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Research Article

On the Convergence of Continuous-Time Waveform Relaxation Methods for Singular Perturbation Initial Value Problems

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This paper extends the continuous-time waveform relaxation method to singular perturbation initial value problems. The sufficient conditions for convergence of continuous-time waveform relaxation methods for singular perturbation initial value problems are given.

1. Introduction

Singular perturbation initial value problems play an important role in the research of various applied sciences, such as control theory, population dynamics, medical science, environment science, biology, and economics [1, 2]. These problems are characterized by a small parameter ϵ multiplying the highest derivatives. Since the classical Lipschitz constant and one-sided Lipschitz constant are generally of size $\mathcal{O}(\epsilon^{-1})$ $(0 < \epsilon \ll 1)$, the classical convergence theory, B-convergence theory cannot be directly applied to singular perturbation initial value problems.

Waveform relaxation methods were introduced by Lelarasmee et al. [3]. In recent years, these methods have been widely applied due to their flexibility, convenience, and efficiency. The problems of convergence of waveform relaxation methods for systems of ordinary differential equations, differential algebraic equations, integral equations, delay differential equations, and fractional differential equations were discussed by main authors (cf. [4–7], and references therein). In't Hout [8] consider the convergence of waveform relaxation methods for stiff nonlinear ordinary differential equations. The monograph of Jiang [9] which entitled "Waveform Relaxation Methods" introduced these methods for various ODEs, differential algebraic equations, integral equations, delay differential equations and fractional differential equations, and PDEs. For more comprehensive survey on these methods and their applications, the reader is refered to the monograph in [9] and the references therein. Natesan et al., Vigo-Aguiar and Natesan [10–14] considers the general second order singular perturbation problems.

In this paper, we apply continuous waveform relaxation methods to single stiff singular perturbation initial value problems and obtain the corresponding convergence results.

In the rest parts of the text, we define the maximum norm as follows:

$$\|x\|_T = \max_{0\le t\le T}\|x\|, \tag{1.1}$$

and the λ norm of exponential type:

$$\|x\|_{\lambda,T} = \max_{0\le t\le T}\left\{e^{-\lambda t}\|x\|\right\}, \tag{1.2}$$

where λ is any given positive number and $\|\cdot\|$ denotes an any given norm in C^n.

2. Convergence Analysis of the First Type

2.1. Linear SPPs

Consider the following linear singular perturbation initial value problem

$$\begin{gathered} \epsilon\frac{dy(t)}{dt} + Ay(t) = b(t), \quad t\in[0,T],\ 0<\epsilon\ll 1, \\ y(0) = y_0, \end{gathered} \tag{2.1}$$

where the constant matrix $A \in R^{n\times n}$ and the input function $b(t) \in R^n$, y_0 is the given initial value, and ϵ is the singular perturbation parameter. The constant matrix A is split by $A = A_1 - A_2$, then the system (2.1) can be written as

$$\begin{gathered} \epsilon\frac{dy(t)}{dt} + A_1y(t) = A_2y(t) + b(t), \quad t\in[0,T],\ 0<\epsilon\ll 1, \\ y(0) = y_0, \end{gathered} \tag{2.2}$$

then we can obtain the following iterate scheme:

$$\begin{gathered} \frac{dy^{(k+1)}(t)}{dt} + \frac{1}{\epsilon}A_1y^{(k+1)}(t) = \frac{1}{\epsilon}A_2y^{(k)}(t) + \frac{b(t)}{\epsilon}, \quad t\in[0,T],\ 0<\epsilon\ll 1, \\ y^{(k+1)}(0) = y_0, \quad k = 1,2,\ldots, \end{gathered} \tag{2.3}$$

here we can choose the initial iterative function $y^{(0)}(t) = y_0$, the above iteration is called a continuous-time waveform relaxation process.

