Generating facets for the cut polytope of a graph by triangular elimination

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June 21, 2006

Abstract

The cut polytope of a graph arises in many fields. Although much is known about facets of the cut polytope of the complete graph, very little is known for general graphs. The study of Bell inequalities in quantum information science requires knowledge of the facets of the cut polytope of the complete bipartite graph or, more generally, the complete k-partite graph. Lifting is a central tool to prove certain inequalities are facet inducing for the cut polytope. In this paper we introduce a lifting operation, named triangular elimination, applicable to the cut polytope of a wide range of graphs. Triangular elimination is a specific combination of zero-lifting and Fourier-Motzkin elimination using the triangle inequality. We prove sufficient conditions for the triangular elimination of facet inducing inequalities to be facet inducing. The proof is based on a variation of the lifting lemma adapted to general graphs. The result can be used to derive facet inducing inequalities of the cut polytope of various graphs from those of the complete graph. We also investigate the symmetry of facet inducing inequalities of the cut polytope of the complete bipartite graph derived by triangular elimination.

1 Introduction

Cut polytope and related polytopes. The cut polytope arises in many fields [12, 13, 16], and the structure of facets of the cut polytope has been intensively studied. For the complete graph with n nodes, a complete list of the facets of the cut polytope $\mathrm{CUT}_n^\square$ is known for $n \leq 7$ [18], as well as many classes of facet producing valid inequalities. The hypermetric inequalities (see Chapter 28 of [16]) and the clique-web inequalities [15] (also Chapter 29 of [16]), an extension of hypermetric inequalities, are examples of such classes. Very little is known about classes of facets for the cut polytope of an arbitrary graph. One such class are the cycle inequalities, which are projections of the triangle inequalities. They were shown to be facet producing by Barahona and Majoub [4]. The structure of facets of the cut polytope is of both theoretical and practical interest. In the branch-and-cut approach to solve the MAX-CUT problem, facets of the cut polytope are the most powerful cutting planes. However, under the reasonable assumption that NP $\neq$ coNP, the complete list of the facets of the cut polytope does not have a compact representation [26], even for the complete graphs [1, 25]. This implies that we cannot hope to enumerate all of its facets, but rather should look for strong valid inequalities.

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A lifting operation is a procedure which converts a given valid inequality of the cut polytope of a small graph to a new valid inequality of the cut polytope of a larger graph, and is an established method for deriving new facets systematically. The most fundamental example of the lifting operations is zero-lifting (for the complete graph [11, 14] and for general graphs [10]). Readers are referred to Section 26.5 and Chapters 28–30 of [16] for more examples of classes of valid inequalities and lifting operations.

The MAX-CUT problem is equivalent to unconstrained quadratic 0-1 programming [19], and the associated boolean quadric polytope is linearly isomorphic to the cut polytope. This linear isomorphism is called the covariance mapping (see Section 5.2 of [16]). The boolean quadric polytope is also known as the correlation polytope, especially in the physics literature.

Relation of the cut polytope to quantum information processing. The polytopes described in the previous section have many applications in quantum physics and quantum information theory [13, 16]. McRae and Davidson [21] showed the power of polytope theory in quantum physics by proving that the possible solutions to some problems arising in quantum physics form a convex polytope and deriving inequalities for such solutions by convex hull algorithms. One of the polytopes discussed there is identical to the boolean quadric polytope.

In quantum information processing, the cut polytope and the boolean quadric polytope arise in relation to Bell inequalities. In this area, Bell inequalities, a generalization of Bell's original inequality [6], are intensively studied [29, 20] to better understand the nonlocality of quantum physics. Bell inequalities deal with probabilities, and the search for explicit formulae for Bell inequalities is related to Boole's problem [7]. It is natural to consider Bell inequalities as inequalities valid for certain convex polytopes [17, 24, 26, 25, 23] much in the same way as considering Boole's problem as a problem about certain convex polytopes [13]. In particular, Bell inequalities involving joint probabilities of two probabilistic events are exactly inequalities valid for the boolean quadric polytope of a graph [26, 25]. To enumerate all the Bell inequalities for a given physical setting, it is sufficient to enumerate the facets of the corresponding polytope by using a convex hull algorithm. Exhaustive enumeration of the Bell inequalities has been performed [27, 9] in physical settings where parameters such as the number of observables and the number of possible outcomes of each observable are small enough.

Bell inequalities for two parties are inequalities valid for the boolean quadric polytope of the complete bipartite graph $\mathrm{K}_{r,s}$, and they correspond to inequalities valid for the cut polytope $\mathrm{CUT}^{\square}(\nabla \mathrm{K}_{r,s})$ via the covariance mapping. $\nabla \mathrm{K}_{r,s}$ denotes the *suspension graph* of $\mathrm{K}_{r,s}$, that is, the graph obtained by adding a new node to $\mathrm{K}_{r,s}$ and connecting it to all the existing nodes, and in other words, it is the complete tripartite graph $\mathrm{K}_{1,r,s}$. Enumeration of the facets of the cut polytope of the complete graph uses symmetry and other structure specific to the cut polytope, and they are often beyond the reach of general convex hull packages. Avis, Imai, Ito and Sasaki [2] proposed an operation named *triangular elimination*, which is a combination of zero-lifting and Fourier-Motzkin elimination (see e.g. [30]) using the triangle inequality. They proved that triangular elimination maps facet inducing inequalities of the cut polytope of the complete graph to facet inducing inequalities of the cut polytope of $\nabla \mathrm{K}_{r,s}$.

The cut polytope of $\nabla \mathrm{K}_{r,s}$ can be projected to the cut polytope of $\mathrm{K}_{r,s}$, and this means that some Bell inequalities for the correlation polytope of $\mathrm{K}_{r,s}$ correspond to inequalities valid for $\mathrm{CUT}^{\square}(\mathrm{K}_{r,s})$ via the covariance mapping. Such Bell inequalities have good properties in relation to quantum games [8]. They correspond to inequalities for correlation functions [3], whose multi-party version is discussed by Werner and Wolf [28] and Żukowski and Bruckner [31].

Our results. In this paper, we generalize triangular elimination introduced in [2] to an operation which maps inequalities valid for the cut polytope $\mathrm{CUT}^{\square}(G)$ to those for $\mathrm{CUT}^{\square}(G')$ for graphs G

and G' satisfying a certain condition. From the viewpoint of combinatorial optimization, triangular elimination is one of the lifting operations on inequalities valid for the cut polytope.

Though the triangular elimination of an inequality is not uniquely defined, all the choices are switching equivalent (Proposition 5) and therefore triangular elimination can be seen as an operation which, given a switching equivalent class of inequalities valid for $\mathrm{CUT}^{\square}(G)$, uniquely produces a switching equivalent class of inequalities valid for $\mathrm{CUT}^{\square}(G')$.

We prove a sufficient condition (Theorem 4) for the triangular elimination of a facet inducing inequality to be facet inducing. The proof is similar to that of the zero-lifting theorem by Deza and Laurent [14, 16], where the lifting lemma used in the course of the proof is replaced with a version adapted to general graphs.

For certain graphs G and G' which do not satisfy the conditions of Theorem 4, we can sometimes perform repeated triangular eliminations on a sequence of graphs starting from G and ending with G'. Using this idea, we prove another sufficient condition in case where $G = \mathrm{K}_n$. This sufficient condition extends Theorem 2.1 in [2]. It provides a method to derive a large number of inequalities which define facets of the cut polytope of the complete k-partite graph. These are relevant to k-party games, in light of the connection between Bell inequalities and quantum games [8].

We also prove a necessary and sufficient condition for the triangular eliminations of two facet inducing inequalities to be equivalent up to permutation and switching in the case $G = \mathrm{K}_n$ and $G' = \mathrm{K}_{r,s}$.

Organization of the paper. The rest of this paper is organized as follows. Section 2 reviews basic notions about the cut polytope. Section 3 gives the definition of triangular elimination for general graphs and proves its basic properties and the main theorem stating a sufficient condition for the triangular elimination of a facet to be a facet. In Section 4, we prove additional properties of triangular elimination from the complete graph. Section 5 states open problems.

2 Preliminaries

We briefly review basic notions about the cut polytope used in later sections. Definitions, theorems and other results stated in this section are from the comprehensive reference [16] on this topic, which readers are referred to for more information. We assume that readers are familiar with basic notions in convex polytope theory such as convex polytope, facet, projection and Fourier-Motzkin elimination. Readers are referred to a textbook [30] for details.

Throughout this paper, we use the following notation on graphs. We denote the edge between two nodes u and v by uv. For a graph $G = (V, E)$ and a node $v \in V$, we denote the neighbourhood of v by $\mathrm{N}_G(v)$.

2.1 Cut polytope and cone

The cut polytope (resp. cut cone) of a graph $G = (V, E)$ is the convex hull (resp. conic hull) of the cut vectors of G. A formal definition is as follows.

