A Variational Boundary Element Formulation for Shear-Deformable Plate Bending Problems

Taha H. A. Naga
Assistant Lecturer
Faculty of Engineering (Shoubra),
Department of Engineering
Mathematics and Physics,
Benha University,
Benha, Egypt

Youssef F. Rashed[1]
Professor
Deputy Secretary General,
Supreme Council of Universities in Egypt;
Department of Structural Engineering,
Cairo University,
Giza, Egypt
e-mail: youssef@eng.cu.edu.eg

This paper presents the derivation of a new boundary element formulation for plate bending problems. The Reissner's plate bending theory is employed. Unlike the conventional direct or indirect formulations, the proposed integral equation is based on minimizing the relevant energy functional. In doing so, variational methods are used. A collocation based series, similar to the one used in the indirect discrete boundary element method (BEM), is used to remove domain integrals. Hence, a fully boundary integral equation is formulated. The main advantage of the proposed formulation is production of a symmetric stiffness matrix similar to that obtained in the finite element method. Numerical examples are presented to demonstrate the accuracy and the validity of the proposed formulation. [DOI: 10.1115/1.4023623]

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1 Introduction

Integral forms in solid mechanics could be categorized into three main categories:

(1) Integral forms based on superposition: In such forms, the effect of several sources (represented by the fundamental solution, single or double-layer potentials) is computed at a series of field points by means of collocation. The indirect boundary element methods [1] or its discrete version (the method of the fundamental solutions, see, for example, Antes [2] and Cho et al. [3]) is a direct implementation of such forms. Higher order derivatives of these integral forms could be found in the literature under the name of dipole formulation as reported by Fam and Rashed [4] for potential problems and by Mohareb and Rashed [5] for Reissner's plates. A continuous collocation version of such integral forms was presented by Mitic and Rashed [6] and Fam and Rashed [7]. A version of this integral form could be formulated based on the displacement discontinuity as in the textbook of Crouch and Starfield [8]. Other alternative indirect boundary element formulations based on similar integral forms could be found in Patton and Perlin [9] and Tran-Cong et al. [10]. It has to be noted that it is difficult to prove uniqueness and convergence of matrix equations generated from these integral forms, as generally speaking, the number and places of potential sources are not unique for a certain problem.

(2) Integral forms based on two states or the virtual work statements: In such forms, a work statement is written between two states, the actual problem state and a virtual state. In the case of considering the direct boundary integral equations (see for example Ref. [11]), the considered virtual state is the fundamental solution state. Higher derivative of such integral forms are given in forms of traction integral equations or the hyper-singular integral equations (Rashed et al. [12,13]). The equivalency of the direct and the indirect integral forms was considered by Brebbia and Butterfield [14] and was written later in the textbook of Brebbia and Walker [15]. The equivalency between the displacement discontinuity method and the hyper-singular integral equation form was reported by Kuhn et al. [16]. A continuous collocation scheme could be used in these integral forms to formulate the so-called Galerkin direct boundary element (see, for example, Perez-Gavilan and Aliabadi [17]). It has to be noted that several virtual cases could be considered, and for each case the virtual work equation could be written to generate the matrix equation [18]. The number of the virtual states and places of the load points in such states is not unique; therefore, it is also difficult to prove the uniqueness and convergence for these integral forms.

(3) Integral forms based on stationary conditions of certain functionals (see, for example, Reddy [19]). In solid mechanics, in particular, one of these functionals is the potential energy functional. Several other functionals could be considered, such as the Reissner functional and the Hu functional [20]. Most of finite element formulations are based on these integral forms [19]. Some boundary element formulations are based on such functionals. For example, potential and elasticity problems were reported by DeFigueiredo [21], Dumont [22,23] and Liu et al. [24]. The only relevant boundary element textbook that considered such formulations is the book of Gaul et al. [25]. The main advantage of such boundary element formulations is they produce symmetric and positive definite stiffness matrices. Moreover, uniqueness and convergence proofs could be justified in terms of energy [26].

A complete description of the historical developments of most of these formulations is given by Cheng and Cheng [27]. Concerning the application of the BEM for thick plate bending problems, the direct boundary element formulation was originally developed by Vander Weeën [28]. Hence, several applications were considered based on this theory; for example, Barcellos and Silva [29] extended the formulation to Mindlin plates. El-Zafrany et al. [30] divided the formulation into kernels for thin and others for thick plates. Ribeiro and Venturini [31] discussed the application of

[1]Corresponding author.

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elastoplastic analysis to the direct formulation. Westphal et al. [32] studied the fundamental solution used in plates. Marczak and Creus [33] considered the evaluation of singular integrals in the direct integral equation formulation. Fernandes and Konda [34] coupled the formulation with beams. To the authors' best knowledge, none of these formulations considered a variational boundary integral formulation for the thick Reissner's plate bending.

In this paper, a new variational boundary element formulation for the Reissner plate bending problems is derived based on minimizing the relevant energy functional. A collocation based series is used to remove involved domain integrals. Numerical examples are presented to demonstrate the accuracy and the validity of the proposed formulation. This formulation could be used to develop boundary element based super finite elements; it also could be linked to finite elements in a straightforward way.

2 Basic Equations

In this section, the basic equations of the Reissner plate bending theory are reviewed. Throughout this paper, the indicial notation is used. Greek indices vary from 1 to 2, whereas Roman indices vary from 1 to 3. The comma subscript is used to denote differentiation; for example, $(\bullet)_{,\alpha}$ stands for the derivative of $(\bullet)$ with respect to the coordinate $(x_\alpha(\mathbf{x}))$ in which $(\mathbf{x})$ is a general point. Consider an arbitrary plate of thickness (h) has a domain (Ω) and boundary (Γ) in the (x_i) space. The $(x_1\text{-}x_2)$ plane is assumed to be located at the middle surface where $(x_3 = 0)$. The notation $\Gamma(\mathbf{x},....)$ denotes the boundary Γ that contain points $(\mathbf{x},....)$; similar notation is also applicable for the domain (Ω). Part of this boundary is $\Gamma_P(\mathbf{x})$, at which traction is prescribed. The generalized strain-displacement relationships are [35,36]

$$\chi_{\alpha\beta}(\mathbf{y}) = \frac{1}{2}\left(u_{\alpha,\beta}(\mathbf{y}) + u_{\beta,\alpha}(\mathbf{y})\right) \tag{1}$$

$$\psi_{3\alpha}(\mathbf{y}) = u_\alpha(\mathbf{y}) + u_{3,\alpha}(\mathbf{y}) \tag{2}$$

where $\chi_{\alpha\beta}(\mathbf{y}), \psi_\alpha(\mathbf{y})$are the flexure and the transverse shear strains at a point $(\mathbf{y} \in \Omega)$, respectively. The generalized stress resultants-strains relationships are [35,36]

$$M_{\alpha\beta}(\mathbf{y}) = D(1-v)\left(\chi_{\alpha\beta}(\mathbf{y}) + \frac{v}{1-v}\chi_{\gamma\gamma}(\mathbf{y})\delta_{\alpha\beta}\right) + \frac{v b_3(\mathbf{y})}{(1-v)\lambda^2}\delta_{\alpha\beta} \tag{3}$$

$$Q_{3\alpha}(\mathbf{y}) = D\frac{1-v}{2}\lambda^2\psi_{3\alpha}(\mathbf{y}) \tag{4}$$

where $M_{\alpha\beta}(\mathbf{y}), Q_{3\alpha}(\mathbf{y})$ are the bending and shear stress resultants, respectively, $\left(D = Eh^3/12(1-v^2)\right)$ is the plate flexural rigidity, (E) is Young's modulus, (v) is Poisson's ratio, $\left(\lambda = \sqrt{10}/h\right)$is the shear factor [36], and $b_3(\mathbf{y})$ is the distributed load per unit area applied perpendicular to the plate domain. Without losing the generality, throughout this paper, $b_3(\mathbf{y})$ is assumed constant. The equilibrium equations are obtained by considering the equilibrium of a differential plate element [35], to give

$$M_{\alpha\beta,\beta}(\mathbf{y}) - Q_{3\alpha}(\mathbf{y}) + b_\alpha(\mathbf{y}) = 0 \tag{5}$$

$$Q_{3\alpha,\alpha}(\mathbf{y}) + b_3(\mathbf{y}) = 0 \tag{6}$$

where $b_\alpha(\mathbf{y})$ is the rotational body loads defined at a domain point $(\mathbf{y})$ in the direction $x_1(\mathbf{y})$ and $x_2(\mathbf{y})$.

The generalized Cauchy equations for boundary tractions are [35]

$$p_\alpha(\mathbf{x}) = M_{\alpha\beta}(\mathbf{x})n_\beta(\mathbf{x}) \tag{7}$$

$$p_3(\mathbf{x}) = Q_{3\beta}(\mathbf{x})n_\beta(\mathbf{x}) \tag{8}$$

where $n_\beta(\mathbf{x})$ are the components of the outward normal vector to the plate boundary (Γ) at point$(\mathbf{x} \in \Gamma)$ The fundamental solutions, due to unit load at point $(\boldsymbol{\xi})$ (the source point), are originally obtained by Vander Weeën [28] and listed in the Appendix for the sake of completeness.

3 Equivalency Between the Stationary Conditions of the Energy Functional and the Governing Partial Differential Equations

The purpose of this section is to demonstrate the equivalency between the stationary conditions for the potential energy functional and the governing partial differential equation (recall Eqs. (5) and (6)). This is mainly done in order to derive the proposed boundary integral equations in Sec. 5 based on such a functional.