For any fixed $k \geq 0$, from (2.3), we have, upon premultiplying by $e^{(s-t)A_1}$ and integrating from 0 to t, as following:

$$y^{(k+1)}(t) = e^{-(A_1 t)/\epsilon} y_0 + \int_0^t e^{(s-t)A_1/\epsilon} \left(\frac{A_2}{\epsilon} y^{(k)}(s) + \frac{b(s)}{\epsilon} \right) ds. \tag{2.4}$$

Let

$$(\mathcal{R}y)(t) = \int_0^t e^{(s-t)A_1/\epsilon} \frac{A_2}{\epsilon} y(s) ds, \tag{2.5}$$

then (2.5) can be written as

$$y^{(k+1)}(t) = \left(\mathcal{R} y^k \right)(t) + \varphi(t), \tag{2.6}$$

where $\varphi(t) = e^{-(A_1 t)/\epsilon} y_0 + \int_0^t e^{(s-t)A_1/\epsilon} (b(s)/\epsilon) ds$. It is easy to see that $\mathcal{R}$ is a Volterra convolution operator with the kernel function $\kappa(t) = e^{-(A_1 t)/\epsilon}(A_2/\epsilon)$:

$$(\mathcal{R}y)(t) = \kappa(t) * y(t) = \int_0^t \kappa(s-t) y(s) ds, \tag{2.7}$$

and $\mathcal{R}$ is the waveform relaxation operator.

Theorem 2.1. *Let the waveform relaxation operator $\mathcal{R}$ be defined in $C([0,T], R^n)$. If the kernel function $\kappa(t)$ is continuous in $[0,T]$ and satisfies $\|\kappa\|_T \leq C$, where C is a constant, then the sequence of functions $y^{(k)}(t)$ defined by* (2.3) *satisfy*

$$\left\| y^{(k)}(t) - y^*(t) \right\|_T \leq \frac{(CT)^k}{k!} \left\| y^{(0)}(t) - y^*(t) \right\|_T, \tag{2.8}$$

where $y^*(t)$ is the exact solution of system (2.1).

Proof. By the norm $\| \cdot \|_T$ in $C([0,T])$, we can obtain

$$\left\| \mathcal{R}^k \right\|_T \leq \int_0^T \left\| \kappa^k(t) \right\| ds, \quad k = 1, 2, \ldots, \tag{2.9}$$

where $\kappa^1(t) = \kappa(t)$, $\kappa^k(t) = \kappa(t) * \kappa^{k-1}(t)$.

In fact, from the given condition $\|\kappa\|_T \leq C$, for any $t \leq T$, we have

$$\left\| \kappa^k(t) \right\| \leq C \int_0^t \left\| \kappa^{k-1}(t) \right\| ds, \tag{2.10}$$

it can be obtained, by induction, that

$$\left\|\kappa^k(t)\right\| \le C\frac{(Ct)^{k-1}}{(k-1)!}, \tag{2.11}$$

so

$$\left\|\mathcal{R}^k\right\|_T \le \frac{(CT)^k}{k!}, \tag{2.12}$$

which complete the proof. □

Finally, we mention that the estimate (2.8) is superlinear convergence estimate, which reveals a rapid convergence behavior when $k \to \infty$.

2.2. Nonlinear SPPs

Consider the following nonlinear singular perturbation initial value problem:

$$\begin{gathered}\epsilon\frac{dy(t)}{dt} = f(y(t),t), \quad t \in [0,T],\ 0<\epsilon \ll 1, \\ y(0) = y_0,\end{gathered} \tag{2.13}$$

where y_0 is the given initial value, ϵ is the singular perturbation parameter. $f : R^n \times [0,T] \to R^n$ is given continuous function mapping, and $y(t)$ is unknown.

The continuous-time Waveform Relaxation algorithm for (2.13) is

$$\begin{gathered}\frac{dy^{(k+1)}(t)}{dt} = \frac{1}{\epsilon}F\left(y^{(k+1)}(t), y^{(k)}(t), t\right), \quad t \in [0,T],\ 0<\epsilon \ll 1, \\ y^{(k+1)}(0) = y_0, \quad k = 1,2,\dots,\end{gathered} \tag{2.14}$$

where the splitting function $F(u,v,t)$ determines the type of the Waveform Relaxation algorithm, and we assume that $F(u,v,t)$ satisfy the following Lipschitz condition

$$\|F(u_1,v_1,t) - F(u_2,v_2,t)\| \le L_1\|u_1-u_2\| + L_2\|v_1-v_2\|. \tag{2.15}$$

By integrating the inequality (2.15) of both side from 0 to t, we have

$$y^{(k+1)}(t) = y_0 + \int_0^t \frac{1}{\epsilon}F\left(y^{(k+1)}(s), y^{(k)}(s), s\right)ds. \tag{2.16}$$

Let $\widehat{y}(t)$ denote the function that iterated by $y(t)$ from one iteration step, like (2.16), denote $\widehat{y}(t) = (\mathcal{R}y)(t)$.