Definition 1 (Cut polyhedra). The *cut polytope* of a graph $G = (V, E)$, denoted $\mathrm{CUT}^{\square}(G)$, is the convex hull of the cut vectors $\boldsymbol{\delta}_G(S)$ of G defined by all the subsets $S \subseteq V$ in the $|E|$-dimensional vector space $\mathbb{R}^E$. The cut vector $\boldsymbol{\delta}_G(S)$ of G defined by $S \subseteq V$ is a vector in $\mathbb{R}^E$ whose uv-coordinate is defined as follows:

$$\delta_{uv}(S) = \begin{cases} 1 & \text{if } |S \cap \{u, v\}| = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } uv \in E.$$

The *cut cone* of G, denoted $\mathrm{CUT}(G)$, is the conic hull of the cut vectors $\boldsymbol{\delta}_G(S) \in \mathbb{R}^E$ of G for all the subsets $S \subseteq V$. If G is the complete graph K_n, we denote $\mathrm{CUT}^\square(\mathrm{K}_n)$ and $\mathrm{CUT}(\mathrm{K}_n)$ also as $\mathrm{CUT}_n^\square$ and CUT_n, respectively.

For a subset F of a set E, the *incidence vector* of F (in E)[1] is the vector $\boldsymbol{x} \in \{0,1\}^E$ defined by $x_e = 1$ for $e \in F$ and $x_e = 0$ for $e \in E \setminus F$. Using this term, the definition of the cut vector can also be stated as follows: $\boldsymbol{\delta}_G(S)$ is the incidence vector of the cut set $\{uv \in E \mid |S \cap \{u,v\}| = 1\}$ in E.

The cut polytope and cone are full-dimensional in $\mathbb{R}^E$ [5]. The following inequalities are the first class of facets of the cut cone of an arbitrary graph.

Theorem 1 ([4]). *(i) For a graph $G = (V,E)$, a cycle $C \subseteq E$ in G and an edge $uw \in C$, the* cycle inequality

$$x_{uw} - \sum_{e \in C \setminus \{uw\}} x_e \le 0 \tag{1}$$

is valid for $\mathrm{CUT}(G)$.

(ii) If C is a chordless cycle in G, then (1) is facet inducing for $\mathrm{CUT}(G)$.

The following proposition follows immediately from the fact that the origin is a vertex of $\mathrm{CUT}^\square(G)$.

Proposition 1. *Inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ is valid (resp. facet inducing) for* $\mathrm{CUT}^\square(G)$ *if and only if it is valid (resp. facet inducing) for* $\mathrm{CUT}(G)$.

2.2 Operations on inequalities

2.2.1 Symmetric transformations

Let $G = (V,E)$ be a graph. The cut polytope $\mathrm{CUT}^\square(G)$ admits two kinds of symmetric transformations, which correspond to operations on valid inequalities which preserve their properties.

Definition 2 (Permutation). Let σ a permutation on V which is an automorphism of G. Then the *σ-permutation* of an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is an inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0$ where $\boldsymbol{a}' \in \mathbb{R}^E$ is defined by $a'_{ij} = a_{\sigma(i)\sigma(j)}$. Such an inequality is said to be *permutation equivalent* to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$.

Definition 3 (Switching). Let S be a subset of V. Then the *S-switching* of an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is an inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0 - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{\delta}_G(S)$ where $\boldsymbol{a}' \in \mathbb{R}^E$ is defined by $a'_{ij} = (-1)^{\delta_{ij}(S)} a_{ij}$. Such an inequality is said to be *switching equivalent* to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$.

Generalizing the cycle inequality in the form of (1), the cycle inequality [4] for the cut polytope $\mathrm{CUT}^\square(G)$ is defined as follows. For a cycle $C \subseteq E$ in G and a subset $F \subseteq C$ with $|F|$ odd,

$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \le |F| - 1. \tag{2}$$

Inequality (2) is switching equivalent to (1), since it is the S-switching of (1) where S is a subset of the nodes in C such that the intersection of C and the cut set defined by S is equal to $F \triangle \{uw\}$. Here $F \triangle \{uw\}$ denotes the symmetric difference of the two sets F and $\{uw\}$.

We say $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ is *permutation-switching equivalent* to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ if they can be transformed to each other by using permutation and/or switching equivalence.

The following proposition is stated as Lemma 26.2.1 and Corollary 26.3.7 in [16].

[1] The set E is sometimes not specified explicitly when E is clear from the context or the choice of E does not make any difference.

Proposition 2. *Let $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ be permutation-switching equivalent inequalities. Then $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is valid (resp. facet inducing) for $\mathrm{CUT}^{\square}(G)$ if and only if $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ is valid (resp. facet inducing) for $\mathrm{CUT}^{\square}(G)$.*

A *root* of an inequality is a cut vector that satisfies it as an equation. The following well-known proposition, which follows from the definition of switching, shows the essential equivalence of the cut cone and polytope.

Proposition 3. *Let $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ be an inequality, valid for $\mathrm{CUT}^{\square}(G)$, which has a root $\boldsymbol{\delta}_G(S)$. Then its S-switching $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is valid for $\mathrm{CUT}(G)$.*

From Theorem 1 and Proposition 2, the following corollary follows immediately.

Corollary 1 ([4]). *1. For a graph $G = (V, E)$, a cycle $C \subseteq E$ in G and a subset $F \subseteq C$ with $|F|$ odd, the cycle inequality (2) is valid for $\mathrm{CUT}^{\square}(G)$.*

2. If C is a chordless cycle, then (2) is facet inducing for $\mathrm{CUT}^{\square}(G)$.

2.2.2 Collapsing

Let uv be an edge of a graph $G = (V, E)$. The intersection of the cut polytope $\mathrm{CUT}^{\square}(G)$ and the hyperplane $x_{uv} = 0$ is linearly isomorphic to the cut polytope $\mathrm{CUT}^{\square}(G/uv)$ where G/uv denotes the contraction of G at the edge uv. We denote by u the node in G/uv representing the edge uv in G. The uv-collapsing of an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is an inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0$ defined by

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } i, j \neq u, \\ a_{uj} & \text{if } i = u,\ uj \in E,\ vj \notin E, \\ a_{vj} & \text{if } i = u,\ uj \notin E,\ vj \in E, \\ a_{uj} + a_{vj} & \text{if } i = u,\ uj, vj \in E, \end{cases}$$

for every edge ij of G/uv.

The following lemma is given as Lemma 26.4.1 (i) in [16].

Lemma 1. *Any collapsing of a valid inequality is valid.*

2.2.3 Lifting operations

The term *lifting* refers to any general operations which derive an inequality valid for a polyhedron P from an inequality valid for a polyhedron $P \cap \{\boldsymbol{x} \mid x_e = 0\}$ for some coordinate e [22]. It is an important way to derive facet inducing inequalities for combinatorial polyhedra. In context of the cut polytope, a lifting operation means an operation which converts an inequality valid for $\mathrm{CUT}^{\square}(G)$ to an inequality valid for $\mathrm{CUT}^{\square}(G')$ where G is obtained by contracting some edges of G'.

Most lifting operations convert an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ to an inequality whose appropriate collapsing is the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$. Such lifting operations are sometimes called *node splitting* (see Section 26.5 of [16]).

The most fundamental lifting operation is zero-lifting. The following definition and theorem about the zero-lifting of inequalities for general graphs are due to De Simone [10].

Definition 4 (Zero-lifting of inequalities). Let $G = (V, E)$ be a subgraph of $G' = (V', E')$. For $\boldsymbol{a} \in \mathbb{R}^E$ and $a_0 \in \mathbb{R}$, the zero-lifting of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is an inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0$ where $\boldsymbol{a}' \in \mathbb{R}^{E'}$ is defined by $a'_{uv} = a_{uv}$ for $uv \in E$ and $a'_{uv} = 0$ for $uv \in E' \setminus E$.

Theorem 2. *Let $G = (V, E)$ be a graph with n nodes ($n \geq 3$) and $G' = (V', E')$ be a graph with $n+1$ nodes $V' = V \cup \{w\}$ such that V induces G in G'. Let $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq a_0$ be the zero-lifting of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ and u be a node of G. Then $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is facet inducing for $\mathrm{CUT}^{\square}(G')$ if the following conditions are met:*

(i) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ *is facet inducing for* $\mathrm{CUT}^{\square}(G)$.

(ii) $\mathrm{N}_{G'}(w) \setminus \{u\} \subseteq \mathrm{N}_G(u)$.

(iii) The support graph $G(\boldsymbol{a})$ of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ has at least three nodes.

Theorem 26.5.1 of [16] is the case of Theorem 2 where G and G' are the complete graphs and $a_0 = 0$. The proof of Theorem 26.5.1 of [16] uses what is called the lifting lemma (Proposition 2.7 of [14] and Lemma 26.5.3 of [16]), which has a wide range of applications.

Lemma 2 (Lifting lemma). *Let $G = (V, E)$ be the complete graph with n nodes $V = \{1, \ldots, n\}$ ($n \geq 3$). Let $G' = (V', E')$ be the complete graph with $n + 1$ nodes $V' = V \cup \{n + 1\}$. Let $\boldsymbol{a} \in \mathbb{R}^E$ and $\boldsymbol{a}' \in \mathbb{R}^{E'}$. Suppose that the following assertions hold.*

(i) The inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq 0$ is facet inducing for $\mathrm{CUT}(G)$ and the inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq 0$ is valid for $\mathrm{CUT}(G')$.

(ii) There exist $|E|-1$ subsets S_j of $V\setminus\{1\}$ such that the cut vectors $\boldsymbol{\delta}_G(S_j)$ are linearly independent roots of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq 0$ and the cut vectors $\boldsymbol{\delta}_{G'}(S_j)$ are roots of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq 0$.

