Φ_1 and Φ_2, we have $\mathrm{l}_{\varphi}^{\Phi_2} \leq p(\mathrm{l}_{\varphi}^{\Phi_1})$.

Definition 2. The system Φ_1 is p-ℓ-equivalent to system Φ_2 ($\Phi_1 \sim_\ell \Phi_2$), if Φ_1 and Φ_2 p-ℓ-simulate each other.

Similarly p-*t*-simulation and p-*t*-equivalence are defined for *t* – complexity.

Definition 3. The system Φ_2 has exponential ℓ-speed-up (*t*-speed-up) over the system Φ_1, if there exists a sequence of such formulas φ_n, provable both in Φ_1 and Φ_2, that $\mathrm{l}_{\varphi_n}^{\Phi_1} > 2^{\theta(\mathrm{l}_{\varphi_n}^{\Phi_2})} \left(t_{\varphi_n}^{\Phi_1} > 2^{\theta(t_{\varphi_n}^{\Phi_2})}\right)$.

In this paper we compare under the p-*t*-simulation relation $S^0\mathcal{F}$ and $\mathcal{F}$, $S_1\mathcal{F}$ and $\mathcal{F}$, and under p-*t* (p-ℓ)-simulation $S_1^0\,\mathcal{F}$ and $\mathcal{F}$.

For proving the main results we use also the notion of *essential subformulas*, introduced in [Chubaryan et al. 2008], and the notion of τ – set of subformulas, introduced in [Cejtin, Chubaryan, 1975].

Let F be some formula and Sf(F) is the set of all non-elementary subformulas of formula F.

For every formula F, for every φ ∈ Sf(F) and for every variable p $(\mathrm{F})_{\varphi}^{p}$ denotes the result of the replacement of the subformulas φ everywhere in F with the variable p. If φ ∉ Sf(F), then $(\mathrm{F})_{\varphi}^{p}$ is F.

We denote by Var(F) the set of variables in F.

Definition 4. Let p be some variable that p ∉ Var(F) and φ ∈ Sf(F) for some tautology F. We say that φ is an essential subformula in F iff $(\mathrm{F})_{\varphi}^{p}$ is non-tautology.

We denote by Essf(F) the set of essential subformulas in F.

If F is minimal tautology, i.e. F is not a substitution of a shorter tautology, then Essf(F) = Sf(F).

The formula φ is called determinative for the $\mathcal{F}$-rule $\frac{A_1, A_2, \ldots, A_m}{B}$ (m ≥ 1) if φ is essential subformula in formula A_1 & (A_2 & … & (A_{m-1} & A_m) …) ⊃ B. By the Dsf(A_1, …, A_m, B) the set of all determinative formulas for rule $\frac{A_1, A_2, \ldots, A_m}{B}$ is denoted.

We say that the formula φ is important for some $\mathcal{F}$-proof (S$\mathcal{F}$ -proof) if φ is essential in some axiom of this proof or φ is determinative for some $\mathcal{F}$-rule.

In [Chubaryan et al. 2008] the following statement is proved.

Proposition 1. Let F be a minimal tautology and φ ∈ Essf(F), then in every S$\mathcal{F}$-proof of F, in which the employed substitution rules are

$$\frac{A_1}{A_1\sigma_1}; \frac{A_2}{A_2\sigma_2}; \ldots ; \frac{A_r}{A_r\sigma_r}$$

either φ must be important for this proof or it must be the result of the successive employment of the substitutions $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_s}$ for $1 \le i_1, i_2, \ldots, i_s \le r$ in any important formula.

τ – set of subformulas for some formula F with the logical connectives &, ∨, ⊃ and ¬ is defined as follows:

τ(F) = {F} ∪ τ_1(F), where

τ_1(F) = ∅, if F is propositional variable

$\tau_1(F_1 \& F_2) = \tau(F_1) \cup \tau(F_2)$, if $F = F_1 \& F_2$

$\tau_1(F_1 \vee F_2) = \tau(F_1) \cap \tau(F_2)$, if $F = F_1 \vee F_2$

$\tau_1(F_1 \supset F_2) = \tau(F_2) \setminus \tau(F_1)$, if $F = F_1 \supset F_2$

$\tau_1(\neg F_1) = \overline{\tau(F_1)}$, if $F = \neg F_1$

In [Nalbandyan, 2010] the following 3 auxiliary statements are proved.