The energy functional for the Reissner's plate bending problems could be obtained as follows [20]:

$$\begin{aligned}\Pi(u_\alpha(\mathbf{y}), u_3(\mathbf{y})) &= \int_{\Omega(\mathbf{y})}\frac{1}{2}\left(M_{\alpha\beta}(\mathbf{y})\chi_{\alpha\beta}(\mathbf{y}) + Q_{3\alpha}(\mathbf{y})\psi_{3\alpha}(\mathbf{y})\right)d\Omega(\mathbf{y}) \\ &- \int_{\Omega(\mathbf{y})}\left(b_\alpha(\mathbf{y})u_\alpha(\mathbf{y}) + b_3(\mathbf{y})u_3(\mathbf{y})\right)d\Omega(\mathbf{y}) \\ &- \int_{\Gamma_P(\mathbf{x})}\left(\bar{p}_\alpha(\mathbf{x})\tilde{u}_\alpha(\mathbf{x}) + \bar{p}_3(\mathbf{x})\tilde{u}_3(\mathbf{x})\right)d\Gamma(\mathbf{x})\end{aligned} \tag{9}$$

with the following subsidiary compatibility conditions:

$$u_\alpha(\mathbf{x}) = \tilde{u}_\alpha(\mathbf{x}) \quad \text{on } \Gamma(\mathbf{x}) \tag{10}$$

$$u_3(\mathbf{x}) = \tilde{u}_3(\mathbf{x}) \quad \text{on } \Gamma(\mathbf{x}) \tag{11}$$

and boundary conditions:

$$\tilde{u}_\alpha(\mathbf{x}) = \bar{u}_\alpha(\mathbf{x}) \quad \text{on } \Gamma_u(\mathbf{x}) \tag{12}$$

$$\tilde{u}_3(\mathbf{x}) = \bar{u}_3(\mathbf{x}) \quad \text{on } \Gamma_u(\mathbf{x}) \tag{13}$$

where

- $u_\alpha(\mathbf{y})$: is the rotations at a domain point $(\mathbf{y})$ in the directions $x_1(\mathbf{y})$ and $x_2(\mathbf{y})$
- $u_3(\mathbf{y})$: is the deflection at a domain point (y) in the direction $x_3(\mathbf{y})$
- $p_\alpha(\mathbf{y})$: is the moment tractions at a domain point $(\mathbf{y})$ in the directions $x_1(\mathbf{y})$ and $x_2(\mathbf{y})$
- $p_3(\mathbf{y})$: is the shear traction at a domain point $(\mathbf{y})$ in the direction $x_3(\mathbf{y})$
- $\tilde{u}_\alpha(\mathbf{x})$: is the rotations defined at a boundary point $(\mathbf{x})$ in the directions $x_1(\mathbf{x})$ and $x_2(\mathbf{x})$
- $\tilde{u}_3(\mathbf{x})$: is the deflection defined at a boundary point $(\mathbf{x})$ in the direction $x_3(\mathbf{x})$
- $\tilde{p}_\alpha(\mathbf{x})$: is the moment tractions defined at a boundary point $(\mathbf{x})$ in the directions $x_1(\mathbf{x})$ and $x_2(\mathbf{x})$
- $\tilde{p}_3(\mathbf{x})$: is the shear traction defined at a boundary point $(\mathbf{x})$ in the direction $x_3(\mathbf{x})$
- $\bar{p}_\alpha(\mathbf{x})$: is the prescribed values of moment tractions defined at boundary point $(\mathbf{x})$ on part of the boundary $\Gamma(_P)$, in the directions x_1 $(\mathbf{x})$ and x_2 $(\mathbf{x})$
- $\bar{p}_3(\mathbf{x})$: is the prescribed value of shear traction defined at a boundary point ($\mathbf{x}$ on part of the $(\mathbf{x})$ $\Gamma(_P)$, in the direction x_3 $(\mathbf{x})$, and
- $b_3(\mathbf{y})$: is the body load defined at a domain point $(\mathbf{y})$ in the direction x_3 $(\mathbf{y})$

Combining the boundary conditions in Eqs. (10) and (11) with the energy functional in Eq. (9) using a set of Lagrange multipliers $\lambda_\alpha(\mathbf{x}), \lambda_3(\mathbf{x})$, the following new functional is formed:

$$\begin{aligned}&\Pi_2(u_\alpha(\mathbf{y}),u_3(\mathbf{y}),\tilde{u}_\alpha(\mathbf{x}),\tilde{u}_3(\mathbf{x}),\lambda_\alpha(\mathbf{x}),\lambda_3(\mathbf{x}))\\&=\int_{\Omega(\mathbf{y})}\frac{1}{2}\left(M_{\alpha\beta}(\mathbf{y})\chi_{\alpha\beta}(\mathbf{y})+Q_{3\alpha}(\mathbf{y})\psi_{3\alpha}(\mathbf{y})\right)d\Omega(\mathbf{y})\\&-\int_{\Omega(\mathbf{y})}(b_\alpha(\mathbf{y})u_\alpha(\mathbf{y})+b_3(\mathbf{y})u_3(\mathbf{x}))d\Omega(\mathbf{y})\\&-\int_{\Gamma_p(\mathbf{x})}(\bar{p}_\alpha(\mathbf{x})\tilde{u}_\alpha(\mathbf{x})+\bar{p}_3(\mathbf{x})\tilde{u}_3(\mathbf{x}))d\Gamma(\mathbf{x})\\&+\int_{\Gamma(\mathbf{x})}\lambda_\alpha(\mathbf{x})(\tilde{u}_\alpha(\mathbf{x})-u_\alpha(\mathbf{x}))d\Gamma(\mathbf{x})\\&+\int_{\Gamma(\mathbf{x})}\lambda_3(\mathbf{x})(\tilde{u}_3(\mathbf{x})-u_3(\mathbf{x}))d\Gamma(\mathbf{x})\end{aligned}\tag{14}$$

with the following admissible boundary conditions:

$$\tilde{u}_\alpha(\mathbf{x})=\bar{u}_\alpha(\mathbf{x})\quad\text{on }\Gamma_u(\mathbf{x})\tag{15}$$

$$\tilde{u}_3(\mathbf{x})=\bar{u}_3(\mathbf{x})\quad\text{on }\Gamma_u(\mathbf{x})\tag{16}$$

Consider the following relationships:

$$\delta M_{\alpha\beta}(\mathbf{y})\chi_{\alpha\beta}(\mathbf{y})=M_{\alpha\beta}(\mathbf{y})\delta\chi_{\alpha\beta}(\mathbf{y})\tag{17}$$

and

$$\delta Q_{3\alpha}(\mathbf{y})\psi_\alpha(\mathbf{y})=Q_{3\alpha}(\mathbf{y})\delta\psi_\alpha(\mathbf{y})\tag{18}$$

where $\delta(\cdot)$ denotes the first variational operator [20]. Substituting Eq. (1) into Eq. (17), gives

$$M_{\alpha\beta}(\mathbf{y})\delta\chi_{\alpha\beta}(\mathbf{y})=\frac{1}{2}M_{\alpha\beta}(\mathbf{y})\delta\left(u_{\alpha,\beta}(\mathbf{y})+u_{\beta,\alpha}(\mathbf{y})\right)\tag{19}$$

and considering the symmetry of moment tensor, then Eq. (19) could be rewritten as follows:

$$M_{\alpha\beta}(\mathbf{y})\delta\chi_{\alpha\beta}(\mathbf{y})=M_{\alpha\beta}(\mathbf{y})\delta u_{\alpha,\beta}(\mathbf{y})\tag{20}$$

From variational calculus, the operator δ can be swapped with the derivative operator; hence Eq. (20) could be rewritten as follows:

$$M_{\alpha\beta}(\mathbf{y})\delta\chi_{\alpha\beta}(\mathbf{y})=M_{\alpha\beta}(\mathbf{y})\frac{\partial}{\partial x_\beta(\mathbf{y})}\delta u_\alpha(\mathbf{y})\tag{21}$$

In a similar way to moments, substituting Eq. (2) into Eq. (18), gives

$$Q_{3\alpha}(\mathbf{y})\delta\psi_{3\alpha}(\mathbf{y})=Q_{3\alpha}(\mathbf{y})\delta\left(u_\alpha(\mathbf{y})+u_{3,\alpha}(\mathbf{y})\right)\tag{22}$$

and, hence,

$$\begin{aligned}Q_{3\alpha}(\mathbf{y})\delta\left(u_\alpha(\mathbf{y})+u_{3,\alpha}(\mathbf{y})\right)&=Q_{3\alpha}(\mathbf{y})\delta u_\alpha(\mathbf{y})\\&+Q_{3\alpha}(\mathbf{y})\frac{\partial}{\partial x_\beta(\mathbf{y})}\delta u_3(\mathbf{y})\end{aligned}\tag{23}$$

Substituting Eqs. (21) and (23) into the first variation of Eq. (14) gives

$$\begin{aligned}&\delta\Pi_2(u_\alpha(\mathbf{y}),u_3(\mathbf{y}),\tilde{u}_\alpha(\mathbf{x}),\tilde{u}_3(\mathbf{x}),\lambda_\alpha(\mathbf{x}),\lambda_3(\mathbf{x}))\\&=\int_{\Omega(\mathbf{y})}M_{\alpha\beta}(\mathbf{y})\frac{\partial}{\partial x_\beta}\delta u_\alpha(\mathbf{y})d\Omega(\mathbf{y})\\&+\int_{\Omega(\mathbf{y})}Q_{3\alpha}(\mathbf{y})\frac{\partial}{\partial x_\beta}\delta u_3(\mathbf{y})d\Omega(\mathbf{y})\\&-\int_{\Omega(\mathbf{y})}(b_\alpha(\mathbf{y})\delta u_\alpha(\mathbf{y})+b_3(\mathbf{y})\delta u_3(\mathbf{y})-Q_{3\alpha}(\mathbf{y})\delta u_\alpha(\mathbf{y}))d\Omega(\mathbf{y})\\&+\int_{\Gamma_P(\mathbf{x})}(\lambda_\alpha(\mathbf{x})-\bar{p}_\alpha(\mathbf{x}))\delta\tilde{u}_\alpha(\mathbf{x})d\Gamma(\mathbf{x})\\&+\int_{\Gamma_P(\mathbf{x})}(\lambda_3(\mathbf{x})-\bar{p}_3(\mathbf{x}))\delta\tilde{u}_3(\mathbf{x})d\Gamma(\mathbf{x})\\&+\int_{\Gamma(\mathbf{x})}\delta\lambda_\alpha(\mathbf{x})(\tilde{u}_\alpha(\mathbf{x})-u_\alpha(\mathbf{x}))d\Gamma(\mathbf{x})\\&+\int_{\Gamma(\mathbf{x})}\delta\lambda_3(\mathbf{x})(\tilde{u}_3(\mathbf{x})-u_3(\mathbf{x}))d\Gamma(\mathbf{x})\\&-\int_{\Gamma(\mathbf{x})}\lambda_\alpha(\mathbf{x})\delta u_\alpha(\mathbf{x})d\Gamma(\mathbf{x})-\int_{\Gamma(\mathbf{x})}\lambda_3(\mathbf{x})\delta u_3(\mathbf{x})d\Gamma(\mathbf{x})\end{aligned}\tag{24}$$