(iii) There exist n subsets T_k of V' with $1 \notin T_k$ and $n+1 \in T_k$ such that the cut vectors $\boldsymbol{\delta}_{G'}(T_k)$ are roots of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq 0$ and the incidence vectors of the sets T_k are linearly independent.

Then the inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq 0$ is facet inducing for $\mathrm{CUT}(G')$.

3 Triangular elimination for general graphs

3.1 Definition and validity

Suppose that we have an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ which is facet inducing for the cut polytope $\mathrm{CUT}^{\square}(G)$ of a graph $G = (V, E)$. We would like to remove an edge uv from G and instead add some nodes and edges, converting the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ to a facet inducing inequality of $\mathrm{CUT}^{\square}(G')$ for the new graph $G' = (V', E')$.

One way to do this is to add a new node w and new edges uw and vw, and add the triangle inequality on u, v and w to eliminate the term x_{uv} from the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$. For simplicity, we restrict ourselves to the case where $a_0 = 0$ and $a_{uv} > 0$. Then the triangle inequality to add is $-a_{uv}x_{uv} + a_{uv}x_{uw} - a_{uv}x_{vw} \leq 0$. This can be seen as a variation of lifting operation since collapsing the node w to v restores the original inequality, though it removes an edge from the underlying graph.

Proposition 4. *Let $G = (V, E)$ be a graph and uv be an edge in G. Let $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq 0$ be a facet inducing inequality of $\mathrm{CUT}(G)$ with $a_{uv} > 0$. Let w be a new node which does not belong to V, and let $G' = (V', E')$ be a graph with $V' = V \cup \{w\}$ and $E' = (E \setminus \{uv\}) \cup \{uw, vw\}$. Then the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} - a_{uv}x_{uv} + a_{uv}x_{uw} - a_{uv}x_{vw} \leq 0$ is facet inducing for $\mathrm{CUT}(G')$.*

Proposition 4 is a special case of Corollary 2.10 (a) of [4]. We will give a direct proof of Proposition 4 here since the proof of Theorem 4 will follow the same steps (though with more complicated details).

Proof. Let $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ be the new inequality. The inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is valid for $\mathrm{CUT}(G'')$, where $G'' = (V', E \cup E')$, since it is the sum of two inequalities $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ and $-a_{uv}x_{uv} + a_{uv}x_{uw} - a_{uv}x_{vw} \le 0$ both of which are valid for $\mathrm{CUT}(G'')$. The inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is also valid for $\mathrm{CUT}(G')$ since it consists of terms corresponding to edges of G', which is a subgraph of G''.

Since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ is facet inducing for $\mathrm{CUT}(G)$, there exist $|E|-1$ subsets $S_1, \dots, S_{|E|-1}$ of $V \setminus \{v\}$ such that the $|E|-1$ cut vectors $\boldsymbol{\delta}_G(S_j)$ are linearly independent roots of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$.

If we collapse the node w to the node v in $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$, we obtain the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$. This implies that $\boldsymbol{\delta}_{G'}(S_j)$ are linearly independent roots of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$. The $|E|-1$ cut vectors $\boldsymbol{\delta}_{G'}(S_j)$ satisfy an equation $x_{vw} = 0$. On the other hand, a cut vector $\boldsymbol{\delta}_{G'}(\{w\})$ is a root of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ with $x_{vw} = 1 \ne 0$. Therefore, the $|E| = |E'|-1$ roots $\boldsymbol{\delta}_{G'}(S_j)$ and $\boldsymbol{\delta}_{G'}(\{w\})$ of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ are linearly independent. This implies that $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is facet inducing for $\mathrm{CUT}(G')$. □

A special case of Theorem 1 (ii) where the graph G is identical to the cycle C may be proved by using Proposition 4 repeatedly as follows.

Corollary 2. *The cycle inequality (1) is facet inducing for* $\mathrm{CUT}(C)$*.*

Proof. The proof is by induction on the length n of the cycle C. If $n = 3$, then the inequality (1) is the triangle inequality and facet inducing for $\mathrm{CUT}(C)$. In case of $n > 3$, we let v be the node in C adjacent of w other than u and apply Proposition 4 with $G = C/vw$, $G' = C$, and $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ is the cycle inequality in C/vw, which is facet inducing for C/vw by the induction hypothesis. □

One question arises here: can we add more edges to G' keeping the property that the new inequality is facet inducing for $\mathrm{CUT}^{\square}(G')$? We will answer this question affirmatively by Theorem 4. The main ingredient of the proof is the notion of triangular elimination, which generalizes the operation described in Proposition 4.

In what follows, we use the following notation and terms. Let $\Delta(u,v;w) = x_{uv} - x_{uw} - x_{vw}$ and $\Delta(u,v,w) = x_{uv} + x_{uw} + x_{vw} - 2$ for any three nodes u, v, w in the graph in question. The notation $\Delta\{u,v,w\}$ ambiguously denotes one of the four triangular forms $\Delta(u,v;w)$, $\Delta(w,v;u)$, $\Delta(u,w;v)$ or $\Delta(u,v,w)$. The *support graph* of a vector $\boldsymbol{a} \in \mathbb{R}^E$ is a subgraph $G(\boldsymbol{a}) = (V(\boldsymbol{a}), E(\boldsymbol{a}))$ of G whose edges are all edges e in G with $a_e \ne 0$ and nodes are all the endpoints of the edges in $E(\boldsymbol{a})$. For a vector $\boldsymbol{a} \in \mathbb{R}^E$, a scalar $a_0 \in \mathbb{R}$ and a subset $F \subseteq E$, we say the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is *completely supported* by F when $E(\boldsymbol{a})$ is included in F.

Definition 5 (Triangular elimination for graphs). Let $G = (V, E)$ be a graph, t be an integer, and let $F = \{u_i v_i \mid i = 1, \dots, t\}$ be any subset of E. The graph $G' = (V', E')$ is a *triangular elimination* of G (with respect to F) if $V' = V \cup \{w_1, \dots, w_t\}$, $E' \supseteq \{w_i u_i, w_i v_i \mid i = 1, \dots, t\}$, and $E' \cap E = E \setminus F$. Here $w_1, \dots, w_t$ are distinct nodes not in V. Node w_i of G' is said to be *associated* with edge $u_i v_i$ of G.

Definition 6 (Triangular elimination for inequalities). Let $G' = (V', E')$ be a triangular elimination of $G = (V, E)$, and suppose we are given $\boldsymbol{a} \in \mathbb{R}^E$, $a_0 \in \mathbb{R}$, $\boldsymbol{a}' \in \mathbb{R}^{E'}$, $a'_0 \in \mathbb{R}$. Then inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ is a *triangular elimination* of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ if for some choices of triangular forms $\Delta_i\{u_i, v_i, w_i\}$, $i = 1, \dots, t$, we have

$$(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} - a'_0 = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} - a_0 + \sum_{i=1}^{t} |a_{u_i v_i}| \Delta_i\{u_i, v_i, w_i\}.$$

The operation in Proposition 4 is the case where $t = 1$, $u_1 = u$, $v_1 = v$, $w_1 = w$, $\Delta_1\{u,v,w\} = \Delta(u,w;v)$, and w has no neighbours other than u and v in G'.

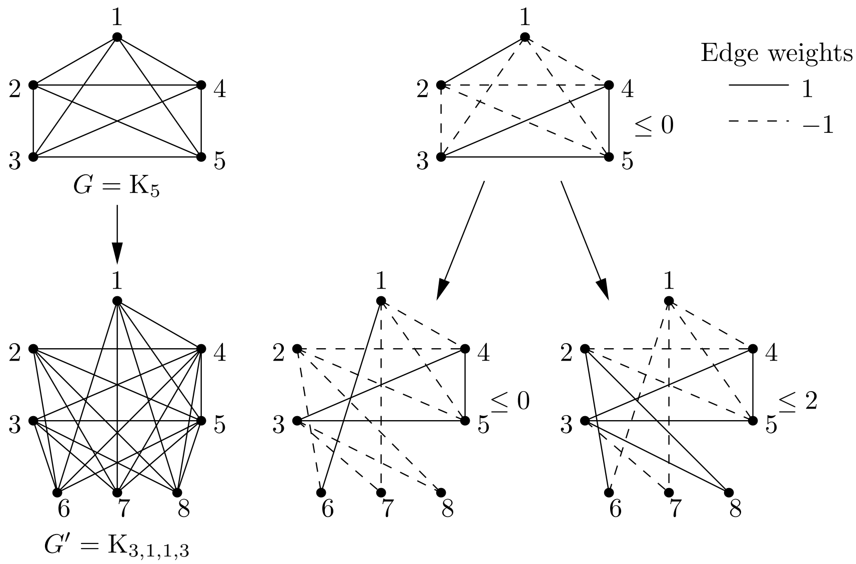

Figure 1: A graph G' and two inequalities obtained as the triangular eliminations of $G = \mathrm{K}_5$ and its facet inducing pentagonal inequality.