1. For every minimal tautology F τ(F) ⊆ Essf(F).
2. For every formula F if subformula φ ∈ τ(F), then every occurrence of φ in F is positive.

The notions of *positive (negative)* occurrence of some subformula in the formula are well known (see, for example [Buss, 1995]), as well as *0-1 numeration* of subformulas in formula.

3. If formula F has only the connectives ⊃ and ¬, then the number of every subformula from the set τ(F) is in the form ($\underbrace{11 \ldots 1}_{n}$) for the corresponding n.

The main results, connected with comparison of different substitution rule modifications, are the followings.

Theorem 1.

1. given arbitrary $k_1 \geq 1$ and $k_2 \geq 1$ $S_{k_1}\mathcal{F} \sim_\ell S_{k_2}\mathcal{F}$ $S_{k_1}\mathcal{F} \sim_t S_{k_2}\mathcal{F}$
2. given arbitrary $k \geq 1$ $S_k\mathcal{F} \sim_\ell S\mathcal{F}$
3. given arbitrary $k \geq 1$ $S\mathcal{F}$ has exponential t -speed-up over the system $S_k\mathcal{F}$
4. given arbitrary $k \geq 1$ $S_k\mathcal{F}$ has exponential t -speed-up over the system $\mathcal{F}$.

Theorem 2.

1. given arbitrary $d_1 \geq 1$ and $d_2 \geq 1$ $S^{d_1}\mathcal{F} \sim_\ell S^{d_2}\mathcal{F}$ $S^{d_1}\mathcal{F} \sim_t S^{d_2}\mathcal{F}$
2. given arbitrary $d \geq 0$ $S^d\mathcal{F} \sim_\ell S\mathcal{F}$
3. given arbitrary $d \geq 0$ $S\mathcal{F}$ has exponential t -speed-up over the system $S^d\mathcal{F}$
4. given arbitrary $d \geq 0$ $S^d\mathcal{F}$ has exponential t -speed-up over the system $\mathcal{F}$.

The proofs of the points 1., 2., 3. for both theorems are given in [Chubaryan et al. 2008] and [Chubaryan, Nalbandyan, 2009], [Chubaryan et al. 2009] accordingly. Note that the proofs of the points 1. and 2. are based on the result of Buss [Buss, 1995], who proved that renaming Frege systems p-ℓ-simulate Frege systems with substitution without any restrictions.

The proof of point 4. for k = 1 from Theorem 1., using the formulas

$\varphi_n = p_1 \supset \overbrace{(p_2 \supset (p_2 \supset \ldots \supset (p_2}^{n} \supset p_1) \ldots)$, is given in [Cejtin, Chubaryan, 1975], where only single substitution rule is considered, therefore the proof for every $k \geq 1$ follows from point 1.

To prove the statement of point 4. from Theorem 2. we show that for the formulas

$$\Psi_n = (p_1 \supset p_1) \,\&\, (p_2 \supset p_2) \,\&\, (p_3 \supset p_3) \,\&\, \ldots \,\&\, (p_n \supset p_n)$$

are true the following results

$$t^{S^0\mathcal{F}}_{\Psi_n} = O(\log_2 n) \text{ and } t^{\mathcal{F}}_{\Psi_n} = \Omega(n).$$

Really, the formula Ψ_n can be derived in $S^0\mathcal{F}$ as follows:

1. $p_1 \supset p_1$
2. $p_2 \supset p_2$ (renaming $\binom{p_2}{p_1}$)