The first two domain integral on the right hand side of Eq. (24) could be converted into boundary integrals as follows [37]:

$$\begin{aligned}&\int_{\Omega(\mathbf{y})}M_{\alpha\beta}(\mathbf{y})\frac{\partial}{\partial x_\beta(\mathbf{y})}\delta u_\alpha(\mathbf{y})d\Omega(\mathbf{y})\\&=\int_{\Gamma(\mathbf{y})}M_{\alpha\beta}(\mathbf{y})n_\beta(\mathbf{y})\delta u_\alpha(\mathbf{y})d\Gamma(\mathbf{y})-\int_{\Omega(\mathbf{y})}M_{\alpha\beta,\beta}(\mathbf{y})\delta u_\alpha(\mathbf{y})d\Omega(\mathbf{y})\end{aligned}\tag{25}$$

and

$$\begin{aligned}&\int_{\Omega(\mathbf{y})}Q_{3\alpha}(\mathbf{y})\frac{\partial}{\partial x_\beta(\mathbf{y})}\delta u_3(\mathbf{y})d\Omega(\mathbf{y})\\&=\int_{\Gamma(\mathbf{y})}Q_{3\alpha}(\mathbf{y})n_\alpha(\mathbf{y})\delta u_3(\mathbf{y})d\Gamma(\mathbf{y})-\int_{\Omega(\mathbf{y})}Q_{3\alpha,\alpha}(\mathbf{y})\delta u_3(\mathbf{y})d\Omega(\mathbf{y})\end{aligned}\tag{26}$$

Substituting Eq. (7) into Eq. (25), the following integral form could be obtained:

$$\begin{aligned}&\int_{\Omega(\mathbf{y})}M_{\alpha\beta}(\mathbf{y})\frac{\partial}{\partial x_\beta(\mathbf{y})}\delta u_\alpha(\mathbf{y})d\Omega(\mathbf{y})\\&=\int_{\Gamma(\mathbf{y})}p_\alpha(\mathbf{y})\delta u_\alpha(\mathbf{y})d\Gamma(\mathbf{y})-\int_{\Omega(\mathbf{y})}M_{\alpha\beta,\beta}(\mathbf{y})\delta u_\alpha(\mathbf{y})d\Omega(\mathbf{y})\end{aligned}\tag{27}$$

Substituting Eq. (8) into Eq. (26), the following integral form could be also obtained:

$$\begin{aligned}&\int_{\Omega(\mathbf{y})}Q_{3\alpha}(\mathbf{y})\frac{\partial}{\partial x_\beta(\mathbf{y})}\delta u_3(\mathbf{y})d\Omega(\mathbf{y})\\&=\int_{\Gamma(\mathbf{x})}p_3(\mathbf{y})\delta u_3(\mathbf{y})d\Gamma(\mathbf{y})-\int_{\Omega(\mathbf{y})}Q_{3\alpha,\alpha}(\mathbf{y})\delta u_3(\mathbf{y})d\Omega(\mathbf{y})\end{aligned}\tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (24) and regrouping terms, gives

$$\begin{aligned}&\delta\Pi_2(u_\alpha(\mathbf{y}),u_3(\mathbf{y}),\tilde{u}_\alpha(\mathbf{x}),\tilde{u}_3(\mathbf{x}),\lambda_\alpha(\mathbf{x}),\lambda_3(\mathbf{x}))\\&=\int_{\Gamma(\mathbf{x},\mathbf{y})}(p_\alpha(\mathbf{y})-\lambda_\alpha(\mathbf{x}))\delta u_\alpha(\mathbf{y})d\Gamma(\mathbf{x},\mathbf{y})\\&+\int_{\Gamma(\mathbf{x},\mathbf{y})}(p_3(\mathbf{y})-\lambda_3(\mathbf{x}))\delta u_3(\mathbf{x})d\Gamma(\mathbf{x},\mathbf{y})\\&+\int_{\Gamma_P(\mathbf{x})}(\lambda_\alpha(\mathbf{x})-\bar{p}_\alpha(\mathbf{x}))\delta\tilde{u}_\alpha(\mathbf{x})d\Gamma(\mathbf{x})\\&+\int_{\Gamma_P(\mathbf{x})}(\lambda_3(\mathbf{x})-\bar{p}_3(\mathbf{x}))\delta\tilde{u}_3(\mathbf{x})d\Gamma(\mathbf{x})\\&+\int_{\Gamma(\mathbf{x})}(\delta\lambda_\alpha(\mathbf{x})(\tilde{u}_\alpha(\mathbf{x})-u_\alpha(\mathbf{x}))d\Gamma(\mathbf{x}))\\&+\int_{\Gamma(\mathbf{x})}(\delta\lambda_3(\mathbf{x})(\tilde{u}_3(\mathbf{x})-u_3(\mathbf{x}))d\Gamma(\mathbf{x}))\\&-\int_{\Omega(\mathbf{y})}\left(M_{\alpha\beta,\beta}(\mathbf{y})-Q_{3\alpha}(\mathbf{y})+b_\alpha(\mathbf{y})\right)\delta u_\alpha(\mathbf{y})d\Omega(\mathbf{y})\\&-\int_{\Omega(\mathbf{y})}\left(Q_{3\alpha,\alpha}(\mathbf{y})+b_3(\mathbf{y})\right)\delta u_3(\mathbf{y})d\Omega(\mathbf{y})\end{aligned}\tag{29}$$

The state of equilibrium is equivalent to the stationary condition for the functional Π_2. This could be determined by setting the first variation of Π_2 to zero (i.e., $\delta\Pi_2$ in Eq. (29) is set to zero), which means

$$M_{\alpha\beta,\beta}(\mathbf{y}) - Q_{3\alpha}(\mathbf{y}) + b_\alpha(\mathbf{y}) = 0 \quad \text{in } \Omega(\mathbf{y}) \tag{30}$$

$$Q_{3\alpha,\alpha}(\mathbf{y}) + b_3(\mathbf{y}) = 0 \quad \text{in } \Gamma(\mathbf{y}) \tag{31}$$

$$\tilde{u}_\alpha(\mathbf{x}) - u_\alpha(\mathbf{x}) = 0 \quad \text{in } \Gamma(\mathbf{x}) \tag{32}$$

$$\tilde{u}_3(\mathbf{x}) - u_3(\mathbf{x}) = 0 \quad \text{in } \Gamma(\mathbf{x}) \tag{33}$$

$$\lambda_\alpha(\mathbf{x}) - \bar{p}_\alpha(\mathbf{x}) = 0 \quad \text{in } \Gamma_P(\mathbf{x}) \tag{34}$$

$$\lambda_3(\mathbf{x}) - \bar{p}_3(\mathbf{x}) = 0 \quad \text{in } \Gamma_P(\mathbf{x}) \tag{35}$$

$$p_\alpha(\mathbf{x}) - \lambda_\alpha(\mathbf{x}) = 0 \quad \text{in } \Gamma(\mathbf{x}) \tag{36}$$

$$p_3(\mathbf{x}) - \lambda_3(\mathbf{x}) = 0 \quad \text{in } \Gamma(\mathbf{x}) \tag{37}$$

From Eqs. (34) and (35), it can be seen that $\lambda_\alpha(\mathbf{x}), \lambda_3(\mathbf{x})$ represent the traction on the boundary, i.e.,

$$\lambda_\alpha(\mathbf{x}) = \bar{p}_\alpha(\mathbf{x}) \tag{38}$$

$$\lambda_3(\mathbf{x}) = \bar{p}_3(\mathbf{x}) \tag{39}$$

It can be noted that Eqs. (30) and (31) are the same as Eqs. (5) and (6); therefore, the stationary condition for the energy functional in Eq. (9) is an integral form equivalent to the governing partial differential equations. In what follows, the following notations are used:

$$\bar{p}_\alpha(\mathbf{x}) = \tilde{p}_\alpha(\mathbf{x}) \tag{40}$$

$$\bar{p}_3(\mathbf{x}) = \tilde{p}_3(\mathbf{x}) \tag{41}$$

Therefore,

$$\lambda_\alpha(\mathbf{x}) = \tilde{p}_\alpha(\mathbf{x}) \tag{42}$$

$$\lambda_3(\mathbf{x}) = \tilde{p}_3(\mathbf{x}) \tag{43}$$

4 The Proposed Variational Formulation

Substituting Eqs. (42) and (43) into Eq. (14) gives a functional equivalent to the energy functional in Eqs. (9) or (14), as follows:

$$\begin{aligned}
&\Pi_3(u_\alpha(\mathbf{y}), u_3(\mathbf{y}), \tilde{u}_\alpha(\mathbf{x}), \tilde{u}_3(\mathbf{x}), \tilde{p}_\alpha(\mathbf{x}), \tilde{p}_3(\mathbf{x}))\\
&= \int_{\Omega(\mathbf{y})} \frac{1}{2}\left(M_{\alpha\beta}(\mathbf{y})\chi_{\alpha\beta}(\mathbf{y}) + Q_{3\alpha}(\mathbf{y})\psi_{3\alpha}(\mathbf{y})\right) d\Omega(\mathbf{y})\\
&- \int_{\Omega(\mathbf{y})} (b_\alpha(\mathbf{y})u_\alpha(\mathbf{y}) + b_3(\mathbf{y})u_3(\mathbf{y})) d\Omega(\mathbf{y})\\
&- \int_{\Gamma_P(\mathbf{x})} (\bar{p}_\alpha(\mathbf{x})\tilde{u}_\alpha(\mathbf{x}) + \bar{p}_3(\mathbf{x})\tilde{u}_3(\mathbf{x})) d\Gamma(\mathbf{x})\\
&+ \int_{\Gamma(\mathbf{x},\mathbf{y})} \tilde{p}_\alpha(\mathbf{x})(\tilde{u}_\alpha(\mathbf{x}) - u_\alpha(\mathbf{x})) d\Gamma(\mathbf{x},\mathbf{y})\\
&+ \int_{\Gamma(\mathbf{x},\mathbf{y})} \tilde{p}_3(\mathbf{x})(\tilde{u}_3(\mathbf{x}) - u_3(\mathbf{y})) d\Gamma(\mathbf{x},\mathbf{y})
\end{aligned} \tag{44}$$