Example 1. To understand how triangular elimination typically works, let us consider another, more explicit, example (see Figure 1). Let $G = (V, E)$ be the complete graph K_5 with nodes labelled as $1, \dots, 5$ and $G' = (V', E')$ be the complete 4-partite graph $\mathrm{K}_{3,1,1,3}$ with vertices partitioned as $\{1, 2, 3\}$, $\{4\}$, $\{5\}$ and $\{6, 7, 8\}$. Then G' is a triangular elimination of G with respect to $F = E \setminus E' = \{12, 13, 23\}$, where $u_1v_1 = 12$, $u_2v_2 = 13$, $u_3v_3 = 23$ and $w_i = 5 + i$ for $i = 1, 2, 3$. Let $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ be the pentagonal inequality $x_{12} + x_{34} + x_{35} + x_{45} - \sum_{1 \leq u \leq 2,3 \leq v \leq 5} x_{uv} \leq 0$. Then one of the triangular eliminations $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq a'_0$ of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is obtained by using triangular forms $\Delta\{1, 2, 6\} = \Delta(1, 6; 2)$, $\Delta\{1, 3, 7\} = \Delta(1, 3; 7)$ and $\Delta\{2, 3, 8\} = \Delta(2, 3; 8)$, and it is $-\sum_{1 \leq u \leq 2,4 \leq v \leq 5} x_{uv} + x_{34} + x_{35} + x_{45} + x_{16} - x_{26} - x_{17} - x_{37} - x_{28} - x_{38} \leq 0$. Another triangular elimination $(\boldsymbol{a}'')^{\mathrm{T}}\boldsymbol{x} \leq a''_0$ of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is obtained by using $\Delta\{1, 2, 6\} = \Delta(2, 6; 1)$, $\Delta\{1, 3, 7\} = \Delta(1, 3; 7)$ and $\Delta\{2, 3, 8\} = \Delta(2, 3, 8)$, and it is $-\sum_{1 \leq u \leq 2,4 \leq v \leq 5} x_{uv} + x_{34} + x_{35} + x_{45} - x_{16} + x_{26} - x_{17} - x_{37} + x_{28} + x_{38} \leq 2$.

Triangular elimination of an inequality can also be seen as a specific combination of zero-lifting operation and Fourier-Motzkin elimination. Let $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ be a valid inequality of $\mathrm{CUT}^{\square}(G)$. Consider a graph $G'' = (V', E \cup E')$ and the zero-lifting of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ to $\mathrm{CUT}^{\square}(G'')$. Then apply Fourier-Motzkin elimination to project out the variables x_{uv} for $uv \in F$, adding triangle inequalities to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$. This gives a triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$.

The "if" part of the following theorem is straightforward from the definitions.

Theorem 3. *Let $G' = (V', E')$ be a triangular elimination of $G = (V, E)$, and let $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq a'_0$ be a triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$. Then $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq a'_0$ is valid for $\mathrm{CUT}^{\square}(G')$ if and only if $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is valid for $\mathrm{CUT}^{\square}(G)$.*

Proof of the "if" part of Theorem 3. Let $G'' = (V', E \cup E')$. The inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq a'_0$ is valid for $\mathrm{CUT}^{\square}(G'')$ since it is a sum of an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ and triangle inequalities all of which are valid for $\mathrm{CUT}^{\square}(G'')$. The inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \leq a'_0$ is also valid for $\mathrm{CUT}^{\square}(G')$ since it consists of terms corresponding to edges of G', which is a subgraph of G''. □

We prove the "only if" part in Section 3.2.

3.2 Switching of the triangular elimination

As is shown in Example 1, Definition 6 allows several choices of $\Delta_i\{u_i, v_i, w_i\}$, and different choices give apparently different inequalities. This may complicate handling of triangular elimination. It turns out that if we deal with equivalence classes of inequalities under switching equivalence instead of the inequalities themselves, triangular elimination is easier to handle.

Proposition 5. *Let G' be a triangular elimination of G. Let $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ be a triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b_0'$ be a triangular elimination of $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ such that the association of nodes in G' with edges in G are the same in both triangular eliminations. If $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is switching equivalent to $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$, then $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ is switching equivalent to $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b_0'$.*

Proof. Let

$$((\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} - a_0') - (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} - a_0) = \sum_{1\le i\le t} |a_{u_iv_i}| \Delta_i\{u_i, v_i, w_i\}.$$

First we prove the proposition when $\boldsymbol{a} = \boldsymbol{b}$ and $a_0 = b_0$. In this case, let

$$((\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} - b_0') - (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} - a_0) = \sum_{1\le i\le t} |a_{u_iv_i}| \tilde{\Delta}_i\{u_i, v_i, w_i\}.$$

For $i = 1, \dots, t$, if $a_{u_iv_i} \ne 0$, then $\Delta_i\{u_i, v_i, w_i\}$ and $\tilde{\Delta}_i\{u_i, v_i, w_i\}$ are either identical or the $\{w_i\}$-switching of each other, by comparing their $x_{u_iv_i}$-coefficient. Let S be the set of w_i for i such that $\Delta_i\{u_i, v_i, w_i\}$ and $\tilde{\Delta}_i\{u_i, v_i, w_i\}$ are the $\{w_i\}$-switching of each other. Then two inequalities $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b_0'$ are the S-switching of each other.

Next we prove the general case. Let S be a subset of V such that $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ are the S-switching of each other. Let $(\boldsymbol{b}'')^{\mathrm{T}}\boldsymbol{x} \le b_0''$ be the S-switching of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$. Then $((\boldsymbol{b}'')^{\mathrm{T}}\boldsymbol{x} - b_0'') - (\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} - b_0) \le 0$ is the S-switching of the inequality $((\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} - a_0') - (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} - a_0) \le 0$ and therefore the S-switching of $\sum_{1\le i\le t}|a_{u_iv_i}|\Delta_i\{u_i, v_i, w_i\} \le 0$. This means that $(\boldsymbol{b}'')^{\mathrm{T}}\boldsymbol{x} \le b_0''$ as well as $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b_0'$ is a triangular elimination of $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$. By the case we already proved, $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b_0'$ and $(\boldsymbol{b}'')^{\mathrm{T}}\boldsymbol{x} \le b_0''$ are switching equivalent. Therefore, $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b_0'$ are switching equivalent. □

Example 2 (continued from Example 1). Both inequalities $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ and $(\boldsymbol{a}'')^{\mathrm{T}}\boldsymbol{x} \le a_0''$ described in Example 1) are the triangular eliminations of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$. By Proposition 5, $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ and $(\boldsymbol{a}'')^{\mathrm{T}}\boldsymbol{x} \le a_0''$ are switching equivalent. In fact, they are the $\{6, 8\}$-switching equivalent of each other.

Proposition 5 essentially states that triangular elimination is well-defined as an operation acting on switching-equivalence classes of inequalities. By Proposition 5, we can freely replace $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ with its switching and we do not need to care the choice of Δ_i when we are interested in switching-invariant properties of the inequalities obtained by triangular elimination such as whether it is valid or not, facet inducing or not, and so on. Any properties of inequalities that we deal with in the rest of the paper are switching-invariant.

By using Proposition 5, we can now complete the proof of Theorem 3.

Proof of the "only if" part of Theorem 3. Suppose that $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ is valid for $\mathrm{CUT}^{\square}(G')$. By Proposition 5, we can assume without loss of generality that for $1 \le i \le t$, $\Delta_i\{u_i, v_i, w_i\} = \Delta(u_i, v_i; w_i)$ (if $a_{u_iv_i} \le 0$) or $\Delta_i\{u_i, v_i, w_i\} = \Delta(u_i, w_i; v_i)$ (if $a_{u_iv_i} \ge 0$). Then the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is obtained from $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ by collapsing the node w_i to v_i for every $1 \le i \le t$. This means that the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is also valid. □

3.3 Facets and triangular elimination

We state and prove a sufficient condition for triangular elimination to be facet preserving. Note the similarity to the conditions in Theorem 2.

Theorem 4. *Let $G' = (V', E')$ be a triangular elimination of $G = (V, E)$, and let $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ be a triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$. Then $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ is facet inducing for $\mathrm{CUT}^{\square}(G')$ if the following conditions apply:*

(i) The inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is facet inducing for $\mathrm{CUT}^{\square}(G)$.

(ii) For $i = 1, \dots, t$, $\mathrm{N}_{G'}(w_i) \setminus \{u_i, v_i\} \subseteq \mathrm{N}_G(u_i) \cap \mathrm{N}_G(v_i)$.

(iii) For $i = 1, \dots, t$, the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is not completely supported by the edge set $\{u_i l, v_i l \mid l \in \mathrm{N}_{G'}(w_i)\}$.

Note that condition (ii) implies that the set $\{w_i \mid i = 1, \dots, t\}$ is an independent set in G'.

Example 3 (continued from Example 1). The inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ described in Example 1 is facet inducing for $\mathrm{CUT}^{\square}(G')$, since the graphs G, G' and the inequalities $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0, (\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0'$ satisfy the conditions in Theorem 4.

We prove Theorem 4 in a similar way to the proof of the zero-lifting theorem, Theorem 26.5.1 of [16], as follows. We first introduce a variation of the lifting lemma, Lemma 2, adapted to graphs other than the complete graphs and contraction of multiple edges. Then to prove one of the preconditions of the lemma, we use a lemma from [16].

First we introduce the variation of the lifting lemma.

Lemma 3. *Let $G' = (V', E')$ be a graph and $H = (V', F)$ be a forest in G' with t edges $F = \{v_1 w_1, \dots, v_t w_t\} \subseteq E'$. Let $G = (V, E)$ be the graph obtained from G' by contracting the edges in H. Let $U_i = \mathrm{N}_{G'}(v_i) \cap \mathrm{N}_{G'}(w_i)$. We require that $|E'| = |E| + |U_1| + \dots + |U_t| + t$. Let $\boldsymbol{a} \in \mathbb{R}^E$ and $\boldsymbol{a}' \in \mathbb{R}^{E'}$. Suppose that the following assertions hold.*

(i) The inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ is facet inducing for $\mathrm{CUT}(G)$ and the inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is valid for $\mathrm{CUT}(G')$.

