Universality of Quantum Critical Dynamics in a Planar Optical Parametric Oscillator

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We analyze the critical quantum fluctuations in a coherently driven planar optical parametric oscillator. We show that the presence of transverse modes combined with quantum fluctuations changes the behavior of the "quantum image" critical point. This zero-temperature nonequilibrium quantum system has the same universality class as a finite-temperature magnetic Lifshitz transition.

The nonequilibrium system consisting of a nonlinear crystal inside a laser driven Fabry-Perot interferometer, that couples subharmonic intracavity modes to a harmonic pump [1], is known as an optical parametric oscillator (OPO). As well as demonstrating quantum squeezing [2] and EPR entanglement in numerous quantum information experiments, this system is widely used in frequency conversion applications. In single-mode experiments, there is a critical point in the phase diagram. This is caused by an increase in the pump intensity, which results in a transition from a disordered (but quantum squeezed) phase below threshold, to an ordered phase with a coherent output above threshold.

When extended to an interferometer with multiple transverse modes, more complex dynamical effects occur due to diffraction of the down-converted light, which are governed by the Swift-Hohenberg equation near threshold [3]. The theory can be quantized [4], and hence includes quantum fluctuations [5]. Experimentally observable [6,7] "quantum images" are evidence for quantum pattern formation with spatio-temporal correlations in the output quadratures. Thus, the system can show both spatial critical fluctuations and nonequilibrium spontaneous pattern formation, which occurs in many fields of physics and other sciences [8].

In this Letter we apply the theory of finite size scaling to solve for the critical quantum dynamical properties, and obtain the universality class this phase-transition corresponds to. We provide a full quantum description of this nonequilibrium system using the positive P representation, focusing on the nature of the critical point and critical fluctuations. This allows us to obtain an analytic solution for the functional distribution of the large critical fluctuations caused by quantum noise in the down-conversion process. There is an unexpected universality property in the solutions. Even though this is a nonequilibrium quantum system of coupled boson fields, we find that in one quadrature, the quantum fluctuations have exactly the same behavior as a classical thermal system of fields at a two-dimensional Lifshitz point, which is a model commonly used to describe the phase transition to a modulated mag-

netic phase; in the complementary quadrature there is strong entanglement.

The unitary evolution of the OPO system can be described by the Hamiltonian [5]

$$\widehat{\mathcal{H}} = \sum_{n=0,1} \int d^2\vec{x} \left\{ \hat{A}_n^\dagger \left[\omega_n - \frac{v^2}{2\omega_n} \nabla^2 \right] \hat{A}_n \right\}$$
$$+ i\hbar \int d^2\vec{x} \hat{A}_0 \{ \mathcal{E}^* e^{2i\omega_L t} - \chi \hat{A}_1^{\dagger 2} \}. \quad (1)$$

The term χ is a coupling parameter that depends on the nonlinear crystal, the frequencies of the field modes are ω_1, $\omega_0 = 2\omega_1$, v is the intracavity group velocity, and $\hat{A}_n$ is the nth photon field. The pump is described by the amplitude $\mathcal{E}$ that could carry a spatial structure—but here we will assume a constant plane wave input. In addition, there are damping effects due to output couplings from the cavity mirrors, which can be well approximated using as a Markovian master equation for the density matrix $\hat{\rho}$, so that

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{i\hbar} [\widehat{\mathcal{H}}, \hat{\rho}] + \sum_{n=0,1} \gamma_n \mathcal{L}_n[\hat{\rho}], \quad (2)$$

where $\mathcal{L}_n[\hat{\rho}] = \int d^2\vec{x} [2\hat{A}_n \hat{\rho} \hat{A}_n^\dagger - \hat{\rho} \hat{A}_n^\dagger \hat{A}_n - \hat{A}_n^\dagger \hat{A}_n \hat{\rho}]$ describes the output coupling from the nth intracavity mode, with damping rate γ_n. This leads to a set of Fokker-Planck equations, mapped from the quantum density matrix, using operator representation theory. These are valid provided boundary terms vanish in the mapping transformation, which we have checked numerically. Using the positive P representation, we derive the following stochastic equations [1,5,6] in a rotating frame at frequency ω_L:

$$\frac{\partial A_1}{\partial t} = -\tilde{\gamma}_1 A_1 + \chi A_1^+ A_0 + i\gamma_1 D\nabla^2 A_1 + \sqrt{\chi A_0} \xi_1(t, \vec{x}),$$
$$\frac{\partial A_0}{\partial t} = -\tilde{\gamma}_0 A_0 + \mathcal{E} - \frac{\chi^*}{2} A_1^2 + \frac{i\gamma_1}{2} D\nabla^2 A_0. \quad (3)$$