From Eq. (1) and taking into consideration the symmetry of the moment stress-resultant tensor, the following identity could be obtained:

$$M_{\alpha\beta}(\mathbf{y})\chi_{\alpha\beta}(\mathbf{y}) = M_{\alpha\beta}(\mathbf{y})u_{\alpha,\beta}(\mathbf{y}) \tag{45}$$

and similarly for the shear stress-resultant tensor,

$$Q_{3\alpha}(\mathbf{y})\psi_{3\alpha}(\mathbf{y}) = Q_{3\alpha}(\mathbf{y})\left(u_\alpha(\mathbf{y}) + u_{3,\alpha}(\mathbf{y})\right) \tag{46}$$

Substituting Eqs. (45) and (46) into Eq. (44) gives

$$\begin{aligned}
&\Pi_3(u_\alpha(\mathbf{y}), u_3(\mathbf{y}), \tilde{u}_\alpha(\mathbf{x}), \tilde{u}_3(\mathbf{x}), \tilde{p}_\alpha(\mathbf{x}), \tilde{p}_3(\mathbf{x}))\\
&= \int_{\Omega(\mathbf{y})} \frac{1}{2}M_{\alpha\beta}(\mathbf{y})u_{\alpha,\beta}(\mathbf{y}) d\Omega(\mathbf{y}) + \int_{\Omega(\mathbf{y})} \frac{1}{2}Q_{3\alpha}(\mathbf{y})\left(u_\alpha(\mathbf{y}) + u_{3,\alpha}(\mathbf{y})\right) d\Omega(\mathbf{y})\\
&- \int_{\Omega(\mathbf{y})} (b_\alpha(\mathbf{y})u_\alpha(\mathbf{y}) + b_3(\mathbf{y})u_3(\mathbf{y})) d\Omega(\mathbf{y})\\
&- \int_{\Gamma_P(\mathbf{x})} (\bar{p}_\alpha(\mathbf{x})\tilde{u}_\alpha(\mathbf{x}) + \bar{p}_3(\mathbf{x})\tilde{u}_3(\mathbf{x})) d\Gamma(\mathbf{x})\\
&+ \int_{\Gamma(\mathbf{x},\mathbf{y})} \tilde{p}_\alpha(\mathbf{x})(\tilde{u}_\alpha(\mathbf{x}) - u_\alpha(\mathbf{y})) d\Gamma(\mathbf{x},\mathbf{y})\\
&+ \int_{\Gamma(\mathbf{x},\mathbf{y})} \tilde{p}_3(\mathbf{x})(\tilde{u}_3(\mathbf{x}) - u_3(\mathbf{y})) d\Gamma(\mathbf{x},\mathbf{y})
\end{aligned} \tag{47}$$

The first two domain integrals on the right hand side of Eq. (47) could be converted into boundary integrals, as follows [37]:

$$\int_{\Omega(\mathbf{y})} \frac{1}{2}M_{\alpha\beta}(\mathbf{y})u_{\alpha,\beta}(\mathbf{y}) d\Omega(\mathbf{y}) = \int_{\Gamma(\mathbf{y})} \frac{1}{2}M_{\alpha\beta}(\mathbf{y})n_\beta(\mathbf{y})u_\alpha(\mathbf{y}) d\Gamma(\mathbf{y}) - \int_{\Omega(\mathbf{y})} \frac{1}{2}M_{\alpha\beta,\beta}(\mathbf{y})u_\alpha(\mathbf{y}) d\Omega(\mathbf{y}) \tag{48}$$

and

$$\int_{\Omega(\mathbf{y})} \frac{1}{2}Q_{3\alpha}(\mathbf{y})u_{3,\alpha}(\mathbf{y}) d\Omega(\mathbf{y}) = \int_{\Gamma(\mathbf{y})} \frac{1}{2}Q_{3\alpha}(\mathbf{y})n_\alpha(\mathbf{y})u_3(\mathbf{y}) d\Gamma(\mathbf{y}) - \int_{\Omega(\mathbf{y})} \frac{1}{2}Q_{3\alpha,\alpha}(\mathbf{y})u_3(\mathbf{y}) d\Omega(\mathbf{y}) \tag{49}$$

Substituting Eqs. (7) and (8) into Eqs. (48) and (49) gives

$$\int_{\Omega(\mathbf{y})} \frac{1}{2}M_{\alpha\beta}(\mathbf{y})u_{\alpha,\beta}(\mathbf{y}) d\Omega(\mathbf{y}) = \int_{\Gamma(\mathbf{y})} \frac{1}{2}p_\alpha(\mathbf{y})u_\alpha(\mathbf{y}) d\Gamma(\mathbf{y}) - \int_{\Omega(\mathbf{y})} \frac{1}{2}M_{\alpha\beta,\beta}(\mathbf{y})u_\alpha(\mathbf{y}) d\Omega(\mathbf{y}) \tag{50}$$

and

$$\int_{\Omega(\mathbf{y})} \frac{1}{2}Q_{3\alpha}(\mathbf{y})u_{3,\alpha}(\mathbf{y}) d\Omega(\mathbf{y}) = \int_{\Gamma(\mathbf{y})} \frac{1}{2}p_3(\mathbf{y})u_3(\mathbf{y}) d\Gamma(\mathbf{y}) - \int_{\Omega(\mathbf{y})} \frac{1}{2}Q_{3\alpha,\alpha}(\mathbf{y})u_3(\mathbf{y}) d\Omega(\mathbf{y}) \tag{51}$$

Substituting Eqs. (50) and (51) into Eq. (47) gives

$$\begin{aligned}
&\Pi_3(u_\alpha(\mathbf{x}), u_3(\mathbf{x}), \tilde{u}_\alpha(\mathbf{x}), \tilde{u}_3(\mathbf{x}), \tilde{p}_\alpha(\mathbf{x}), \tilde{p}_3(\mathbf{x}))\\
&= \int_{\Gamma(\mathbf{y})} \frac{1}{2}u_i(\mathbf{y})p_i(\mathbf{y}) d\Gamma(\mathbf{y}) - \int_{\Omega(\mathbf{y})} \frac{1}{2}M_{\alpha\beta,\beta}(\mathbf{y})u_\alpha(\mathbf{y}) d\Omega(\mathbf{y})\\
&- \int_{\Omega(\mathbf{y})} \frac{1}{2}Q_{3\alpha,\alpha}(\mathbf{y})u_3(\mathbf{y}) d\Omega(\mathbf{y}) - \int_{\Omega(\mathbf{y})} b_i(\mathbf{y})u_i(\mathbf{y}) d\Omega(\mathbf{y})\\
&- \int_{\Gamma_p(\mathbf{x})} \tilde{u}_i(\mathbf{x})\bar{p}_i(\mathbf{x}) d\Gamma(\mathbf{x}) + \int_{\Gamma(\mathbf{x})} \tilde{p}_i(\mathbf{x})\tilde{u}_i(\mathbf{x}) d\Gamma(\mathbf{x})\\
&- \int_{\Gamma(\mathbf{x},\mathbf{y})} \tilde{p}_i(\mathbf{x})u_i(\mathbf{y}) d\Gamma(\mathbf{x},\mathbf{y})
\end{aligned} \tag{52}$$

The first four integrals in Eq. (52) involve the domain variablesu$_i(\mathbf{x})$, $p_i(\mathbf{x})$. The following two integrals involve the boundary variables $\tilde{p}_i(\mathbf{x}), \tilde{u}_i(\mathbf{x})$, and the last integral involves both the domain and boundary variables.

5 The Proposed Boundary Element Model

In this section, a new variational boundary element formulation for the Reissner plate bending model is obtained by representing the three independent field variables $u_i, \tilde{p}_i, \tilde{u}_i$ via approximate schemes. Hence variational principles are used to minimize the functionalΠ_3. The stationary condition (that corresponds to the equilibrium condition) for such a functional represents an approximate integral equation of the problems.

5.1 Approximation of the Domain Variables. The purpose of this section is to approximate the domain terms $(u_i(\mathbf{y}), p_i(\mathbf{y})$ in which $\mathbf{y} \in \Omega(\mathbf{y})$in the first four integrals in Eq. (52). As in the indirect boundary element or the super-position formulation for Reissner's plate bending problems [5], the rotation and the displacement components vector at any point $(\mathbf{y})$ inside the domain Ω could be approximated via a collection series. This series contains the product of fundamental solution $(U^*_{ki}(\mathbf{y}, \xi_n))$ and an unknown set of fictitious concentrated tractions $(\gamma_k(\xi_n))$ located at a set of arbitrary source points (ξ_n), as follows:

$$u_i(\mathbf{y}) = U^*_{ki}(\mathbf{y}, \xi_n)\gamma_k(\xi_n) \tag{53}$$

where the subscript (n) denotes an arbitrary set of source points (its number could be taken later as the number of boundary nodes N) in which the fictitious tractions are applied along the direction (x_k).