(ii) There exist $|E|-1$ subsets $\tilde{S}_j$ of V such that the cut vectors $\boldsymbol{\delta}_G(\tilde{S}_j)$ are linearly independent roots of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ and the cut vectors $\boldsymbol{\delta}_{G'}(S_j)$ are roots of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$, where $S_j = \tilde{S}_j \cup \{w_i \mid v_i \in \tilde{S}_j\}$.

(iii) For $1 \le i \le t$, there exist $|U_i| + 1$ subsets T_{ik} of V' with $v_i \notin T_{ik}$, $w_i \in T_{ik}$ and $\delta_{v_l w_l}(T_{ik}) = 0$ for $1 \le l \le t$, $l \ne i$ such that the cut vectors $\boldsymbol{\delta}_{G'}(T_{ik})$ are roots of $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ and the incidence vectors of the sets $T_{ik} \cap (U_i \cup \{w_i\})$ are linearly independent.

Then the inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is facet inducing for $\mathrm{CUT}(G')$.

Note that Lemma 2 is a special case of this lemma with $t = 1$, $G = \mathrm{K}_n$, $G' = \mathrm{K}_{n+1}$, $v_1 = 1$ and $w_1 = n + 1$. The proof is similar to the latter half of the proof of Theorem 26.5.1 of [16], though our proof is a little more complicated because we cannot use a correlation cone instead of the cut cone $\mathrm{CUT}(G')$.

Remark 1. The same remark on node splitting as that given below Lemma 26.5.3 of [16] applies for Lemma 3. That is, if the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ comes from $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ by collapsing the nodes w_i to the corresponding nodes v_i, then the assertion (ii) is implied by the assertion (i).

Proof. Note that $|E'| - 1 = (|E| - 1) + (|U_1| + 1) + \dots + (|U_t| + 1)$. We show that $|E'| - 1$ cut vectors $\boldsymbol{\delta}_{G'}(S_j)$ and $\boldsymbol{\delta}_{G'}(T_{ik})$ are linearly independent. Let us consider the $|E'| \times (|E'| - 1)$ matrix M, whose columns are these $|E'| - 1$ cut vectors. We prove that M has full column rank. The rows of M are indexed by the edges in E', which can be grouped as $E' = I \cup \bigcup_{1 \le i \le t} (J_i \cup K_i \cup L_i)$:

- I consists the edges in E' which do not belong to any of the following groups,
- $J_i = \{v_i u \mid u \in U_i\}$,
- $K_i = \{w_i u \mid u \in U_i\}$,
- $L_i = \{v_i w_i\}$.

Note that some edges in E' may belong to more than one set. In that case, we consider that M contains the corresponding rows twice. We can do so since this does not change the rank of M. Then the matrix M is of the form:

$$M = \begin{array}{c} \\ (I) \\ (J_1) \\ (J_2) \\ \vdots \\ (J_t) \\ (K_1) \\ (L_1) \\ (K_2) \\ (L_2) \\ \vdots \\ (K_t) \\ (L_t) \end{array} \begin{array}{c} \begin{array}{ccccc} (S_j) & (T_{1k}) & (T_{2k}) & \cdots & (T_{tk}) \end{array} \\ \begin{pmatrix} X_0 & X_1 & X_2 & \cdots & X_t \\ Y_{01} & Y_{11} & Y_{21} & \cdots & Y_{t1} \\ Y_{02} & Y_{12} & Y_{22} & \cdots & Y_{t2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{0t} & Y_{1t} & Y_{2t} & \cdots & Y_{tt} \\ Y_{01} & \mathbf{1} - Y_{11} & Y_{21} & \cdots & Y_{t1} \\ 0 & \mathbf{1} & 0 & \cdots & 0 \\ Y_{02} & Y_{12} & \mathbf{1} - Y_{22} & \cdots & Y_{t2} \\ 0 & 0 & \mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{0t} & Y_{1t} & Y_{2t} & \cdots & \mathbf{1} - Y_{tt} \\ 0 & 0 & 0 & \cdots & \mathbf{1} \end{pmatrix} \end{array},$$

where $\mathbf{1}$ denotes the all-ones matrix. To prove that M has full column rank, we transform M by reversible linear operations on its row vectors as follows: subtract the rows corresponding to the edge $v_i u$ in J_i from the rows corresponding to the edge $w_i u$ in K_i, subtract the row L_i from each row in K_i, and divide the rows in K_i by -2. Then we obtain:

$$M' = \begin{array}{c} \\ (I) \\ (J_1) \\ (J_2) \\ \vdots \\ (J_t) \\ (K_1) \\ (L_1) \\ (K_2) \\ (L_2) \\ \vdots \\ (K_t) \\ (L_t) \end{array} \begin{array}{c} \begin{array}{ccccc} (S_j) & (T_{1k}) & (T_{2k}) & \cdots & (T_{tk}) \end{array} \\ \begin{pmatrix} X_0 & X_1 & X_2 & \cdots & X_t \\ Y_{01} & Y_{11} & Y_{21} & \cdots & Y_{t1} \\ Y_{02} & Y_{12} & Y_{22} & \cdots & Y_{t2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{0t} & Y_{1t} & Y_{2t} & \cdots & Y_{tt} \\ 0 & Y_{11} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & 0 & \cdots & 0 \\ 0 & 0 & Y_{22} & \cdots & 0 \\ 0 & 0 & \mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Y_{tt} \\ 0 & 0 & 0 & \cdots & \mathbf{1} \end{pmatrix} \end{array}.$$

The leftmost $|E| - 1$ columns of M' have full column rank by assertion (ii). The kth column of the square submatrix $\binom{Y_{ii}}{\mathbf{1}}$ of order $|U_i| + 1$ of M' is the incidence vector of the set $T_{ik} \cap (U_i \cup \{w_i\})$. This implies that $\binom{Y_{ii}}{\mathbf{1}}$ has full rank by assertion (iii). Therefore, M' (and also M) has full column rank. $\square$

The other lemma which we use is Lemma 26.5.2 (ii) of [16]. In [16] the graph G is restricted to the complete graph, but this restriction is not relevant.

Lemma 4 ([16]). *Let $G=(V,E)$ be a graph and $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}\le 0$ be an inequality inducing a facet F of* $\mathrm{CUT}(G)$. *Let D be a subset of E and F' be the projection of $F\subseteq\mathbb{R}^E$ to $\mathbb{R}^D$. If there exists an edge $e\in E\setminus D$ with $a_e\neq 0$, then F' is full-dimensional. Otherwise, the dimension of F' is $|D|-1$.*

Using Lemmas 3 and 4, we prove Theorem 4.

Proof of Theorem 4. By Propositions 3 and 5, we can assume without loss of generality that a_0 is equal to zero and that for $1\le i\le t$, $\Delta_i\{u_i,v_i,w_i\}$ is equal to $\Delta(u_i,v_i;w_i)$ (if $a_{u_iv_i}\le 0$) or $\Delta(u_i,w_i;v_i)$ (if $a_{u_iv_i}\ge 0$). These assumptions imply that neither $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}\le a_0$ nor $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x}\le a'_0$ has a nonzero constant term. By Proposition 1, this means that $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}\le 0$ is facet inducing for the cut cone $\mathrm{CUT}(G)$, and that $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x}\le 0$ is valid for the cut cone $\mathrm{CUT}(G')$.

We prove that $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x}\le 0$ induces a facet of $\mathrm{CUT}(G')$ by using Lemma 3.

The assertion (i) holds by Theorem 3. The assertion (ii) holds since the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}\le 0$ comes from the inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x}\le 0$ by collapsing the nodes w_i to the corresponding nodes v_i. All we need to do is to check the assertion (iii).

Let $1\le i\le t$ and $m=|U_i|$. We define $D_i=\{u_iv_i\}\cup\{u_il,v_il\in E\mid l\in U_i\}$. We construct $m+1$ subsets T_{ik} of V' satisfying the assertion (iii).

Let F be the facet of $\mathrm{CUT}(G)$ induced by $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}\le 0$. By Lemma 4 with $D=D_i$, the projection F_i of F to $\mathbb{R}^{D_i}$ is full-dimensional. This means that we have $|D_i|=2m+1$ subsets $\tilde{T}_{ik}$ of V with $v_i\notin\tilde{T}_{ik}$ such that $\boldsymbol{\delta}_G(\tilde{T}_{ik})$ are roots of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}\le 0$ and the $2m+1$ cut vectors $\boldsymbol{\delta}_{D_i}(\tilde{T}_{ik})$ are linearly independent.