Theorem 2.2. *Assume that the splitting function $F(u, v, t)$ in WR iteration process* (2.14) *is Lipschitz continuous with respect to u and v, then the continuous-time Waveform Relaxation algorithm* (2.14) *is convergent.*

Proof. We introduce another continuous function $z(t)$, and denote $\hat{z}(t) = (\mathcal{R}z)(t)$, then

$$\begin{aligned}\|\hat{y}(t) - \hat{z}(t)\| &= \frac{1}{\epsilon}\left\|\int_0^t \left(F(\hat{y}(s), y(s), s) - F(\hat{z}(s), z(s), s)\right) ds\right\| \\ &\leq \frac{1}{\epsilon}\int_0^t \|F(\hat{y}(s), y(s), s) - F(\hat{z}(s), z(s), s)\| ds.\end{aligned} \tag{2.17}$$

Equations (2.15)–(2.17) yield

$$\|\hat{y}(t) - \hat{z}(t)\| \leq \frac{L_1}{\epsilon}\int_0^t \|\hat{y}(t) - \hat{z}(s)\| ds + \frac{L_2}{\epsilon}\int_0^t \|y(s) - z(s)\| ds. \tag{2.18}$$

From (2.18), we have, upon premultiplying by $e^{-\lambda t}$, the following:

$$\begin{aligned}e^{-\lambda t}\|\hat{y}(t) - \hat{z}(t)\| &\leq \frac{L_1}{\epsilon}e^{-\lambda t}\int_0^t \|\hat{y}(s) - \hat{z}(s)\| ds + \frac{L_2}{\epsilon}e^{-\lambda t}\int_0^t \|y(s) - z(s)\| ds \\ &\leq \frac{L_1}{\epsilon}e^{-\lambda t}\int_0^t e^{\lambda s}\left(e^{-\lambda s}\|\hat{y}(s) - \hat{z}(s)\|\right) ds + \frac{L_2}{\epsilon}e^{-\lambda t}\int_0^t e^{\lambda s}\left(e^{-\lambda s}\|y(s) - z(s)\|\right) ds \\ &\leq \left(\frac{L_1}{\epsilon}\max_{0\leq s\leq t}\left\{e^{-\lambda s}\|\hat{y}(s) - \hat{z}(s)\|\right\} + \frac{L_2}{\epsilon}\max_{0\leq s\leq t}\left\{e^{-\lambda s}\|y(s) - z(s)\|\right\}\right)e^{-\lambda t}\int_0^t e^{\lambda s} ds,\end{aligned} \tag{2.19}$$

because of $e^{-\lambda t}\int_0^t e^{\lambda s} ds \leq 1/\lambda$, and from the definition of λ norm of the exponential type, we have

$$\|\hat{y}(t) - \hat{z}(t)\|_{\lambda,T} \leq \frac{L_1}{\lambda\epsilon}\|\hat{y}(s) - \hat{z}(s)\|_{\lambda,T} + \frac{L_2}{\lambda\epsilon}\|y(s) - z(s)\|_{\lambda,T}. \tag{2.20}$$

It is easy to obtain

$$\|\hat{y}(t) - \hat{z}(t)\|_{\lambda,T} \leq \frac{L_2}{\lambda\epsilon - L_1}\|y(s) - z(s)\|_{\lambda,T}. \tag{2.21}$$

We can choose large enough λ such that $\gamma = L_2/(\lambda\epsilon - L_1) < 1$. Thus, the Waveform Relaxation operator $\mathcal{R}$ is a contractive operator under this norm. From the contractive mapping principle, we can derive that the continuous-time Waveform Relaxation algorithm (2.14) is convergent. □