In a similar way, the traction components at any point $(\mathbf{y})$ inside the domain Ω could be approximated via a collection series containing the products of fundamental solution $\left(P^*_{kl}(\mathbf{y}, \xi_n)\right)$ and the same unknown fictitious concentrated tractions $(\gamma_k(\xi_n))$, which are located at the same set of points (ξ_n), as follows:

$$p_i(\mathbf{y}) = P^*_{ki}(\mathbf{y}, \xi_n)\gamma_k(\xi_n) \tag{54}$$

Using the representation given in Eqs. (53) and (54), the first integral on the right hand side of Eq. (52) could be rewritten as follows:

$$\int_{\Gamma(\mathbf{y})} \frac{1}{2} u_i(\mathbf{y}) p_i(\mathbf{y}) d\Gamma(\mathbf{y}) = \frac{1}{2}\gamma_k(\xi_n)\left[\int_{\Gamma(\mathbf{y})} U^*_{ki}(\mathbf{y}, \xi_n) P^*_{mi}(\mathbf{y}, \xi_n) d\Gamma(\mathbf{y})\right]\gamma_m(\xi_n) \tag{55}$$

or in matrix form

$$\int_{\Gamma(\mathbf{y})} \frac{1}{2} u_i(\mathbf{y}) p_i(\mathbf{y}) d\Gamma(\mathbf{y}) = \frac{1}{2}\{\gamma\}^T_{1\times 3N}[\mathbf{F}]_{3N\times 3N}\{\gamma\}_{3N\times 1} \tag{56}$$

where

$$[\mathbf{F}]_{3N\times 3N} = \int_{\Gamma(\mathbf{y})} U^*_{ki}(\mathbf{y}, \xi_n) P^*_{mi}(\mathbf{y}, \xi_n) d\Gamma(\mathbf{y}) \tag{57}$$

in which (N) is the number of boundary points.

The second and third domain integrals on the right hand side of Eq. (52) are set to zeros. This is done by making use of considered approximations in Eqs. (53) and (54) and placing the source points (ξ_n) outside the plate boundary. Noting that in this case (recall Eqs. (15) and (16), $b_\alpha(\mathbf{y}) = \delta(\mathbf{y}, \xi_n)e_\alpha$ and $b_3(\mathbf{y}) = \delta(\mathbf{y}, \xi_n)e_3$, in which $\delta(\mathbf{y}, \xi_n)$is the Dirac δ distribution and e_i is a unit vector along the x_i direction; therefore,

$$\int_{\Omega(\mathbf{y})} \frac{1}{2} M_{\alpha\beta,\beta}(\mathbf{y}) u_\alpha(\mathbf{y}) d\Omega(\mathbf{y}) = 0 \tag{58}$$

and

$$\int_{\Omega(\mathbf{y})} \frac{1}{2} Q_{3\alpha,\alpha}(\mathbf{y}) u_3(\mathbf{y}) d\Omega(\mathbf{y}) = 0 \tag{59}$$

The last domain integral in Eq. (52) could be represented as follows:

$$\int_{\Omega(\mathbf{x})} b_i(\mathbf{y}) u_i(\mathbf{y}) d\Omega(\mathbf{y}) = \gamma_k(\xi_n)\left[\int_{\Omega(\mathbf{y})} U^*_{ki}(\mathbf{y}, \xi_n) b_i(\mathbf{y}) d\Omega(\mathbf{y})\right] \tag{60}$$

$$= \{\gamma\}^T_{1\times 3N}\{B\}_{3N\times 1} \tag{61}$$

where

$$\{\mathbf{B}\}_{3N\times 1} = \int_{\Omega(\mathbf{y})} U^*_{ki}(\mathbf{y}, \xi_n) b_i(\mathbf{y}) d\Omega(\mathbf{y}) \tag{62}$$

It has to be noted that the vector $\{\mathbf{B}\}$ in Eq. (62) is similar to the one that appears in the classical direct boundary element method [18] and could be transformed to the boundary using similar ways as those given by Rashed and Brebbia [37].

5.2 Approximation of the Boundary Variables. In this paper, the boundary displacement and traction vectors denoted by $(\tilde{u}_i)$ and $(\tilde{p}_i)$are approximated using constant boundary elements; therefore,

$$\tilde{u}_i(\mathbf{x}) = u_i(\mathbf{x_e}) \quad \forall\ \mathbf{x} \text{ in } \Gamma_e \tag{63}$$

$$\tilde{p}_i(\mathbf{x}) = p_i(\mathbf{x_e}) \quad \forall\ \mathbf{x} \text{ in } \Gamma_e \tag{64}$$

where $u_i(\mathbf{x_e})$ and $p_i(\mathbf{x_e})$ are vectors whose components are nodal $(\mathbf{x_e})$ values for boundary displacements and boundary tractions, respectively. It has to be noted that higher order boundary elements such as linear or quadratic elements could be used. However, the accuracy of constant elements, as it will be seen in the example Sec. 8, is very good in most of the cases.

Using the representation given in Eqs. (63) and (64), the fifth integral of Eq. (52) could be rewritten as follows:

$$\int_{\Gamma_p(\mathbf{x})} \tilde{u}_i(\mathbf{x})\bar{p}_i(\mathbf{x}) d\Gamma(\mathbf{x}) = \sum_{\text{elements}(\Gamma_e)} u_i(\mathbf{x_e}) \int_{\Gamma_e(\mathbf{x_e})} \bar{p}_i(\mathbf{x_e}) d\Gamma(\mathbf{x_e}) \tag{65}$$

$$= \{\mathbf{u}\}^T_{1\times 3N}\{\bar{\mathbf{P}}\}_{3N\times 1} \tag{66}$$

where

$$\{\bar{\mathbf{P}}\}_{3N\times 1} = \int_{\Gamma_e} \bar{p}_i d\Gamma(\mathbf{x_e}) \tag{67}$$

in which (N) is the number of the used boundary elements. The sixth integral of Eq. (52) could be rewritten as follows:

$$\int_{\Gamma(\mathbf{x})} \tilde{p}_i(\mathbf{x})\tilde{u}_i(\mathbf{x}) d\Gamma(\mathbf{x}) = \sum_{\text{elements}(\Gamma_e)} p_i(\mathbf{x_e})\left[\int_{\Gamma_e(\mathbf{x_e})} d\Gamma(\mathbf{x_e})\right] u_i(\mathbf{x_e}) \tag{68}$$

$$= \{\mathbf{p}\}^T_{1\times 3N}[\mathbf{L}]_{3N\times 3N}\{\mathbf{u}\}_{3N\times 1} \tag{69}$$

where

$$[\mathbf{L}]_{3N\times 3N} = \int_{\Gamma_e} d\Gamma(\mathbf{x_e}) \tag{70}$$

The last integral of Eq. (52) could be approximated as follows:

$$\int_{\Gamma(\mathbf{x})} \tilde{p}_i(\mathbf{x}) u_i(\mathbf{x}) d\Gamma(\mathbf{x})$$

$$= \sum_{\text{elements}(\Gamma_e)} p_i(\mathbf{x_e}) \left[\int_{\Gamma_e(\mathbf{x_e})} U^*_{ki}(\mathbf{x_e}, \xi_n) d\Gamma(\mathbf{x_e}) \right] \gamma_k(\xi_n) \quad (71)$$

$$= \{\mathbf{p}\}^T_{1\times 3N} [\mathbf{G}]^T_{3N\times 3N} \{\gamma\}_{3N\times 1} \quad (72)$$

$$[\mathbf{G}]_{3N\times 3N} = \int_{\Gamma(\mathbf{y})} U^*_{ki}(\mathbf{x_e}, \xi_n) d\Gamma(\mathbf{y}) \quad (73)$$

It has to be noted that the matrix $[\mathbf{G}]$ in Eq. (73) is similar to the one that appears in the classical direct boundary element method [36].

6 Final System of Equations

Using the approximations in Eqs. (56), (58), (59), (61), (66), (69), and (72), Eq. (52) could be rewritten as follows:

$$\Pi_3 = \frac{1}{2}\{\gamma\}^T [\mathbf{F}]\{\gamma\} - \{\mathbf{u}\}^T \{\bar{\mathbf{P}}\} + \{\mathbf{p}\}^T [\mathbf{L}]\{\mathbf{u}\} - \{\mathbf{p}\}^T [\mathbf{G}]^T \{\gamma\} - \{\gamma\}^T \{\mathbf{B}\} \quad (74)$$

The final system of algebraic equations could be obtained by computing the stationary conditions associate with Π_3 in Eq. (74). This can be obtained by taking the first variation of Eq. (74) as follows:

$$\delta\Pi_3 = \frac{1}{2}\{\delta\gamma\}^T [\mathbf{F}]\{\gamma\} + \frac{1}{2}\{\gamma\}^T [\mathbf{F}]\{\delta\gamma\} - \{\delta\mathbf{u}\}^T \{\bar{\mathbf{P}}\} + \{\delta\mathbf{p}\}^T [\mathbf{L}]\{\mathbf{u}\} + \{\mathbf{p}\}^T [\mathbf{L}]\{\delta\mathbf{u}\} - \{\delta\mathbf{p}\}^T [\mathbf{G}]^T \{\gamma\} - \{\mathbf{p}\}^T [\mathbf{G}]^T \{\delta\gamma\} - \{\delta\gamma\}^T \{\mathbf{B}\} \quad (75)$$

Rearranging Eq. (75) gives

$$\delta\Pi_3 = \{\delta\gamma\}^T \Big([\mathbf{F}]\{\gamma\} - [\mathbf{G}]\{\mathbf{p}\} - \{\mathbf{B}\} \Big) + \{\delta\mathbf{u}\}^T \Big([\mathbf{L}]^T \{\mathbf{p}\} - \{\bar{\mathbf{p}}\} \Big) + \{\delta\mathbf{p}\}^T \Big([\mathbf{L}]\{\mathbf{u}\} - [\mathbf{G}]^T \{\gamma\} \Big) \quad (76)$$