Here we write the two-dimensional Laplacian causing diffraction, as $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The complex relaxation rates are $\tilde{\gamma}_i = \gamma_i(1 + i\Delta_i)$. The relative detunings

between the pump laser at $2\omega_L$, and the modes supported by the cavity are $\Delta_0 = (\omega_0 - 2\omega_L)/\gamma_0$, and $\Delta_1 = (\omega_1 - \omega_L)/\gamma_1$. The diffraction rate is defined as $D = v^2/(2\gamma_1\omega_1)$. The stochastic field ξ_1 which describes quantum noise is real and Gaussian, with correlations of $\langle \xi_1(t) \rangle = 0$ and $\langle \xi_1(\vec{x}, t)\xi_1(\vec{x}', t') \rangle = \delta^2(\vec{x} - \vec{x}')\delta(t - t')$.

In addition, there are equations that correspond to the hermitian conjugate fields. As elsewhere in this Letter, we obtain these by conjugating the constant terms and replacing the stochastic and noise fields according to: $A_i \rightarrow A_i^+$, $\xi_1 \rightarrow \xi_1^+$, where ξ_1, ξ_1^+ are independent real Gaussian noises. These two noise fields are sufficient to generate all quantum effects, and are physically caused by the discrete nature of the photon pairs produced in down-conversion. The c-number fields $A_i(t, \vec{x})$, $A_i^+(t, \vec{x})$ are therefore not complex conjugate, although they are stochastically equivalent in terms of normally ordered operator moments to photon operator fields $\hat{A}_i(t, \vec{x})$, $\hat{A}_i^\dagger(t, \vec{x})$. Thus, for example, the photon number density is $\langle \hat{A}_i^\dagger(t, \vec{x})\hat{A}_i(t, \vec{x}') \rangle = \langle A_i^+(t, \vec{x})A_i(t, \vec{x}') \rangle$.

If we remove the transverse modes from the above equations we return to the well-known single-mode OPO theory. This system has a quantum critical point—a phase transition in the infinite volume limit, where the quantum fluctuations are reduced below the vacuum level for the squeezed quadrature and become huge for the unsqueezed quadrature [9]. In a recent analysis [10] of this problem near the critical point, going beyond the linear theory, we obtained a scaling law for the squeezing quadrature spectrum near threshold, and the parameters for the optimum squeezing.

The introduction of transverse modes generates a spatial structure in the subharmonic field with an intensity correlation function known as a "quantum image" [5], since it is supported by quantum fluctuations. To treat this problem analytically, we can perform an adiabatic elimination of the stable pump mode in the limit of $\gamma_0 \gg \gamma_1$ and $\Delta_0 \rightarrow 0$. That is, we assume that the pump mode has a short relaxation time.

Neglecting pump diffraction—which is negligible in the critical regime—we obtain an adiabatic solution for the pumped field: $\bar{A}_0 = (\mathcal{E} - \chi^* A_1^2/2)/\gamma_0$, together with a similar equation for the conjugate term. This solution takes into account the depletion of the pumping mode that supplies energy for down-converted light, and leads to an adiabatic equation for the down-converted field:

$$\frac{\partial A_1}{\partial t} = -\tilde{\gamma}_1 A_1 + \frac{\chi}{\gamma_0}\left(\mathcal{E} - \frac{\chi^*}{2}A_1^2\right)A_1^+ + i\gamma_1 D\nabla^2 A_1 + \sqrt{\chi \bar{A}_0}\xi_1(t, \vec{x}). \quad (4)$$