3. Convergence Analysis of the Second Type

3.1. Linear SPPs

Consider the following linear singular perturbation initial value problem

$$\begin{gathered} x'(t) + Ax(t) + By(t) = r(t), \quad t \in [0, T], \\ \epsilon \frac{dy(t)}{dt} + Cx(t) + Dy(t) = s(t), \quad 0 < \epsilon \ll 1, \\ x(0) = x_0, \qquad y(0) = y_0, \end{gathered} \tag{3.1}$$

where x_0 and y_0 are the given initial value, ϵ is the singular perturbation parameter, $r(t) \in R^{n_1}$ and $s(t) \in R^{n_2}$ are given functions. The constant matrices $A \in R^{n_1 \times n_1},\ B \in R^{n_1 \times n_2},\ C \in R^{n_2 \times n_1}$, $D \in R^{n_2 \times n_2}$ are split by $A = A_1 - A_2,\ B = B_1 - B_2,\ C = C_1 - C_2,\ D = D_1 - D_2$ respectively, $x(t) \in R^{n_1}$ and $y(t) \in R^{n_2}$ are unknowns. Then the system (3.1) can be written as

$$\begin{gathered} x'(t) + A_1 x(t) + B_1 y(t) = A_2 x(t) + B_2 y(t) + r(t), \quad t \in [0, T], \\ \epsilon \frac{dy(t)}{dt} + C_1 x(t) + D_1 y(t) = C_2 x(t) + D_2 y(t) + s(t), \quad 0 < \epsilon \ll 1, \\ x(0) = x_0, \qquad y(0) = y_0. \end{gathered} \tag{3.2}$$

The continuous-time Waveform Relaxation algorithm for (3.1) is as follows:

$$\begin{gathered} \frac{dx^{(k+1)}(t)}{dt} + A_1 x^{(k+1)}(t) + B_1 y^{(k+1)}(t) = A_2 x^{(k)}(t) + B_2 y^{(k)}(t) + r(t), \quad t \in [0, T], \\ \frac{dy^{(k+1)}(t)}{dt} + \frac{C_1}{\epsilon} x^{(k+1)}(t) + \frac{D_1}{\epsilon} y^{(k+1)}(t) = \frac{C_2}{\epsilon} x^{(k)}(t) + \frac{D_2}{\epsilon} x^{(k)}(t) + \frac{1}{\epsilon} s(t), \quad 0 < \epsilon \ll 1, \\ x^{(k+1)}(0) = x_0, \quad y^{(k+1)}(0) = y_0, \quad k = 1, 2, \ldots, \end{gathered} \tag{3.3}$$

The matrix form of (3.3) reads

$$\frac{d}{dt} \begin{pmatrix} x^{(k+1)}(t) \\ y^{(k+1)}(t) \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ \frac{C_1}{\epsilon} & \frac{D_1}{\epsilon} \end{pmatrix} \begin{pmatrix} x^{(k+1)}(t) \\ y^{(k+1)}(t) \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ \frac{C_2}{\epsilon} & \frac{D_2}{\epsilon} \end{pmatrix} \begin{pmatrix} x^{(k)}(t) \\ y^{(k)}(t) \end{pmatrix} + \begin{pmatrix} r(t) \\ \frac{s(t)}{\epsilon} \end{pmatrix}. \tag{3.4}$$

Solve the equations (3.4), we can derive

$$\begin{aligned} \begin{pmatrix} x^{(k+1)}(t) \\ y^{(k+1)}(t) \end{pmatrix} = {} & \exp\left(-\begin{pmatrix} A_1 & B_1 \\ \frac{C_1}{\epsilon} & \frac{D_1}{\epsilon} \end{pmatrix} t \right) \begin{pmatrix} x^{(k+1)}(0) \\ y^{(k+1)}(0) \end{pmatrix} \\ & + \int_0^t \exp\left(\begin{pmatrix} A_1 & B_1 \\ \frac{C_1}{\epsilon} & \frac{D_1}{\epsilon} \end{pmatrix} (s - t) \right) \left(\begin{pmatrix} A_2 & B_2 \\ \frac{C_2}{\epsilon} & \frac{D_2}{\epsilon} \end{pmatrix} \begin{pmatrix} x^{(k)}(s) \\ y^{(k)}(s) \end{pmatrix} + \begin{pmatrix} r(s) \\ \frac{s(s)}{\epsilon} \end{pmatrix} \right) ds. \end{aligned} \tag{3.5}$$