The functional Π_3 is stationary when its first variation $\delta\Pi_3$ vanishes for any arbitrary values of $(\delta\gamma(\xi_n), \delta u(\mathbf{x})$ and $\delta p(\mathbf{x}))$. Therefore, the corresponding generalized Euler's equations are

$$[\mathbf{F}]\{\gamma\} - [\mathbf{G}]\{\mathbf{P}\} - \{\mathbf{B}\} = 0 \quad (77)$$

$$[\mathbf{L}]^T \{\mathbf{p}\} - \{\bar{\mathbf{P}}\} = 0 \quad (78)$$

$$[\mathbf{L}]\{\mathbf{u}\} - [\mathbf{G}]^T \{\gamma\} = 0 \quad (79)$$

The unknown vectors $\{\gamma\}$ and $\{\mathbf{P}\}$are expressed in terms of the vector $\{\mathbf{P}\}$ to obtain a final system of equations involving only the boundary unknown vector $\{\mathbf{P}\}$. Provided that the matrix $[\mathbf{G}]$ is not singular [36], Eq. (79) could be rewritten as follows:

$$\{\gamma\} = \left[[\mathbf{G}]^T \right]^{-1} [\mathbf{L}]\{\mathbf{u}\} \quad (80)$$

Substituting Eq. (80) into Eq. (77) gives

$$\{\mathbf{P}\} = [\mathbf{G}]^{-1} [\mathbf{F}] \left[[\mathbf{G}]^T \right]^{-1} [\mathbf{L}]\{\mathbf{u}\} - [\mathbf{G}]^{-1} \{\mathbf{B}\} \quad (81)$$

Substituting Eq. (81) into Eq. (78) gives

$$[\mathbf{L}]^T [\mathbf{G}]^{-1} [\mathbf{F}] \left[[\mathbf{G}]^T \right]^{-1} [\mathbf{L}]\{\mathbf{u}\} - [\mathbf{L}]^T [\mathbf{G}]^{-1} \{\mathbf{B}\} - \{\bar{\mathbf{P}}\} = 0 \quad (82)$$

Introducing the following definitions,

$$[\mathbf{R}] = \left[[\mathbf{G}]^T \right]^{-1} [\mathbf{L}] \quad (83)$$

and

$$[\mathbf{R}]^T = [\mathbf{L}]^T [\mathbf{G}]^{-1} \quad (84)$$

hence, Eq. (82) could be rewritten as follows:

$$[\mathbf{R}]^T [\mathbf{F}][\mathbf{R}]\{\mathbf{u}\} - [\mathbf{R}]^T \{\mathbf{B}\} - \{\bar{\mathbf{P}}\} = 0 \quad (85)$$

Defining

$$[\mathbf{K}] = [\mathbf{R}]^T [\mathbf{F}][\mathbf{R}] \quad (86)$$

and

$$\{\mathbf{Q}\} = [\mathbf{R}]^T \{\mathbf{B}\} + \{\bar{\mathbf{P}}\} \quad (87)$$

hence, Eq. (85) could be rewritten as follows:

$$[\mathbf{K}]_{3N\times 3N} \{\mathbf{u}\}_{3N\times 1} = \{\mathbf{Q}\}_{3N\times 1} \quad (88)$$

It has been noted that the obtained $[\mathbf{K}]$ or the stiffness matrix is symmetric, positive definite, and similar to the one obtained from the finite element method [38]. The vectors $\{\mathbf{u}\}$ and $\{\mathbf{Q}\}$ are the corresponding vectors of boundary displacements and forces.

7 Solution at Internal Points

After solving Eq. (88), the vector $\{\gamma\}$ is computed from Eq. (80). Hence, the internal displacement vector at any point $(\mathbf{y})$ inside the domain (Ω) is computed using Eq. (53) as follows:

$$u_\alpha(\mathbf{y}) = U^*_{k\alpha}(\mathbf{y}, \xi_n)\gamma_k(\xi_n) \quad (89)$$

$$u_3(\mathbf{y}) = U^*_{k3}(\mathbf{y}, \xi_n)\gamma_k(\xi_n) \quad (90)$$

Stress resultants at any point $(\mathbf{y})$ inside the domain (Ω) are computed using Eqs. (3) and (4) after carrying out relevant derivatives as follows:

$$u_{\alpha,\gamma}(\mathbf{y}) = U^*_{k\alpha,\gamma}(\mathbf{y}, \xi_n)\gamma_k(\xi_n) \quad (91)$$

and

$$u_{3,\gamma}(\mathbf{y}) = U^*_{k3,\gamma}(\mathbf{y}, \xi_n)\gamma_k(\xi_n) \quad (92)$$

Expanding the index (k) to (β) and (3) gives

$$u_{\alpha,\gamma}(\mathbf{y}) = U^*_{\beta\alpha,\gamma}(\mathbf{y}, \xi_n)\gamma_\beta(\xi_n) + U^*_{3\alpha,\gamma}(\mathbf{y}, \xi_n)\gamma_3(\xi_n) \quad (93)$$

and

$$u_{3,\gamma}(\mathbf{y}) = U^*_{\beta 3,\gamma}(\mathbf{y}, \xi_n)\gamma_\beta(\xi_n) + U^*_{33,\gamma}(\mathbf{y}, \xi_n)\gamma_3(\xi_n) \quad (94)$$

The new derivatives $U^*_{\alpha\beta,\gamma}, U^*_{3\alpha,\gamma}, U^*_{\beta 3,\gamma}, U^*_{33,\gamma}$ are given in the Appendix. It has to be noted that unlike the direct boundary element method [28], all relevant derivatives herein are carried out with respect to the coordinate of the field point $(x_\gamma(\mathbf{y}))$.

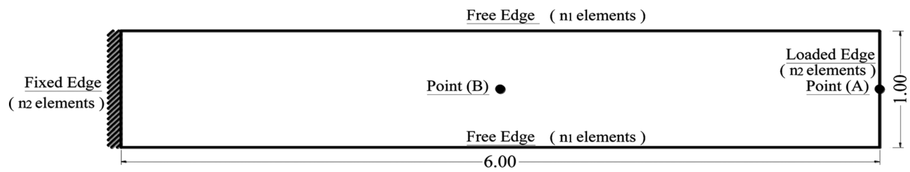

Fig. 1 Cantilever plate subject to edge load

8 Numerical Examples

In this section, four examples are presented to demonstrate the accuracy and the validity of the proposed formulation. Constant elements are used to approximate boundary values. The sources points (ξ_n) are chosen outside the plate boundary by a distance equal to the boundary element length and changes according to the used discretization. This is to avoid computing singular integrals. The chosen number of source points is the same as the number of the used boundary elements. A fifth example is carried out to study the effect of different locations of source points. It was found that source locations do not greatly affect the solution accuracy as far as they are located in range between 0.50 to 1.50 of the boundary element length.

8.1 Cantilever Plate Subjected to Edge Load. The 0.3 m thickness plate shown in Fig. 1 is subjected to constant edge load of intensity −6 t/m along its free edge. The Young's modulus for the plate material is $E = 2.5 \times 10^7$ t/m^2 and Poisson's ratio ν is set to zero to allow comparison against the analytical solutions of the beam theory. The results for the generalized displacements at the free edge middle point (A) are evaluated using different meshes and presented in Table 1 together with analytical values and

Table 1 Results of the generalized displacements at point (A)

	u_1(A) (analytical value = 1.92×10^{-3} rad)				u_3(A) (analytical value = -7.68×10^{-3} m)			
$n_1 \times n_2$ mesh	Present BEM	Error %	Conventional BEM [11]	Error %	Present BEM	Error %	Conventional. BEM [11]	Error %
1 × 1	1.91336×10^{-3}	0.35	1.52811×10^{-3}	20.41	-7.58322×10^{-3}	1.26	-1.41511×10^{-3}	81.57
2 × 2	1.91243×10^{-3}	0.39	1.56182×10^{-3}	18.66	-7.65727×10^{-3}	0.30	-3.41002×10^{-3}	55.60
3 × 2	1.9179×10^{-3}	0.11	1.63808×10^{-3}	14.68	-7.67487×10^{-3}	0.07	-4.98529×10^{-3}	35.09
4 × 4	1.91627×10^{-3}	0.19	1.59018×10^{-3}	17.18	-7.66863×10^{-3}	0.15	-6.02196×10^{-3}	21.59
5 × 5	1.91567×10^{-3}	0.23	1.64894×10^{-3}	14.12	-7.67144×10^{-3}	0.11	-6.73783×10^{-3}	12.27
6 × 5	1.91660×10^{-3}	0.18	1.72293×10^{-3}	10.26	-7.67622×10^{-3}	0.05	-7.18664×10^{-3}	6.42

Table 2 Results of the generalized displacements at point (B)

	u_1(B) (analytical value = 1.44×10^{-3} rad)				u_3(B) (analytical value = -2.40×10^{-3} m)			
$n_1 \times n_2$ mesh	Present BEM	Error %	Conventional. BEM [11]	Error %	Present BEM	Error %	Conventional. BEM [11]	Error %
1 × 1	1.39831×10^{-3}	2.9	0.12244×10^{-3}	91.5	-2.33448×10^{-3}	2.73	-0.2427×10^{-3}	89.89
2 × 2	1.43873×10^{-3}	0.09	1.57199×10^{-3}	9.17	-2.37845×10^{-3}	0.90	-1.21049×10^{-3}	49.56
3 × 2	1.44045×10^{-3}	0.03	0.69934×10^{-3}	51.43	-2.39972×10^{-3}	0.01	-1.66951×10^{-3}	30.44
4 × 4	1.43349×10^{-3}	0.45	1.55858×10^{-3}	8.23	-2.39643×10^{-3}	0.15	-1.97834×10^{-3}	17.57
5 × 5	1.4342×10^{-3}	0.40	1.16345×10^{-3}	19.20	-2.39623×10^{-3}	0.16	-2.16139×10^{-3}	9.94
6 × 5	1.43781×10^{-3}	0.15	1.52797×10^{-3}	6.11	-2.39829×10^{-3}	0.07	-2.28711×10^{-3}	4.70