Next, we introduce dimensionless variables $\tau = t/t_0$ and $\vec{r} = \vec{x}/x_0$, with a corresponding down-converted field $\alpha = x_0 A_1$. Because of critical slowing down, the characteristic length x_0 and time t_0 scale as $t_0 = 1/(g\gamma_1)$ and $x_0^2 =$

$D/\sqrt{g_c}$, where the effective nonlinear coefficient is $g_c = |\chi|^{4/3}/[8D\gamma_0\gamma_1]^{2/3}$, and we will assume that $g_c \ll 1$. The dimensionless driving field is $\tilde{\mu} = \chi\mathcal{E}/[\gamma_1\gamma_0] = \mu + i\theta$. We also introduce appropriately scaled noise fields with $\xi(\tau, \vec{r}) = x_0\sqrt{t_0}\xi_1(t, \vec{x})$, and the corresponding hermitian conjugate terms.

With these definitions, we find that:

$$\frac{\partial \alpha}{\partial \tau} = \frac{1}{g_c}[-(1 + i\Delta_1)\alpha + (\tilde{\mu} - 4g_c^2\alpha^2)\alpha^+]$$
$$+ \frac{1}{\sqrt{g_c}}[i\nabla_r^2\alpha + \xi\sqrt{(\tilde{\mu} - 4g_c^2\alpha^2)}]. \quad (5)$$

This equation includes both the down-conversion term proportional to α_1^+, which generates a squeezed signal—together with a nonlinear saturation term proportional to $-\alpha_1^2\alpha_1^+$, which limits the down-converted amplitude, and leads to finite size critical fluctuations.

Critical fluctuations are most usefully analyzed with scaled quadratures that correspond to experimentally accessible homodyne detection. These are defined as

$$X(\tau, \vec{r}) = \sqrt{g_c}[\alpha_1(\tau, \vec{r}) + \alpha_1^+(\tau, \vec{r})],$$
$$Y(\tau, \vec{r}) = i[\alpha_1^+(\tau, \vec{r}) - \alpha_1(\tau, \vec{r})]. \quad (6)$$

Similarly, there are quadrature noise fields defined as $\xi_x(\tau, \vec{r}) = \xi(\tau, \vec{r}) + \xi^+(\tau, \vec{r})$, and $\xi_y(\tau, \vec{r}) = i\sqrt{g_c}[\xi^+(\tau, \vec{r}) - \xi(\tau, \vec{r})]$. The resulting quantum dynamical equations for these signal field quadratures are

$$\frac{\partial X}{\partial \tau} = -[\gamma_x + X^2 + g_cY^2]X - [\gamma_{xy} + \nabla^2]Y + \xi_x,$$
$$g_c\frac{\partial Y}{\partial \tau} = -[\gamma_y + g_cX^2 + g_c^2Y^2]Y - [\gamma_{yx} - \nabla^2]X + \xi_y. \quad (7)$$

The linear decay matrix that couples the X and Y quadratures is given by

$$\begin{bmatrix} \gamma_x & \gamma_{xy} \\ \gamma_{yx} & \gamma_y \end{bmatrix} = \begin{bmatrix} (1 - \mu)/g_c & -(\theta + \Delta_1)/\sqrt{g_c} \\ (\Delta_1 - \theta)/\sqrt{g_c} & (1 + \mu) \end{bmatrix}. \quad (8)$$

We assume also that close to threshold, and for small enough detunings, $\gamma_x = O(1)$ and $\gamma_{xy} = -(\theta + \Delta_1)/\sqrt{g_c} = O(1)$. We can always choose quadrature phases so that $\theta = \Delta_1 + O(g_c)$. With this choice, Y is mainly coupled to the X quadrature via the diffraction term, which couples noise from the critical fluctuations back into the squeezed quadrature. This implies that $\tilde{\mu} = 1 + O(g_c)$ and $\gamma_y = 2 + O(\sqrt{g_c})$, so that the noise correlations are given by

$$\langle \xi_x(\tau, \vec{r})\xi_x(\tau', \vec{r}') \rangle = 2\delta(\tau - \tau')\delta^2(\vec{r} - \vec{r}') + O(g_c). \quad (9)$$

We can now perform a second type of adiabatic elimination, which is valid in a neighborhood of the critical point.