3. $(p_1 \supset p_1) \& (p_2 \supset p_2)$ $(\frac{A,B}{A \& B}$ rule)
4. $(p_3 \supset p_3) \& (p_4 \supset p_4)$ (renaming $\begin{pmatrix} p_3 & p_4 \\ p_1 & p_2 \end{pmatrix}$)
5. $((p_1 \supset p_1) \& (p_2 \supset p_2)) \& ((p_3 \supset p_3) \& (p_4 \supset p_4))$ $(\frac{A,B}{A \& B}$ rule)

⋮

2k + 1. $(p_1 \supset p_1) \& \ldots\ldots\ldots \& (p_{2^k} \supset p_{2^k})$ $(\frac{A,B}{A \& B}$ rule)

2k + 2. $(p_{2^k+1} \supset p_{2^k+1}) \& \ldots\ldots\ldots \& (p_{2^{k+1}} \supset p_{2^{k+1}})$ (renaming $\begin{pmatrix} p_{2^k+i} \\ p_i \end{pmatrix}$)

2(k+1) + 1. $(p_1 \supset p_1) \& \ldots\ldots\ldots \& (p_{2^{k+1}} \supset p_{2^{k+1}})$ $(\frac{A,B}{A \& B}$ rule)

On the other hand τ – set of Ψ_{2^k} has 2^{k+1} formulas, and therefore from above auxiliary statements 1. and 2., follows that the number of steps in $\mathcal{F}$ - proof of Ψ_{2^k} must be no less than $c \cdot 2^k$ for some c, depending only from choice of system $\mathcal{F}$.

Note that if we compare the sizes of $\mathcal{F}$ - proof and $S_1\mathcal{F}$ - proof for φ_n and the sizes of $\mathcal{F}$ - proof and $S^0\mathcal{F}$ - proof for Ψ_n, then we obtain only polynomial increase.

3. Main result

Here we will consider the Frege system augmented with double restrictions substitution rule(single renaming). According to above notations, such system must be denoted by $S_1^0\mathcal{F}$.

In [Cook, Reckhow, 1979] it is proved that every two Frege systems are polynomially equivalent both by size and by length, therefore without loss of generality we assume that $\mathcal{F}$ is a Frege system, whose language contains only the connectives ⊃ and ¬.

The axiom-schemas are:

1. $A \supset (B \supset A)$
2. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3. $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$

and inference rule is Modus ponens.

The main result of the paper is the following statement.

Main Theorem.

$$S\mathcal{F} \sim_\ell \mathcal{F}$$

First we will prove that $S_1^0\mathcal{F} \sim_\ell \mathcal{F}$ and obtain the statement of the Main Theorem as a corollary.

Let us recall some notions in addition.

Note that every Φ-proof has an associated graph with nodes, labeled by formulas, and edges from A to B if formula B is the result of applying of some inference rule to A (perhaps with another formulas). A proof is said to be tree-like if the associated graph is tree. It is obvious, that any derivation can be made tree-like by merely repeating parts of the derivation if need be. It is also obvious that by such natural transformation we have tree-like proof, the steps of which can be much more than the steps in original proof-sequence, nevertheless in [Krajicek, 1994] a transformation of proof-sequence into tree-like proof is suggested such that the following statement is hold:

for every Frege system there exists a polynomial p() such that for every tautology φ if n is the steps of its proof in the sequence form, then steps of its tree-like proof is no more than p(n).

Without making some important corrections in the proof of this statement, we prove

Lemma 1. There exists a polynomial p() such that for every tautology φ if n is the number of steps of its $S_1^0\mathcal{F}$-proof in the sequence form, then the number of steps of its tree-like $S_1^0\mathcal{F}$-proof is no more than p(n).

In [Cejtin, Chubaryan, 1995] a natural method of transformation of a given S$\mathcal{F}$-proof into $\mathcal{F}$-proof is described. This method is following: let some formula ψ of S$\mathcal{F}$-proof be inferred from φ by substitution rule, i.e. there is a substitution σ such that ψ = φσ. To prove the formula ψ in $\mathcal{F}$ we have to repeat the proof of φ, applying the substitution σ to all formulas of this proof.