Table 3 Results of the bending moment stress resultant at point (B)

	M_{11}(B) (analytical value = 18 t·m/m)			
$n_1 \times n_2$ mesh	Present BEM	Error %	Conventional BEM [11]	Error %
1 × 1	18.71	3.94	2.04	88.67
2 × 2	18.66	3.69	6.59	63.37
3 × 2	17.68	1.75	11.38	36.79
4 × 4	17.87	0.71	12.49	30.58
5 × 5	18.11	0.63	18.15	0.86
6 × 5	18.14	0.77	15.81	12.15

Table 4 Results of the shear stress resultant at point (B)

	Q_{13}(B) (analytical value = –6 t)			
$n_1 \times n_2$ mesh	Present BEM	Error %	Conventional BEM [11]	Error %
1 × 1	–3.71	38.12	3.14	152.35
2 × 2	–6.96	15.97	–57.02	850.41
3 × 2	–6.81	13.50	7.42	223.69
4 × 4	–4.83	19.43	–42.33	605.57
5 × 5	–5.06	15.60	9.70	261.72
6 × 5	–6.55	9.23	–27.38	356.40

Table 5 Results of the generalized displacements at point (A)

$n_1 \times n_2$ mesh	u_1(A) (analytical value = 1.28×10^{-3} rad) Present BEM	Error %	Conventional BEM [11]	Error %	u_3(A) (analytical value = -5.76×10^{-3} m) Present BEM	Error %	Conventional BEM [11]	Error %
1 × 1	1.26816×10^{-3}	0.93	1.01370×10^{-3}	20.81	-5.64952×10^{-3}	1.92	-1.00462×10^{-3}	82.56
2 × 2	1.27143×10^{-3}	0.67	1.00632×10^{-3}	21.38	-5.73442×10^{-3}	0.44	-2.50394×10^{-3}	56.53
3 × 2	1.2779×10^{-3}	0.16	1.06059×10^{-3}	17.14	-5.75549×10^{-3}	0.08	-3.67736×10^{-3}	36.16
4 × 4	1.27632×10^{-3}	0.29	1.03202×10^{-3}	19.37	-5.74907×10^{-3}	0.19	-4.43966×10^{-3}	22.92
5 × 5	1.27626×10^{-3}	0.29	1.07582×10^{-3}	15.95	-5.7522×10^{-3}	0.14	-4.97647×10^{-3}	13.60

Table 6 Results of the generalized displacements at point (B)

$n_1 \times n_2$ mesh	u_1(B) (analytical value = 1.12×10^{-3} rad) Present BEM	Error %	Conventional BEM [11]	Error %	u_3(B) (analytical value = -2.040×10^{-3} m) Present BEM	Error %	Conventional BEM [11]	Error %
1 × 1	1.07725×10^{-3}	3.82	0.09471×10^{-3}	91.54	-1.93166×10^{-3}	5.31	-0.22147×10^{-3}	89.14
2 × 2	1.11787×10^{-3}	0.19	1.19357×10^{-3}	6.57	-2.0178×10^{-3}	1.09	-0.96925×10^{-3}	52.49
3 × 2	1.12031×10^{-3}	0.03	0.53248×10^{-3}	52.46	-2.04252×10^{-3}	0.12	-1.38588×10^{-3}	32.06
4 × 4	1.11374×10^{-3}	0.56	1.18846×10^{-3}	6.11	-2.03938×10^{-3}	0.03	-1.64105×10^{-3}	19.56
5 × 5	1.1145×10^{-3}	0.49	0.88918×10^{-3}	20.61	-2.03912×10^{-3}	0.04	-1.80521×10^{-3}	11.51
8 × 5	1.14125×10^{-3}	1.9	1.20826×10^{-3}	7.88	-2.11183×10^{-3}	3.52	-2.14121×10^{-3}	4.96

Table 7 Results of the moment stress resultants at point (B)

$n_1 \times n_2$ mesh	M_{11}(B) (analytical value = 9 t·m/m) Present BEM	Error %	Conventional BEM [11]	Error %
1 × 1	10.87	20.82	0.84	90.66
2 × 2	9.713	7.92	2.968	67.02
3 × 2	8.700	3.33	5.319	40.90
4 × 4	8.834	1.85	5.991	33.44
5 × 5	9.144	1.60	8.877	1.36
8 × 5	8.970	0.33	8.865	1.50

Table 8 Results of the shear stress resultant at point (B)

$n_1 \times n_2$ mesh	Q_{13}(B) (analytical value = –6 t) Present BEM	Error %	Conventional BEM [11]	Error %
1 × 1	–3.81	36.54	2.12	135.28
2 × 2	–6.89	14.83	–43.78	629.62
3 × 2	–6.75	12.49	4.70	178.37
4 × 4	–4.92	17.99	–32.58	443.07
5 × 5	–5.14	14.37	6.00	199.92
8 × 5	–5.88	2.07	–15.15	152.51

results obtained from the conventional direct boundary element method [11]. Results for the generalized displacements and stresses results at internal point (B) are presented in Tables 2, 3, and 4. The Error % presented in Tables 1–4 is computed as the absolute value of (the numerical value – the analytical value)/ the analytical value × 100%.

It can be seen from Tables 1 to 4 that results for the present variational formulation is accurate with respect to the analytical values even with coarse discretization. It has to be noted that, despite the accuracy superiority of the present variational boundary element over the conventional direct boundary element method, values for shear forces might need the use of higher order boundary elements, which will be considered as future work.

8.2 Cantilever Plate Subjected to Domain Load. The same plate considered in the previous example is reconsidered herein. In this example, the plate is loaded with uniform domain load of intensity $-2\,t/m^2$. The results for the generalized displacements at the free edge middle point (point (A)) are evaluated using different meshes and presented in Table 5. Analytical values and values computed based on the conversional direct boundary element method [11] are also presented in Table 5. Results for the generalized displacements and stress resultants at internal point (B) are presented in Tables 6–8.

Similar conclusions to those of the previous example could be obtained from Tables 5–8.

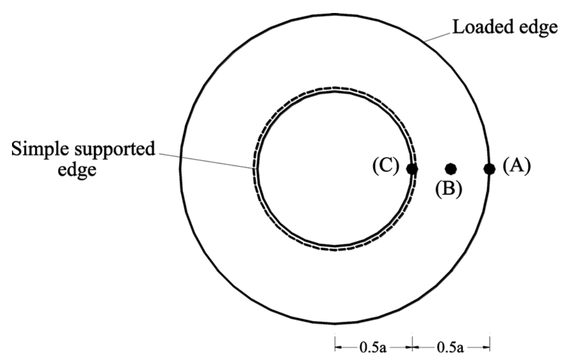

Fig. 2 Annular plate under edge load

8.3 Annular Thin Plate. The annular plate shown in Fig. 2 has an outer radius (a), inner radius (0.5 a), and thickness (0.02 a). The Poisson's ratio is taken to be 0.3. The inner boundary is simply supported and the outer one is free. The plate is loaded by edge load (q) along its free outer boundary. The solution of this plate using the convention direct boundary elements with constant and quadratic elements is given by Rashed [11]. The present variational boundary element results for the deflection at points (A) and (B) and the shear forces at the support point (C) are presented in Table 9. Such results are compared against the conventional

Table 9 Results of the generalized displacements at points (A), (B) and results of the shear stress resultant at point (C)

		$u_3(A)\times\frac{8D}{qa^3}$	$u_3(B)\times\frac{8D}{qa^3}$	$Q(C)\times\frac{-1}{2q}$
Analytical values		3.0935	1.6106	1.0000
Present BEM	18 constant elements	3.0721	1.6190	0.9890
	32 constant elements	3.0881	1.6129	1.0373
	64 constant elements	3.0926	1.6115	1.0081
Conventional direct. [11]	32 constant elements	2.6758	1.4608	1.0089
	64 constant elements	2.9120	1.5347	1.0045
	128 constant elements	3.0323	1.5840	1.0017
	16 quadratic elements	3.0139	1.6395	0.9974
	32 quadratic elements	3.0974	1.6131	0.9998

direct boundary element method results and analytical results in the same Table 9.

It can be seen from Table 9 the accuracy superiority of the present formulation results with constant elements even when compared to the conventional direct formulation with quadratic elements.

8.4 Assembled Plate Bending Problem. The former three examples demonstrate the accuracy and the validity of the present variational boundary element formulation. The purpose of this example is to demonstrate the capability of the present formulation in generating stiffness matrix, load vector, and carrying out the solution procedure similar to those of the finite element method [38]. The same plate in the first example is reconsidered herein. In this example, the plate is subjected to a constant edge load of -1 t/m at its free edge. The Young's modulus for the material is $E = 2.21 \times 10^6$ t/m^2. The results are evaluated using two ways. In the first way (way 1) the whole plate is considered as one domain (similar solution to that of Sec. 8.1). The alternative way (way 2) is to divide the plate into two subdomains (Part 1 and Part 2) as shown in Figure 3. The boundary element meshes used for the two parts are presented in Figure 3. The degrees of freedom associated with the discretization in Figure 3 is demonstrated in Figure 4.