We show that u_i belongs to exactly $m+1$ out of the $2m+1$ sets $\tilde{T}_{ik}$. To show this by contradiction, first suppose that u_i belongs to at most m of them. This means that at least $m+1$ of them does not contain u_i and that the intersection $F_i\cap\{\boldsymbol{x}\in\mathbb{R}^{D_i}\mid x_{u_iv_i}=0\}$ has a dimension at least $m+1$. However, this intersection is contained in the intersection $\mathrm{CUT}(D_i)\cap\{\boldsymbol{x}\in\mathbb{R}^{D_i}\mid x_{u_iv_i}=0\}$,[2] whose dimension is m, a contradiction. Thus u_i belongs to at least $m+1$ out of the $2m+1$ subsets. On the other hand, suppose that u_i belongs to at least $m+2$ of them. If $u_i\in\tilde{T}_{ik}$, then $\boldsymbol{\delta}_{D_i}(\tilde{T}_{ik})$ satisfies equations $x_{u_iv_i}=x_{u_il}+x_{v_il}$ for all $l\in U_i$. This implies that the $(m+1)$-dimensional subspace of $\mathbb{R}^{D_i}$ defined by $x_{u_iv_i}=x_{u_il}+x_{v_il}$ for $l\in U_i$ contains $m+2$ linearly independent vectors, a contradiction. Thus u_i belongs to exactly $m+1$ out of the $2m+1$ sets $\tilde{T}_{ik}$. As a result, we can assume $u_i\in\tilde{T}_{i1},\dots,\tilde{T}_{i,m+1}$ and $u_i\notin\tilde{T}_{i,m+2},\dots,\tilde{T}_{i,2m+1}$ without loss of generality. We define $m+1$ subsets T_{ik} of V' as follows. If $a_{u_iv_i}\le 0$, then let $T_{ik}=\tilde{T}_{ik}\cup\{w_l\mid v_l\in\tilde{T}_{ik}\}\cup\{w_i\}$ for $1\le k\le m+1$. Otherwise, let $T_{ik}=\tilde{T}_{i,m+1+k}\cup\{w_l\mid v_l\in\tilde{T}_{i,m+1+k}\}\cup\{w_i\}$ for $1\le k\le m$ and $T_{i,m+1}=\{w_i\}$.

Now we prove that the incidence vectors of $m+1$ sets $T_{ik}\cap(U_i\cup\{w_i\})$ are linearly independent. Let M be the $(2m+1)\times(m+1)$ matrix whose kth column vector is the cut vector $\boldsymbol{\delta}_{D_i}(T_{ik})$, and M' be the square matrix of order $m+1$ whose kth column vector is the incidence vector of $T_{ik}\cap(U_i\cup\{w_i\})$. We prove that M' is nonsingular. The matrix M' is of the form $M'=\binom{X}{\mathbf{1}}$, where the bottommost row corresponds to the node w_i. The rows of M correspond to the edges in D_i which are grouped as $D_i=J\cup K\cup L$: $J=\{u_il\mid l\in U_i\}$, $K=\{v_il\mid l\in U_i\}$ and $L=\{u_iv_i\}$. If $a_{u_iv_i}\le 0$, then the matrix M is given by:

$$M=\begin{matrix}(J)\\(K)\\(L)\end{matrix}\begin{pmatrix}\mathbf{1}-X\\X\\\mathbf{1}\end{pmatrix},$$

[2] Here we denote by $\mathrm{CUT}(D_i)$ the cut cone of the graph with edges D_i and nodes V or any subset of V that contains $U_i\cup\{u_i,v_i\}$. We justify this slight abuse of the notation by the fact that adding isolated nodes to a graph does not change the cut cone.

and it has full column rank by assumption. Without decreasing its rank, we can transform M to M' by reversible linear operations on rows and removing all-zero rows. This means M' is nonsingular. Similarly, if $a_{u_i v_i} > 0$, then the matrix M is given by:

$$M = \begin{matrix} (J) \\ (K) \\ (L) \end{matrix} \begin{pmatrix} X \\ X \\ \mathbf{0} \end{pmatrix}.$$

Its leftmost m columns are linearly independent and its rightmost column is the all-zero vector by assumption. By a similar argument as above, the leftmost m columns of X are linearly independent. This implies the $m+1$ column vectors of X are affinely independent, or equivalently the matrix M' is nonsingular. This means the assertion (iii) is satisfied. □

To make Theorem 4 easier to use, we show that condition (iii) in Theorem 4 holds for any facet inducing inequalities except for the triangle inequality and the inequality of the forms $x_e \geq 0$ and $x_e \leq 1$.

Proposition 6. *Let $G = (V, E)$ be a graph and $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ be a facet inducing inequality of $\mathrm{CUT}^{\square}(G)$. Let $u_1v_1, \ldots, u_tv_t$ be t distinct edges of G. If the support graph of the vector $\boldsymbol{a}$ has more than three nodes, then the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is not completely supported by the edge set $\{u_il, v_il \mid l \in \mathrm{N}_G(u_i) \cap \mathrm{N}_G(v_i)\}$ for any $i = 1, \ldots, t$.*

To prove Proposition 6, we need the following lemma.

Lemma 5. *Let $G = (V, E)$ be a graph, $\boldsymbol{a} \in \mathbb{R}^E$ be a vector, and $a_0 \in \mathbb{R}$ be a scalar. Suppose the following assumptions hold.*

(i) G contains a triangle on nodes l, u, v as a subgraph.

(ii) At least one of a_{lu} and a_{lv} is nonzero.

(iii) For any node $i \in \mathrm{N}_G(l) \setminus \{u, v\}$, $a_{li} = 0$.

Then the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is not facet inducing for $\mathrm{CUT}^{\square}(G)$ unless it is a triangle inequality on l, u, v.

Proof. The proof is by contradiction. Suppose the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is facet inducing for $\mathrm{CUT}^{\square}(G)$ but it is not a triangle inequality on l, u, v.

First we consider the case where $a_{lu} = -\lambda \leq 0$ and $a_{lv} = -\mu \leq 0$. Without loss of generality, we assume that $\lambda \leq \mu$. Then the inequality

$$\lambda x_{uv} - \lambda x_{lu} - \mu x_{lv} = \lambda(x_{uv} - x_{lu} - x_{lv}) - (\mu - \lambda)x_{lv} \leq 0 \tag{3}$$

is valid for $\mathrm{CUT}^{\square}(G)$. By assumption (ii), λ and μ are not both zero, and the left hand side of the inequality (3) is not identically zero.

The inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is the sum of (3) and an inequality

$$\sum_{ij \in E \setminus \{lu, lv, uv\}} a_{ij}x_{ij} + (a_{uv} - \lambda)x_{uv} \leq a_0. \tag{4}$$

By assumption (iii), the node l is not used in the inequality (4). The inequality (4) comes from the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ by collapsing the node l to the node v, and is therefore valid for $\mathrm{CUT}^{\square}(G)$. Therefore, the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is a sum of two valid inequalities. By our assumption that the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq a_0$ is facet inducing for $\mathrm{CUT}^{\square}(G)$, the inequality (4) is identically zero (especially

$a_0 = 0$) and the inequality (3) is facet inducing for $\mathrm{CUT}^{\square}(G)$. The inequality (3) is facet inducing only if $\lambda = \mu$, and if this holds, then the inequality (3) is a triangle inequality on l, u, v. This means that $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is the triangle inequality. This contradicts our assumption.

Now we consider the cases where at least one of a_{lu} or a_{lv} is positive. Switching the inequality on an appropriate subset of $\{u, v\}$, we can make both a_{lu} and a_{lv} nonpositive. This reduces general cases to the case where $a_{lu} \le 0$ and $a_{lv} \le 0$ hold. $\square$

Proof of Proposition 6. Suppose the contrary: the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is completely supported by the edge set $\{u_i l, v_i l \mid l \in \mathrm{N}_G(u_i) \cap \mathrm{N}_G(v_i)\}$. Since $G(\boldsymbol{a})$ has more than three nodes, there exists a node $l \in U_i \setminus \{u_i, v_i\}$ such that at least one of $a_{u_i l}$ and $a_{v_i l}$ is nonzero. By Lemma 5 with $u = u_i$ and $v = v_i$, the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is not facet inducing for $\mathrm{CUT}^{\square}(G)$, a contradiction. $\square$

4 Triangular elimination from K_n

Triangular elimination from the complete graph to another graph is useful because much is known about facets of the cut polytope of the complete graph.

4.1 Facets and triangular elimination from K_n

Theorem 4 provides a sufficient condition for an inequality obtained by triangular elimination to be facet inducing. We prove another sufficient condition when G is the complete graph.

Theorem 5. *Let $G = (V, E)$ be the complete graph on n nodes with $n \ge 5$. Let $V = V_1 \cup \cdots \cup V_m$ be a partition of V to m disjoint sets of nodes. We denote by $E_l = \{u_{l1}v_{l1}, \dots, u_{lt_l}v_{lt_l}\}$ the set of edges in the clique on V_l, where $t_l = |E_l| = \binom{|V_l|}{2}$. Let $F = E_1 \cup \cdots \cup E_m$. Let $G' = (V', E')$ be a graph with $n + \sum_{1 \le l \le m} t_l$ nodes. n nodes in G' are labelled by V, and we group the other nodes into m sets $W_1, \dots, W_m$ with $|W_l| = t_l$. We denote the nodes in W_l by $w_{l1}, \dots, w_{lt_l}$. If the following conditions apply, then G' is a triangular elimination of G with respect to F associating node w_{li} with edge $u_{li}v_{li}$, and the triangular elimination of any non-triangle facet inducing inequality for $\mathrm{CUT}^{\square}(G)$ is facet inducing.*

(i) The subgraph of G' induced by V is the complete m-partite graph $\mathrm{K}_{|V_1|,\dots,|V_m|}$ whose nodes are partitioned as $V_1, \dots, V_m$.