This takes into account the fact that the fluctuations in the X quadrature become very slow near threshold, while the Y quadrature still responds on fast time scales of order $1/\gamma_1$. To leading order we can drop terms of $O(\sqrt{g_c})$ where $g_c \ll 1$, and approximate the above equations as follows:

$$\frac{\partial X}{\partial \tau} = -\gamma_x X - \gamma_{xy} Y - X^3 - \nabla^2 Y + \xi_x,$$
$$0 = -2Y + \nabla^2 X. \quad (10)$$

We can therefore eliminate the fast or noncritical quadrature variable Y, by writing the steady state solution of the Y quadrature as $Y \simeq \nabla^2 X/2$. This produces a reduced equation for the critical quadrature variable X, which is valid near threshold:

$$\frac{\partial X}{\partial \tau} = -\gamma_x X - X^3 - \frac{\gamma_{xy}}{2} \nabla^2 X - \frac{1}{2} \nabla^4 X + \xi_x. \quad (11)$$

The above Langevin equation is a Ginzburg-Landau equation describing the critical quadrature dynamics. Unlike the usual application of this equation, we note that the system is a nonequilibrium one. The noise term ξ_x is of quantum origin rather than thermal origin, and is present at zero temperature. It is possible to write an equivalent functional Fokker-Planck equation for the probability density $P[X]$,

$$\frac{\partial P}{\partial \tau} = \frac{\delta}{\delta X}\left[\left(\gamma_x + X^2 + \frac{\gamma_{xy}}{2}\nabla^2 + \frac{1}{2}\nabla^4\right)X + \frac{\delta}{\delta X}\right]P, \quad (12)$$

and look for the equilibrium distribution in the form $P[X] = N \exp(-V[X])$, where $V(X)$ is a potential functional. Making this substitution, the solution for the distribution $P[X]$ is given by

$$P \propto e^{\{-\int d^2\vec{r}(2\gamma_x X^2 + X^4 - \gamma_{xy}[\nabla X]^2 + [\nabla^2 X]^2)/4\}}. \quad (13)$$

This expression is exactly the same as the Ginzburg-Landau free energy of a next nearest neighbor magnetic interaction, where X plays the role of an order parameter. That is, we have been able to map this problem into a soluble magnetic phase-transition equation with a Lifshitz point [11]. The phase diagram of this optical system should therefore have two ordered phases, one of them a spatially modulated phase associated with a pattern formation. This generic behavior is known to occur in an OPO, from previous analysis [3].

In this analogy, the "optical paramagnetic phase" corresponds to a random photon emission from the OPO operating below threshold, the "optical ferromagnetic phase" to a continuum emission uniformly distributed in the transverse plane parallel to the cavity operating above threshold. In the "optical ferromagnetic modulated phase," we have a continuum emission but with modulated quadrature in this plane. At the Lifshitz point, all three phases coexist.

The line $\gamma_x = 0$ is the line of the second-order phase transition between order-disorder (coherent-incoherent)

states. In the incoherent phase below threshold, $\gamma_x > 0$, and in the uniform coherent phase above threshold $\gamma_x < 0$, as expected in the single-mode case. If γ_x vanishes we have a Lifshitz point over the line $\gamma_x = 0$, and thus a triple point characterizing the coexistence of the three phases. This holds in the case of perfect tuning of the signal field inside the cavity, so that $\Delta_1 = 0$.

In condensed matter physics, the nature of the Lifshitz point [12,13] is crucially dependent on the order parameter and spatial dimension. Depending on the dimensionality of the order parameter, the system may or may not have a true phase transition in two dimensions. Our system has a one-dimensional real order parameter and two spatial dimensions with transverse modes, so we expect that this system should have a true phase transition in the infinite volume limit at finite temperature. According to the Mermin-Wagner theorem [14], increasing the order parameter dimension (as in type II down-conversion) would result in phase fluctuations that completely destroy any long range order. Similarly, reducing the spatial dimension would result in a continuous transition without a threshold.