As the sequence of successive substitution is closed under composition, then described transformation method must be applied in the case when both formulas φ and φσ are used for the inference of some next formulas in the given S$\mathcal{F}$-proof, and therefore, as it is pointed in introduction, the number of steps of $\mathcal{F}$-proof can be much more than the number of steps of S$\mathcal{F}$-proof, but if S$\mathcal{F}$-proof is in tree-like form, then the number of formulas in corresponding $\mathcal{F}$-proof is no more than in S$\mathcal{F}$-proof, so we obtain the following statements.

Lemma 2. $S_1^0\mathcal{F} \sim_t \mathcal{F}$

Now we must compare the size of the proofs for arbitrary formula in $S_1^0\mathcal{F}$ and in $\mathcal{F}$.

Let us recall the notion of right-chopping proof, introduced in [Nurijanyan, 1981]. For Intuitionistic and Minimal (Johansson's) Logic there is proved the following statements:

If the axiom $F_1 \supset (F_2 \supset (\dots \supset (F_m \supset G) \dots)$ and the formulas $F_1, F_2, \dots, F_m$ are used from the minimal (by steps) derivation of formula G by the successive applying of the rule modus ponens, then $m \le 2$, i.e. the length of branch, going to right and upwards from every node of the corresponding graph, is no more than 2. Such graph and hence the corresponding proof are called right-chopping.

The analogous statement for classical Hilbert style systems is not valid, but using 1) the method of Nurijanyan, 2) proved in [Cejtin, Chubaryan, 1975] fact that every formula from τ – set of arbitrary derivable formula must be an element from τ – set at least of one axiom from this derivation, and 3) that the τ – sets of our axioms are the following:

1. $\tau(A \supset (B \supset A)) = \{A \supset (B \supset A), B \supset A\}$
2. $\tau((A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))) = \{(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C)),$
   $(A \supset (B \supset C)) \supset (A \supset C), A \supset C\}$
3. $\tau((\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)) = \{(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A), (\neg A \supset \neg B) \supset A)\}$,

we obtain the following statements.

Lemma 3. Every $\mathcal{F}$-proof of a formula φ can be transformed into right-chopping proof of φ, the t-complexity of which is no more than t-complexity of original proof.

Note that the depth of occurrence of each formula from τ – set of any above axioms is no more than 2.

In [Buss, 1995] it is showed that a Frege proof for a formula can be transformed into new one, where the symbol-size is related to line-size and the depth of the original proof. Recall that the depth of the proof is the maximum and/or depth of a formula ψ, occurring in the proof.

More precise result is the following:

a depth d Frege proof with m lines can be transformed into a depth d Frege proof with $O(m^d)$ symbols.

Using 1) the main idea of the proof of this result, 2) above remark about depth of formulas from τ – set of axioms, 3) the possibility of evaluating of the sizes for interpolants and two contrary formulas, deduced from counter-factural hypothesis in right-chopping proof, we obtain the following statement.

Lemma 4. If t is the number of steps in right-chopping $\mathcal{F}$-proof of tautology φ, then the size of this proof is no more than $t^3 \cdot |\varphi|$.

Now we can proof the following

Main Lemma. $S_1^0\mathcal{F} \sim_\ell \mathcal{F}$

Really, let we have some $S_1^0\mathcal{F}$ proof of arbitrary tautology φ with the size L. It is obvious that $|\varphi| < L$ and t-complexity of this proof also is < L. At first we transform this proof into tree-like proof, then into $\mathcal{F}$-proof, and finally into right-chopping $\mathcal{F}$-proof.

Using the statements of above Lemmas, we can state that there is a polynomial p(), depending only on the choice of Frege system such that $\ell_\varphi^{\mathcal{F}}$ = O(p(L)).

So $\mathcal{F}$ p-ℓ-simulates $S_1^0\mathcal{F}$. The reverse p-ℓ-simulation is obvious.

Now the proof of Main Theorem follows from the results of Theorem 1, Theorem 2 and Main Lemma.

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