(ii) For $l = 1, \dots, m$, W_l is an independent set in G'.

(iii) For $l = 1, \dots, m$ and $i = 1, \dots, t_l$, $u_{li}w_{li}, v_{li}w_{li} \in E'$.

Proof. By conditions (i) and (iii), it is straightforward to check that G' is a triangular elimination of G with respect to F.

Let $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ be a facet inducing inequality of $\mathrm{CUT}^{\square}(G)$ which is not the triangle inequality, and $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ be a triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$. Similarly to the proof of Theorem 4, without loss of generality, we assume $a_0 = a'_0 = 0$ and that no triangle forms used in the process of the triangular elimination has a nonzero constant term.

The idea is to apply Theorem 4 m times to convert $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ of $\mathrm{CUT}^{\square}(G)$ to $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ of $\mathrm{CUT}^{\square}(G^{(m)})$ where G' is a subgraph of $G^{(m)}$, and then project the resulting facet to a facet of $\mathrm{CUT}^{\square}(G')$ by using Lemma 4.

First we define intermediate graphs $G^{(l)} = (V^{(l)}, E^{(l)})$ and inequalities $(\boldsymbol{a}^{(l)})^{\mathrm{T}}\boldsymbol{x} \le 0$ for $l = 0, 1, \dots, m$. Let $G^{(0)} = G$ and $\boldsymbol{a}^{(0)} = \boldsymbol{a}$. For $l = 1, \dots, m$, $G^{(l)} = (V^{(l)}, E^{(l)})$ is defined by $V^{(l)} = V^{(l-1)} \cup W_l$ and $E^{(l)} = (E^{(l-1)} \setminus E_l) \cup \{vw \mid v \in V^{(l-1)}, w \in W_l\}$. Then $G^{(l)}$ is a triangular elimination of $G^{(l-1)}$ with respect to E_l where node $w_{li} \in W_l$ of $G^{(l)}$ is associated with edge $u_{li}v_{li} \in E_l$ of $G^{(l-1)}$.

Let $(\boldsymbol{a}^{(l)})^{\mathrm{T}}\boldsymbol{x} \le 0$ be a triangular elimination of $(\boldsymbol{a}^{(l-1)})^{\mathrm{T}}\boldsymbol{x} \le 0$.

$(\boldsymbol{a}^{(0)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is facet inducing for $\mathrm{CUT}^{\square}(G^{(0)})$, and the support graph of $(\boldsymbol{a}^{(0)})^{\mathrm{T}}\boldsymbol{x} \le 0$ has more than three nodes. Since triangular elimination never decreases the number of nodes of the support graph of an inequality, the support graph of $(\boldsymbol{a}^{(l)})^{\mathrm{T}}\boldsymbol{x} \le 0$ has more than three nodes for $l = 1, \dots, m$. By applying Proposition 6 and Theorem 4 m times, $(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is facet inducing for $\mathrm{CUT}^{\square}(G^{(m)})$.

$(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is a triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ since

$$(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = \sum_{1 \le l \le m} ((\boldsymbol{a}^{(l)})^{\mathrm{T}}\boldsymbol{x} - (\boldsymbol{a}^{(l-1)})^{\mathrm{T}}\boldsymbol{x})$$

is the sum of the triangular forms used in m applications of triangular elimination. This combined with Proposition 5 implies that $(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is switching equivalent to $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$.

By conditions (i) and (ii), G' is a subgraph of $G^{(m)}$. The support graph of $(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is a subgraph of a graph $G'' = (V', E'')$ obtained from $\mathrm{K}_{|V_1|,\dots,|V_m|}$ by adding nodes in $W_1 \cup \dots \cup W_m$ and edges $u_{li}w_{li}, v_{li}w_{li}$ for $l = 1, \dots, m$ and $i = 1, \dots, t_l$. By conditions (i) and (iii), this support graph is a subgraph of G'. By Lemma 4, the dimension of the face of $\mathrm{CUT}^{\square}(G')$ defined by $(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is $|E'| - 1$, which implies that $(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is facet inducing for $\mathrm{CUT}^{\square}(G')$. Since $(\boldsymbol{a}^{(m)})^{\mathrm{T}}\boldsymbol{x} \le 0$ is switching equivalent to $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$, $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is also facet inducing for $\mathrm{CUT}^{\square}(G')$. □

Remark 2. If $a_{u_{li}v_{li}} = 0$ for some edge $u_{li}v_{li} \in F$, then the associated node w_{li} is not used in the triangular elimination $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$, and the triangular elimination becomes facet inducing for $\mathrm{CUT}^{\square}(G' - w_{li})$, where $G' - w_{li}$ denotes a graph obtained by removing node w_{li} and edges incident to it from G'.

Corollary 3. *Let $G = (V, E)$, V_l, E_l, F, W_l and V' as stated in Theorem 5. We partition V' into k ($m \le k \le 2m$) disjoint sets $V'_1, \dots, V'_k$, and let $G' = (V', E')$ be the complete k-partite graph with vertices partitioned into the sets $V'_1, \dots, V'_k$. If the following conditions are satisfied, then G' is a triangular elimination of G, and the triangular elimination of any non-triangle facet inducing inequality for $\mathrm{CUT}^{\square}(G)$ is facet inducing.*

(i) For $l = 1, \dots, m$, V_l and W_l are completely contained in some V'_i and V'_j, respectively, and $i \ne j$.

(ii) For $1 \le l < l' \le m$, V_l and $V_{l'}$ are contained in different sets V'_i and V'_j ($i \ne j$).

Theorem 2.1 of [2] is the special case of Corollary 3 with $m = k = 3$, $|V_3| = 1$ (which implies $W_3 = \varnothing$), and G' is the complete tripartite graph with nodes partitioned into three sets $V_1 \cup W_2$, $V_2 \cup W_1$ and V_3, except that Theorem 2.1 of [2] also deals with the triangular elimination of the triangle inequality.

4.2 Triangular elimination from K_n to $\mathrm{K}_{r,s}$ and equivalence of inequalities

Here we focus on the case $m = k = 2$ in Corollary 3, and we consider how Proposition 5 extends to include permutation equivalence of inequalities. Before that, we restate Corollary 3 in this case.

Corollary 4. *Let $G = (V, E)$ be the complete graph on $n = p + q \ge 5$ nodes $\mathrm{A}_1, \dots, \mathrm{A}_p, \mathrm{B}_1, \dots, \mathrm{B}_q$, and let $G' = (V', E')$ be the complete bipartite graph $\mathrm{K}_{r,s}$ with $r = p + \binom{q}{2}$, $s = q + \binom{p}{2}$ where the nodes are partitioned into $\{\mathrm{A}_i \mid 1 \le i \le p\} \cup \{\mathrm{A}_{jj'} \mid 1 \le j < j' \le q\}$ and $\{\mathrm{B}_j \mid 1 \le j \le q\} \cup \{\mathrm{B}_{ii'} \mid 1 \le i < i' \le p\}$. Then G' is a triangular elimination of G with respect to $F = \{\mathrm{A}_i\mathrm{A}_{i'} \mid 1 \le i < i' \le p\} \cup \{\mathrm{B}_j\mathrm{B}_{j'} \mid 1 \le j < j' \le q\}$, and the triangular elimination of any non-triangle facet inducing inequality for $\mathrm{CUT}^{\square}(G)$ is facet inducing.*

Even if two facet inducing inequalities $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ of $\mathrm{CUT}^{\square}(G)$ are equivalent up to permutation and switching, their triangular eliminations to $\mathrm{CUT}^{\square}(G')$ are generally not, since different edges in G may be treated in different ways in the course of triangular elimination. However, if we consider triangular elimination from $\mathrm{CUT}^{\square}(\mathrm{K}_n)$ to $\mathrm{CUT}^{\square}(\mathrm{K}_{r,s})$ as described in Corollary 4, then we know exactly when the triangular eliminations of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ are equivalent up to permutation and switching.

Theorem 6. *Let $n = p + q \ge 5$, $r = p + \binom{q}{2}$ and $s = q + \binom{p}{2}$, and label the nodes of K_n and $\mathrm{K}_{r,s}$ as described in Corollary 4. For two non-triangle facet inducing inequalities $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ of $\mathrm{CUT}^{\square}(\mathrm{K}_n)$ and their respective triangular eliminations $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b'_0$ to $\mathrm{CUT}^{\square}(\mathrm{K}_{r,s})$, the following two conditions are equivalent.*

(a) The two inequalities $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ can be transformed to each other by applying some combination of the switching operation, the permutation operation within $\{\mathrm{A}_1, \dots, \mathrm{A}_p\}$ or within $\{\mathrm{B}_1, \dots, \mathrm{B}_q\}$, and if $p = q$, the permutation operation swapping A_i and B_i for all i.

(b) The two inequalities $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b'_0$ are permutation-switching equivalent.