A numerical simulation of the Ginzburg-Landau Eqs. (11) with a continuously scanned input shows that the subharmonic quadrature correlations appear to have a true critical point in two transverse dimensions. This result is shown in Fig. 1, which graphs $\langle|X(k)|^2\rangle$ around $k = 0$, as a function of the driving field γ_x near threshold.

While these large fluctuations are occurring, we note that there are still strong nonclassical correlations in the squeezed quadrature. This can be seen by analyzing the relevant equations to the next order in g_c, which we also simplify by using a Gaussian factorization [15]:

$$g_c \frac{\partial Y}{\partial \tau} \approx -\gamma_y Y + \nabla^2 X - g_c Y\langle X^2\rangle - \tilde{\gamma}_{yx} X + \xi_y, \quad (14)$$

where $\tilde{\gamma}_{yx} = \gamma_{yx} + 2g_c\langle XY\rangle$. In general, one can always choose an optimum local oscillator phase so that $\theta = k^2 + \Delta_1 + 2g_c\langle XY\rangle$, in order to minimize the feedback of critical fluctuations into the squeezed quadrature at a given
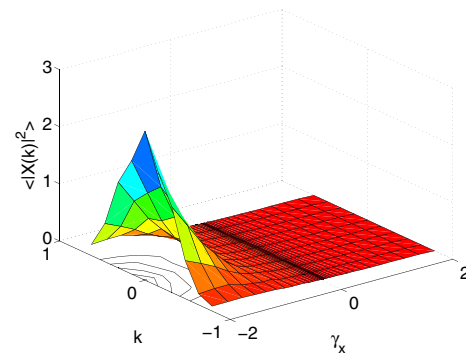

FIG. 1 (color online).   Variance of the quadrature Fourier component $\langle|X(k)|^2\rangle$ as a function of γ_x. Results obtained on a 100×100 lattice with a 40×40 domain size, using periodic boundary conditions and averaging over 100 stochastic trajectories.
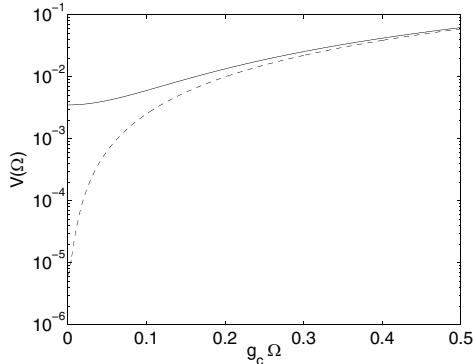
FIG. 2.   Squeezing spectrum as a function of frequency, with
(solid line) and without (dotted line) nonlinear corrections. The
parameters used are $\gamma_x = 0.5$, $g_c = 0.01$.

transverse momentum k. This leads to Fourier solutions
which show that entanglement [16] between the modes of
momentum k and $-k$ can still occur at small enough wave
vectors, resulting in a universal squeezing spectrum as a
function of frequency:

$$V(\Omega) = 1 - \frac{1 - g_c(\langle X^2 \rangle + \gamma_x)}{(g_c \Omega/2)^2 + (1 + g_c(\langle X^2 \rangle - \gamma_x)/2)^2}. \quad (15)$$

This result differs from the linearized predictions of
earlier treatments [5]. A graph of the resulting spectrum
in the Gaussian approximation is shown in Fig. 2, com-
pared to the linearized squeezing spectrum, showing large
differences near threshold.

In summary, we have shown that the planar nonequilib-
rium OPO with quantum noise can be mapped to a mag-
netic phase transition in two dimensions. Since the present
case has a scalar, real order parameter it is analogous to the
uniaxial ($m = 1$) magnetic order parameter case, which is
known to have a thermal equilibrium Lifshitz-point phase-
transition at finite temperature [13]. This demonstrates a
striking resemblance between known thermal equilibrium
phase transitions, and a quantum nonequilibrium system in
which quantum noise replaces thermal noise. In this system
there are also quantum correlations of the emitted photons,
causing quantum squeezing and entanglement. Never-
theless, this highly nonclassical behavior is found only in
the squeezed (Y) quadrature which has no critical slowing

down—and coexists with a rather classical and universal
critical fluctuation field in the conjugate (X) quadrature.

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