Denote $\varepsilon_x^k(t) = x^k(t) - x(t)$, $\varepsilon_y^k(t) = y^k(t) - y(t)$, where $x(t)$ and $y(t)$ are the exact solutions of (3.1). From (3.2) and (3.5), we can obtain

$$\begin{pmatrix} \varepsilon_x^{k+1}(t) \\ \varepsilon_y^{k+1}(t) \end{pmatrix} = \int_0^t \exp\left(\begin{pmatrix} A_1 & B_1 \\ \dfrac{C_1}{\epsilon} & \dfrac{D_1}{\epsilon} \end{pmatrix}(s-t)\right)\begin{pmatrix} A_2 & B_2 \\ \dfrac{C_2}{\epsilon} & \dfrac{D_2}{\epsilon} \end{pmatrix}\begin{pmatrix} \varepsilon_x^k(s) \\ \varepsilon_y^k(s) \end{pmatrix} ds, \tag{3.6}$$

then (3.6) can be written as

$$\begin{pmatrix} \varepsilon_x^{k+1}(t) \\ \varepsilon_y^{k+1}(t) \end{pmatrix} = \left(\mathcal{R}\begin{pmatrix} \varepsilon_x^k(s) \\ \varepsilon_y^k(s) \end{pmatrix}\right)(t), \tag{3.7}$$

clearly, $\mathcal{R}$ is a Volterra convolution operator with the kernel function

$$\begin{gathered} \kappa(t) = \exp\left(-\begin{pmatrix} A_1 & B_1 \\ \dfrac{C_1}{\epsilon} & \dfrac{D_1}{\epsilon} \end{pmatrix}t\right)\begin{pmatrix} A_2 & B_2 \\ \dfrac{C_2}{\epsilon} & \dfrac{D_2}{\epsilon} \end{pmatrix}, \\ (\mathcal{R}y)(t) = \kappa(t) * y(t) = \int_0^t \kappa(s-t)y(s)ds, \end{gathered} \tag{3.8}$$

$\mathcal{R}$ is the Waveform Relaxation operator.

Theorem 3.1. *Let the waveform relaxation operator $\mathcal{R}$ be defined in $C([0,T], R^n)$. If the kernel function $\kappa(t)$ is continuous in $[0,T]$ and satisfies $\|\kappa\|_T \le M$, where M is a constant, then the sequence of functions $(\varepsilon_x^{k+1}(t), \varepsilon_y^{k+1}(t))^T$ defined by (3.6) satisfy*

$$\begin{pmatrix} \left\|\varepsilon_x^k(t)\right\| \\ \left\|\varepsilon_y^k(t)\right\| \end{pmatrix} \le \frac{T^k}{k!}M^k \begin{pmatrix} \max\limits_{0<s<T}\left\|\varepsilon_x^0(s)\right\| \\ \max\limits_{0<s<T}\left\|\varepsilon_y^0(s)\right\| \end{pmatrix}. \tag{3.9}$$

Proof. Taking the norm in both side of (3.6) which reads

$$\begin{pmatrix} \left\|\varepsilon_x^{k+1}(t)\right\| \\ \left\|\varepsilon_y^{k+1}(t)\right\| \end{pmatrix} \le \int_0^t \exp\left(\left\|\begin{matrix} A_1 & B_1 \\ \dfrac{C_1}{\epsilon} & \dfrac{D_1}{\epsilon} \end{matrix}\right\|(s-t)\right)\left\|\begin{matrix} A_2 & B_2 \\ \dfrac{C_2}{\epsilon} & \dfrac{D_2}{\epsilon} \end{matrix}\right\|\begin{pmatrix} \left\|\varepsilon_x^k(s)\right\| \\ \left\|\varepsilon_y^k(s)\right\| \end{pmatrix} ds. \tag{3.10}$$

From the induction of (3.10) and the condition $\|\kappa\|_T \le M$, we can derive

$$\begin{pmatrix} \left\|\varepsilon_x^1(t)\right\| \\ \left\|\varepsilon_y^1(t)\right\| \end{pmatrix} \le M \begin{pmatrix} \max\limits_{0<s<T}\left\|\varepsilon_x^0(s)\right\| \\ \max\limits_{0<s<T}\left\|\varepsilon_y^0(s)\right\| \end{pmatrix} t. \tag{3.11}$$