Remark 3. Condition (a) implies condition (b) even if $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ are the triangle inequality, but the converse does not hold. Here is a counterexample: let $p = 2$ and $q = 3$. Let $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ be the triangle inequalities $-x_{\mathrm{A}_1\mathrm{A}_2} - x_{\mathrm{A}_1\mathrm{B}_1} + x_{\mathrm{A}_2\mathrm{B}_1} \le 0$ and $-x_{\mathrm{A}_1\mathrm{B}_1} + x_{\mathrm{A}_1\mathrm{B}_2} - x_{\mathrm{B}_1\mathrm{B}_2} \le 0$, respectively, and $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b'_0$ be inequalities $-x_{\mathrm{A}_1\mathrm{B}_1} + x_{\mathrm{A}_2\mathrm{B}_1} - x_{\mathrm{A}_1\mathrm{B}_{12}} - x_{\mathrm{A}_2\mathrm{B}_{12}} \le 0$ and $-x_{\mathrm{A}_1\mathrm{B}_1} + x_{\mathrm{A}_1\mathrm{B}_2} - x_{\mathrm{A}_{12}\mathrm{B}_1} - x_{\mathrm{A}_{12}\mathrm{B}_2} \le 0$. Note that the condition (a) does not hold since $p \ne q$, whereas $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b'_0$ are permutation-switching equivalent and the condition (b) is satisfied.

Now we give a proof of Theorem 6.

Proof. First we prove (a) $\implies$ (b). If $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ are switchings of each other, then their triangular eliminations are also switching of each other by Proposition 5.

If $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is transformed to $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ by swapping A_i and $\mathrm{A}_{i'}$, then the triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is transformed to the triangular elimination of $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ by swapping A_i and $\mathrm{A}_{i'}$ and swapping $\mathrm{B}_{ii''}$ and $\mathrm{B}_{i'i''}$ for all $i'' \ne i, i'$, if we also apply this permutation to Δ_r for $1 \le r \le t$.

If $p = q$ and $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is transformed to $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ by swapping A_i and B_i for $1 \le i \le p$ at the same time, then the triangular elimination of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ is transformed to the triangular elimination of $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ by swapping A_i and B_i for $1 \le i \le p$ and swapping $\mathrm{A}_{ii'}$ and $\mathrm{B}_{ii'}$ for $1 \le i < i' \le p$ at the same time.

Next we prove (b) $\implies$ (a). Let $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a'_0$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b'_0$ be the triangular eliminations of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le b_0$ from $\mathrm{CUT}^{\square}(\mathrm{K}_n)$ to $\mathrm{CUT}^{\square}(\mathrm{K}_{r,s})$, respectively. We require that the triangle inequality $\Delta(u_i, v_i, w_i)$ was not used in triangular elimination to produce these two inequalities. Then $a'_0 = a_0$ and $b'_0 = b_0$. In addition, this requirement guarantees $a_{\mathrm{A}_i\mathrm{A}_{i'}} = \max\{a'_{\mathrm{A}_i\mathrm{B}_{ii'}}, a'_{\mathrm{A}_{i'}\mathrm{B}_{ii'}}\}$ and $a_{\mathrm{B}_j\mathrm{B}_{j'}} = \max\{a'_{\mathrm{A}_{jj'}\mathrm{B}_j}, a'_{\mathrm{A}_{jj'}\mathrm{B}_{j'}}\}$, and similar equations for the vectors $\boldsymbol{b}$ and $\boldsymbol{b}'$. By Proposition 5, we only need to consider the case where $a_0 = b_0 = 0$ and $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le a_0$ and $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le b_0$ are equivalent up to permutation.

The key to proving the assertion (b) $\implies$ (a) is that from Lemma 5, we can distinguish the nodes A_i from the nodes $\mathrm{A}_{jj'}$ by examining the inequality $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$.

We prove the assertion (a) holds by case analysis on the permutation used to transform $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ to $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le 0$. The automorphism group of $\mathrm{K}_{r,s}$ is generated by permutations within $\{\mathrm{A}_1, \dots, \mathrm{A}_p, \mathrm{A}_{12}, \dots, \mathrm{A}_{q-1,q}\}$, permutations within $\{\mathrm{B}_1, \dots, \mathrm{B}_q, \mathrm{B}_{12}, \dots, \mathrm{B}_{p-1,p}\}$, and if $r = s$, the permutation τ_0 which swaps A_i and B_i for $1 \le i \le p$ and swaps $\mathrm{A}_{ii'}$ and $\mathrm{B}_{ii'}$ for $1 \le i < i' \le p$ at the same time. Since $p + q \ge 5$, $r = s$ if and only if $p = q$.

If $p = q$ and $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is transformed to $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le 0$ by the permutation τ_0, then $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ is transformed to $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le 0$ by swapping A_i and B_i for $1 \le i \le p$ at the same time. Therefore, from now on, we can assume that $(\boldsymbol{a}')^{\mathrm{T}}\boldsymbol{x} \le 0$ is transformed to $(\boldsymbol{b}')^{\mathrm{T}}\boldsymbol{x} \le 0$ by a permutation τ which permutes nodes within $\{\mathrm{A}_1, \dots, \mathrm{A}_p, \mathrm{A}_{12}, \dots, \mathrm{A}_{q-1,q}\}$ and nodes within $\{\mathrm{B}_1, \dots, \mathrm{B}_q, \mathrm{B}_{12}, \dots, \mathrm{B}_{p-1,p}\}$.

Recall that the support graph of a vector $\boldsymbol{a}$ is a subgraph $G(\boldsymbol{a}) = (V(\boldsymbol{a}), E(\boldsymbol{a}))$ of G with all the edges e in G with $a_e \ne 0$ as its edges and all the endpoints of edges in $E(\boldsymbol{a})$ as its nodes. From Lemma 5, all the nodes in $G(\boldsymbol{a})$ have a degree more than two. Since triangular elimination does not change the degree of existing nodes in the support graph, the nodes A_i and B_j, if present in $G(\boldsymbol{a}')$, have degree more than two in $G(\boldsymbol{a}')$. On the other hand, from the definition of triangular elimination, the nodes $\mathrm{A}_{jj'}$ and $\mathrm{B}_{ii'}$, if present in $G(\boldsymbol{a}')$, have degree equal to two. Therefore, we can partition the nodes of $\mathrm{K}_{r,s}$ into three groups: V_1 consists of those which do not appear in $G(\boldsymbol{a}')$, V_2 consists of those with degree equal to two, and V_3 consists of those with degree more than two. The nodes A_i belong to V_1 or V_3, and the nodes $\mathrm{A}_{jj'}$ belong to V_1 or V_2. The same argument applies to $G(\boldsymbol{b}')$, and we partition the nodes of $\mathrm{K}_{r,s}$ into W_1, W_2 and W_3 in a parallel way. The permutation τ maps V_1 to W_1, V_2 to W_2 and V_3 to W_3, respectively. We define a permutation σ on $\{\mathrm{A}_1, \dots, \mathrm{A}_p, \mathrm{B}_1, \dots, \mathrm{B}_q\}$ as follows. If $\mathrm{A}_i \in V_3$, then let $\sigma(\mathrm{A}_i) = \tau(\mathrm{A}_i)$. If $\mathrm{B}_j \in V_3$, then let $\sigma(\mathrm{B}_j) = \tau(\mathrm{B}_j)$. The rest of σ is defined so that σ maps the nodes in V_1 of the form A_i to the nodes in W_1 of the form A_i, and the nodes in V_1 of the form B_j to the nodes in W_1 of the form B_j.

We show that σ maps $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ to $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le 0$. All we have to prove is that for all edges uv in K_n, we have $a_{uv} = b_{\sigma(u)\sigma(v)}$. If u belongs to V_1, then $\sigma(u) \in W_1$, and we have $a_{uv} = b_{\sigma(u)\sigma(v)} = 0$. Since the same applies for the case $v \in V_1$, we only need to consider the case where both u and v belongs to V_3. In this case, $\sigma(u) = \tau(u)$ and $\sigma(v) = \tau(v)$. If $u = \mathrm{A}_i$ and $v = \mathrm{B}_j$, then $a_{\mathrm{A}_i\mathrm{B}_j} = a'_{\mathrm{A}_i\mathrm{B}_j} = b'_{\tau(\mathrm{A}_i)\tau(\mathrm{B}_j)} = b_{\tau(\mathrm{A}_i)\tau(\mathrm{B}_j)} = b_{\sigma(\mathrm{A}_i)\sigma(\mathrm{B}_j)}$. If $u = \mathrm{A}_i$ and $v = \mathrm{A}_{i'}$, then $a_{\mathrm{A}_i\mathrm{A}_{i'}} = \max\{a'_{\mathrm{A}_i\mathrm{B}_{ii'}}, a'_{\mathrm{A}_{i'}\mathrm{B}_{ii'}}\} = \max\{b'_{\tau(\mathrm{A}_i)\tau(\mathrm{B}_{ii'})}, b'_{\tau(\mathrm{A}_{i'})\tau(\mathrm{B}_{ii'})}\} = b_{\tau(\mathrm{A}_i)\tau(\mathrm{A}_{i'})} = b_{\sigma(\mathrm{A}_i)\sigma(\mathrm{A}_{i'})}$. The same applies to the case where $u = \mathrm{B}_j$ and $v = \mathrm{B}_{j'}$. Therefore, $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \le 0$ is transformed to $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} \le 0$ by the permutation σ. □

5 Concluding remarks

Theorems 4 and 5 are sufficient conditions for a triangular elimination of a facet inducing inequality to be facet inducing. An open problem is: what are necessary and sufficient conditions on graphs G and G' for a triangular elimination of a non-triangle facet inducing inequality to be facet inducing? Extending Theorem 6 to general graphs is another open problem.

Acknowledgments

The third author is supported by the Grant-in-Aid for JSPS Fellows. We would like to thank an anonymous referee for pointing out several shortcomings in the original manuscript.

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