The stiffness matrix for each subdomain (super finite elements) is computed from Eq. (86), and the associated traction vectors are also computed. In a similar way to the well-known assembly procedure of the finite element method [38], the assembled stiffness matrix and load vector for the two subdomains could be computed. Hence, boundary conditions could be applied in a similar way to the finite element procedures. Then the unknown boundary displacements of each subdomain could be computed. The results for the generalized displacements at the point (A) are computed and listed in Table 9. Analytical values based on the beam theory are also given in Table 10. Results for the generalized

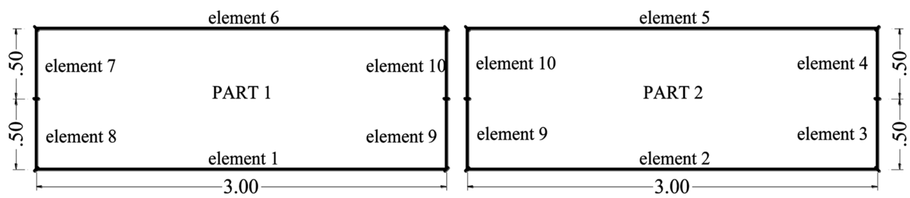

Fig. 3 The cantilever plate divided into two parts

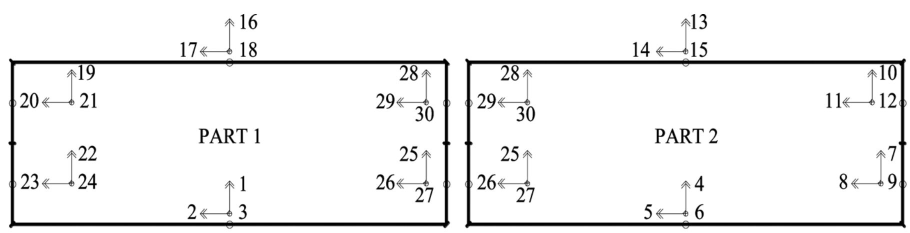

Fig. 4 The considered degrees of freedom for each part

Table 10 Results of the generalized displacements at point (A)

$u_1(A)$ (analytical value $= 3.62 \times 10^{-3}$ rad)				$u_3(A)$ (analytical value $= -1.45 \times 10^{-2}$ m)			
Single domain (2×2 elements) way(1)	Error %	Two sub domains way(2)	Error %	Single domain (2×2 elements) way(1)	Error %	Two sub domains way(2)	Error %
3.606×10^{-3}	0.394	3.599×10^{-3}	0.576	-1.444×10^{-2}	0.296	-1.437×10^{-2}	0.755

Table 11 Results of the generalized displacements at point (B)

$u_1(A)$ (analytical value $= 2.71 \times 10^{-3}$ rad)				$u_3(B)$ (analytical value $= -4.52 \times 10^{-3}$ m)			
Single domain (2×2 elements) way(1)	Error %	Two sub domains way(1)	Error %	Single domain (2×2 elements) way(1)	Error %	Two sub domains way(1)	Error %
2.71×10^{-3}	0.116	2.70×10^{-3}	0.552	-4.48×10^{-3}	0.888	-4.49×10^{-3}	0.766

Table 12 Results of the shear and moment stress-resultants at point (C)

Q_{13}(B) (analytical value = 1 t)				M_{11}(B) (analytical value = –6 t·m/m)			
Single domain (2 × 2 elements) way(1)	Error %	Two sub domains way(1)	Error %	Single domain (2 × 2 elements) way(1)	Error %	Two sub domains way(1)	Error %
0.9982	0.1830	0.9981	0.1917	–5.9922	0.1308	–5.9743	0.4285

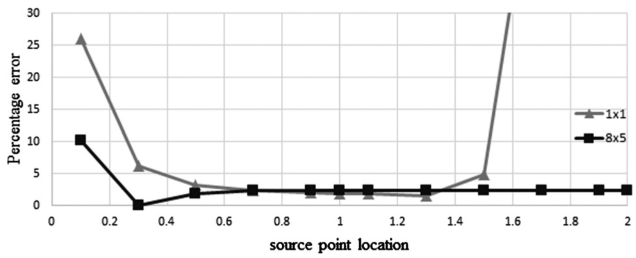

Fig. 5 Effect of source point location on results

displacements at the internal point (B) are presented in Table 11. Shear and moment stress-resultants at the support point (C) are presented in Table 12.

It has to be noted that the results for the assembled plate (way 2) is a bit less accurate than those of the single domain plate (way 1). This is predictable, because in the assembled plate (way 2), the values of generalized displacements and stress resultants are forced to vary constantly along the connecting line (elements 9, 10). This is less accurate than (way 1), in which such values are left to vary according to the governing partial differential equation. However, using the assembled plate solution (way 2) demonstrates the capability of changing the thickness or/and material properties for any of the plate subdomains. Moreover, the solution using (way 2) is similar to the solution in the finite element method. Therefore, this solution strategy (way 2) could be considered as a possible extension of the finite element method or an alternative procedure to the subregion technique in the boundary element method.

8.5 Effect of Source Point Location. The same plate considered in Sec. 8.2 is reconsidered herein. Two boundary discretizations were considered (1×1) and (8×5). The effect of source point location is investigated. The source point location is varied from 0.1 to 2 times the length of the used boundary element.

Figure 5 demonstrates the percentage error in the results of the generalized displacements at point (A) by changing the source point locations.

9 Conclusions

In this paper, a variational boundary element formulation of Reissner's plate bending problems was derived. The formulation was based on minimizing the relevant energy functional. A collocation based series is used to remove domain integrals. Hence, a fully boundary integral equation is formulated. The formulation was transformed into matrix equations using constant boundary elements and was implemented into a computer code. Several examples with different boundary conditions were tested. It was demonstrated that the present formulation results were more accurate compared to results obtained from the conventional direct boundary elements, even with a fewer number of elements. In addition, the present formulation produces a symmetric stiffness matrix similar to that obtained from the finite element method. Therefore, such formulation is very suitable to be coupled with boundary and finite elements or to produce a new family of super finite elements, which will be considered in future research.

Appendix

The expressions for the fundamental solution kernels U^*_{ij} are [28]

$$U^*_{\alpha\beta} = \frac{1}{8\pi D(1-v)}\left[(8B(\lambda r) - (1-v)(2\ln(\lambda r) - 1))\delta_{\alpha\beta}\right) - (8A(\lambda r) + 2(1-v))r_{,\alpha}r_{,\beta}\Big] \quad (A1)$$

$$U^*_{\alpha 3} = -U^*_{3\alpha} = \frac{1}{8\pi D}(2\ln(\lambda r) - 1)r r_{,\alpha} \quad (A2)$$

$$U^*_{33} = \frac{1}{8\pi D(1-v)\lambda^2}\left[(1-v)(\lambda r)^2(\ln(\lambda r) - 1) - 8\ln(\lambda r)\right] \quad (A3)$$

where

$$r = \sqrt{[x_1(\mathbf{x}) - x_1(\boldsymbol{\xi})]^2 + [x_2(\mathbf{x}) - x_2(\boldsymbol{\xi})]^2} \quad (A4)$$

$$r_{,\alpha} = \frac{\partial r}{\partial x_\alpha(\mathbf{x})} \quad (A5)$$

$$A(\lambda r) = K_0(\lambda r) + 2(\lambda r)^{-1}\left[K_1(\lambda r) - (\lambda r)^{-1}\right] \quad (A6)$$

$$B(\lambda r) = K_0(\lambda r) + (\lambda r)^{-1}\left[K_1(\lambda r) - (\lambda r)^{-1}\right] \quad (A7)$$

where K_0, K_1 are modified Bessel functions [39].

The expressions for the fundamental solution kernels P^*_{ij} are [28]

$$P^*_{\gamma\alpha} = \frac{-1}{4\pi r}\Big[(4A(\lambda r) + 2K_1(\lambda r) + 1 - v)\left(\delta_{\alpha\beta}r_{,n} + r_{,\alpha}n_\gamma\right) + (4A(\lambda r) + 1 - v)r_{,\gamma}n_\alpha - 2(8A(\lambda r) + 2(\lambda r)K_1(\lambda r) + 1 - v)r_{,\alpha}r_{,\gamma}r_{,n}\Big] \quad (A8)$$

$$P^*_{\gamma 3} = \frac{\lambda^2}{2\pi}\left(B(\lambda r)n_\gamma - A(\lambda r)r_{,\gamma}r_{,n}\right) \quad (A9)$$

$$P^*_{3\alpha} = \frac{-(1-v)}{8\pi}\left(\left(\frac{2(1+v)}{(1-v)}\ln(\lambda r) - 1\right)n_\alpha + 2r_{,\alpha}r_{,n}\right) \quad (A10)$$

$$P^*_{33} = \frac{-1}{2\pi r}r_{,n} \quad (A11)$$

where

$$r_{,n} = r_{,\alpha}n_\alpha \quad (A12)$$

The expressions for the fundamental solution kernel derivatives $U^*_{ij,k}$ are

$$U^*_{\beta\alpha,\gamma} = \frac{1}{8\pi D(1-v)}\Big\{\left(8B_{,\gamma}(\lambda r) - 2(1-v)\frac{r_{,\gamma}}{r}\right)\delta_{\alpha\beta} - 8A_{,\gamma}(\lambda r)r_{,\alpha}r_{,\beta} - [8A(\lambda r) + 2(1-v)]\left[\delta_{\alpha\gamma}r_{,\beta} + \delta_{\beta\gamma}r_{,\alpha} - 2r_{,\alpha}r_{,\beta}r_{,\gamma}\right]\Big\} \quad (A13)$$

$$U^*_{\beta 3,\gamma} = -U^*_{3\alpha,\gamma} = \frac{1}{8\pi D}\left(2r_{,\alpha}r_{,\gamma} + (2\ln(\lambda r) - 1)\delta_{\alpha\beta}\right) \quad (A14)$$

$$U^*_{33,\gamma} = \frac{1}{8\pi D(1-\nu)}\left((1-\nu)(2\ln(\lambda r)-1) - \frac{8}{(\lambda r)^2}\right) r r_{,\gamma} \quad \text{(A15)}$$

where

$$A_{,\gamma}(\lambda r) = -\lambda r_{,\gamma} K_1(\lambda r) \quad \text{(A16)}$$

$$B_{,\gamma}(\lambda r) = -\lambda r_{,\gamma}\left(K_0(\lambda r) + \frac{K_1(\lambda r)}{\lambda r}\right) \quad \text{(A17)}$$

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