Moreover, we can derive

$$\begin{pmatrix} \|\varepsilon_x^k(t)\| \\ \left\|\varepsilon_y^k(t)\right\| \end{pmatrix} \leq \frac{T^k}{k!} M^k \begin{pmatrix} \max\limits_{0<s<T} \|\varepsilon_x^0(s)\| \\ \max\limits_{0<s<T} \left\|\varepsilon_y^0(s)\right\| \end{pmatrix}, \tag{3.12}$$

thus, $(\|\varepsilon_x^k(t)\|, \|\varepsilon_y^k(t)\|)^T \to 0$ as $k \to \infty$ which complete the proof. □

3.2. Nonlinear SPPs

Consider the following nonlinear singular perturbation initial value problem:

$$\begin{aligned} x'(t) &= f(x(t), y(t), t), \quad t \in [0, T], \\ \epsilon \frac{dy(t)}{dt} &= g(x(t), y(t), t), \quad 0 < \epsilon \ll 1, \\ x(0) &= x_0, \qquad y(0) = y_0, \end{aligned} \tag{3.13}$$

where x_0 and y_0 are given initial values, ϵ is the singular perturbation parameter, $f : R^{n_1} \times R^{n_2} \times [0, T] \to R^{n_1}$ and $g : R^{n_1} \times R^{n_2} \times [0, T] \to R^{n_2}$ are given continuous function mappings.

The continuous-time waveform relaxation algorithm for (3.13) is

$$\begin{aligned} x^{(k+1)}(t) &= F\left(x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t\right), \quad t \in [0, T], \\ \frac{dy^{(k+1)}(t)}{dt} &= \frac{1}{\epsilon} G\left(x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t\right), \quad 0 < \epsilon \ll 1, \\ x^{(k+1)}(0) &= x_0, \qquad y^{(k+1)}(0) = y_0, k = 1, 2, \ldots, \end{aligned} \tag{3.14}$$

where the splitting functions $F(u_1, u_2, u_3, u_4, t)$ and $G(u_1, u_2, u_3, u_4, t)$ determine the type of the waveform relaxation algorithm. we can derive from (3.14) that

$$\begin{aligned} x^{(k)}(t) &= F\left(x^{(k)}(t), x^{(k-1)}(t), y^{(k)}(t), y^{(k-1)}(t), t\right), \quad t \in [0, T], \\ \frac{dy^{(k)}(t)}{dt} &= \frac{1}{\epsilon} G\left(x^{(k)}(t), x^{(k-1)}(t), y^{(k)}(t), y^{(k-1)}(t), t\right), \quad 0 < \epsilon \ll 1, \\ x^{(k)}(0) &= x_0, \quad y^{(k)}(0) = y_0, k = 1, 2, \ldots. \end{aligned} \tag{3.15}$$

Denote

$$\begin{aligned} \varepsilon_x^{k+1}(t) &= x^{k+1}(t) - x^k(t), \\ \varepsilon_y^{k+1}(t) &= y^{k+1}(t) - y^k(t). \end{aligned} \tag{3.16}$$

Theorem 3.2. *Assume that the matrices* $(\partial F / \partial u_i)$ *and* $(\partial G / \partial u_i)$ $(i = 1, 2, 3, 4)$ *of the splitting functions* $F(u_1, u_2, u_3, u_4, t)$ *and* $G(u_1, u_2, u_3, u_4, t)$ *are continuous, then the continuous-time waveform relaxation algorithm* (3.14) *is convergent.*

Proof. Subtracting (3.14) from (3.15), we have

$$\begin{aligned}\frac{d\varepsilon_x^{k+1}(t)}{dt} &= \frac{\partial F}{\partial u_1}\varepsilon_x^{k+1}(t) + \frac{\partial F}{\partial u_2}\varepsilon_x^{k}(t) + \frac{\partial F}{\partial u_3}\varepsilon_y^{k+1}(t) + \frac{\partial F}{\partial u_4}\varepsilon_y^{k}(t),\\ \frac{d\varepsilon_y^{k+1}(t)}{dt} &= \frac{1}{\epsilon}\left(\frac{\partial G}{\partial u_1}\varepsilon_x^{k+1}(t) + \frac{\partial G}{\partial u_2}\varepsilon_x^{k}(t) + \frac{\partial G}{\partial u_3}\varepsilon_y^{k+1}(t) + \frac{\partial G}{\partial u_4}\varepsilon_y^{k}(t)\right),\end{aligned} \tag{3.17}$$

the matrix form of (3.17) reads

$$\frac{d}{dt}\begin{pmatrix}\varepsilon_x^{k+1}(t)\\ \varepsilon_y^{k+1}(t)\end{pmatrix} = \begin{pmatrix}\dfrac{\partial F}{\partial u_1} & \dfrac{\partial F}{\partial u_3}\\ \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_1} & \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_3}\end{pmatrix}\begin{pmatrix}\varepsilon_x^{k+1}(t)\\ \varepsilon_y^{k+1}(t)\end{pmatrix} + \begin{pmatrix}\dfrac{\partial F}{\partial u_2} & \dfrac{\partial F}{\partial u_4}\\ \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_2} & \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_4}\end{pmatrix}\begin{pmatrix}\varepsilon_x^{k}(t)\\ \varepsilon_y^{k}(t)\end{pmatrix}. \tag{3.18}$$

Denote

$$\begin{aligned}\varepsilon^k(t) &= \left(\varepsilon_x^k(t), \varepsilon_y^k(t)\right)^T,\\ A(t) &= \begin{pmatrix}\dfrac{\partial F}{\partial u_1} & \dfrac{\partial F}{\partial u_3}\\ \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_1} & \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_3}\end{pmatrix},\\ B(t) &= \begin{pmatrix}\dfrac{\partial F}{\partial u_2} & \dfrac{\partial F}{\partial u_4}\\ \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_2} & \dfrac{1}{\epsilon}\dfrac{\partial G}{\partial u_4}\end{pmatrix}.\end{aligned} \tag{3.19}$$

Then, we can derive

$$\frac{d\varepsilon^{k+1}(t)}{dt} = A(t)\varepsilon^{k+1}(t) + B(t)\varepsilon^k(t). \tag{3.20}$$

Assume that the basis matrix $\phi(t)$ satisfies

$$\frac{d\phi(t)}{dt} = A(t)\phi(t), \tag{3.21}$$

then the solution of (3.20) can be written as

$$\varepsilon^{k+1}(t) = \phi(t)\int_0^t \phi^{-1}(s)B(s)\varepsilon^k(s)ds. \tag{3.22}$$

From (3.22), we have, upon taking the norm in both side and premultiplying by $e^{-\lambda t}$, that

$$e^{-\lambda t}\left\|\varepsilon^{k+1}(t)\right\| = e^{-\lambda t}\left\|\int_0^t \phi(t)\phi^{-1}(s)B(s)\varepsilon^k(s)ds\right\|, \tag{3.23}$$

furthermore,

$$\max_{0\le t\le T}\left\{e^{-\lambda t}\left\|\varepsilon^{k+1}(t)\right\|\right\} = \max_{0\le t\le T}\left\{e^{-\lambda t}\left\|\int_0^t \phi(t)\phi^{-1}(s)B(s)\varepsilon^k(s)ds\right\|\right\}, \tag{3.24}$$

so

$$\begin{aligned}\left\|\varepsilon^{k+1}(t)\right\|_{\lambda,T} &\le \max_{0\le t\le T}\left\{e^{-\lambda t}\left\|\int_0^t \phi(t)\phi^{-1}(s)B(s)e^{\lambda s}e^{-\lambda s}\varepsilon^k(s)ds\right\|\right\} \\ &\le \max_{0\le t\le T}\left\{e^{-\lambda t}\int_0^t \left\|\phi(t)\phi^{-1}(s)B(s)\right\|e^{\lambda s}e^{-\lambda s}\left\|\varepsilon^k(s)\right\|ds\right\} \\ &\le M\left\|\varepsilon^k(t)\right\|_{\lambda,T}\max_{0\le t\le T}\left\{e^{-\lambda t}\int_0^t e^{\lambda s}ds\right\} \\ &\le \frac{M}{\lambda}\left\|\varepsilon^k(t)\right\|_{\lambda,T},\end{aligned} \tag{3.25}$$

where $M = \max_{0\le t\le T,\ 0\le s\le t}\{\|\phi(t)\phi^{-1}(s)B(s)\|\}$, and we can choose large enough λ such that $M/\lambda < 1$, then the iterative error sequences $\{\varepsilon^k(t)\}$ are convergent. □

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