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Gossip Consensus and Averaging Algorithms with Quantization

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Abstract— **We study distributed consensus problems of multi-agent systems on directed networks and subject to quantized information flow. For the communication among component agents, particular attention is given to the gossip type, which models their asynchronous behaviors; for quantization effects, each agent's state is abstracted to be an integer. The central question investigated is how to design distributed algorithms and what connectivity of networks that together lead to consensus. This investigation is carried out for both general consensus and average consensus; for each case, a class of algorithms is proposed, under which a necessary and sufficient graphical condition is derived to guarantee the corresponding consensus. In particular, the obtained graphical condition ensuring average consensus is weaker than those in the literature for either real-valued or quantized states, in the sense that it does not require symmetric (or balanced) network topologies.**

Index Terms— **Quantized consensus, average consensus, gossip randomization.**

I. Introduction

Distributed consensus problems of multi-agent systems are of current research vitality in systems control. The problem can be described as follows: Consider a system of networked agents each possessing a numerical value, termed *state*; the agents communicate only with their neighbors and update their own states accordingly, in such a way that they eventually 'agree' on *some* common state. Such problems arise naturally in motion coordination of multi-vehicle systems [2], and are also closely related to oscillator synchronization [16] and leader election [13]. In some other applications, the *average* value of the total state sum may be of particular interest; examples include information fusion in sensor networks and load balancing in processor networks [10]. Thus being a special form of general consensus problems, *average consensus* further requires that the agreed, common state be the average of the initial states of all agents.

Substantial work on both general and average consensus problems has been carried out in recent years, which may be categorized in terms of distinct assumptions on state information and network types. Early efforts focused primarily on *real-valued* states and *deterministic* (but possibly time-varying) networks; references include [2], [12], [15]. This basic setup has then been extended in two different directions. One concerns *quantized* state information in deterministic networks, due to practical considerations of agents' physical memories being of finite capacity and digital communication channels of limited data rate [5], [7], [14]. The other direction adopts *randomized* time-varying networks with real-valued states, a model that potentially captures a variety of random behaviors exhibited in realistic networks [3], [17], [18]; see also [8]. In the foregoing literature, particular attention has been given to designing local control strategies for individual agents, finding conditions on graphs/matrices that guarantee consensus, and characterizing the tradeoffs between information flow and system performance. For graph models, we note that both *directed* and *undirected* (or bidirected) have been extensively investigated.

The objective of this paper is to study both general and average consensus problems in the setup where the states are quantized and the networks are randomized. As to quantization effects, following [10] we assume at the outset that the states are *integer-valued*, an abstraction that subsumes a class of quantization effects (e.g., uniform quantization). We note that most work dealing with quantization has concentrated on the scenario where the agents can transmit only quantized (integer) values but store real values (see, e.g., [5], [6], [9], [11]); by contrast, our assumption captures finite capacity constraints in both communication channels and physical memories, as in [10], [14]. On the other hand, for network randomization we employ the *gossip* type [3], [6], [10]. This type specifies that, at each time instant, exactly *one* agent updates its state based on the information transmitted from only *one* of its neighbors. Although less general than the random networks considered in [17], the gossip type instead captures asynchronous behaviors of component agents, a particularly important aspect in distributed systems. In addition to the adopted setting for states and networks, we focus solely on directed graphs, which is distinct from many related works [6], [9]–[11], [18] that assume only undirected graphs. As also argued in [15], directed networks potentially require less amount of information flow and could perform more robustly against link failures when compared to their undirected counterparts.

We emphasize that the central investigation in this paper is to derive connectivity conditions on graphs that ensure general/average consensus. Our contributions are summarized as follows. First, for general consensus we present a necessary and sufficient condition on the network topology that guarantees convergence to some common state, thereby extending the results in [2], [12], [17] from real-valued to quantized states. Second, for average consensus we propose a novel class of algorithms and derive a necessary and sufficient graphical condition ensuring convergence to the true (quantized) average. This result extends the one in [10] from undirected to directed graphs; the extension is challenging

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because with directed graphs of gossip type, the state sum, and hence the average, need not be invariant at each iteration. Also, the graphical condition we find is weaker than those for both real-valued and quantized states in [5], [15], [17], since we do not require maintaining symmetric (or balanced) topologies in random time-varying networks. As a tradeoff, however, the convergence rate of the proposed algorithm may not be fast. Lastly, our result is *scalable* compared to [6], [7], [14] in the sense that the true average is always achieved regardless of the number of agents.

The rest of the paper is organized as follows. First, we formulate both general and average consensus problems in Section II, and then present their solutions in Sections III and IV, respectively. Finally, we state our conclusion in Section V.

II. Problem Formulation

For a network of n (>1) agents, we model their communication structure by *directed graphs* (or simply *digraphs*) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, called *communication digraphs*. Each node in $\mathcal{V} = \{1, ..., n\}$ stands for an agent, and a directed edge (i, j) in $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, pointing from i to j, indicates that i is a neighbor of j and thus j communicates to i. Notice that the information flow over the edge (i, j) is only from j to i. Owing to quantization in information flow, we assume that at time $k \in \mathbb{Z}_+$ (nonnegative integers), each agent has an integer-valued state $x_i(k) \in \mathbb{Z},\ i \in \mathcal{V}$; the aggregate state is denoted by $x(k) = [x_1(k) \cdots x_n(k)]^T$. Define the minimum and maximum states by $m(k) := \min_{i \in \mathcal{V}} x_i(k),\ M(k) := \max_{i \in \mathcal{V}} x_i(k)$. We will design algorithms with which every agent updates its state such that all $x_i(k)$ eventually converge to a common value.

To address asynchronous communication between the agents, we adopt the gossip type of randomized networks. Specifically, at each time instant k exactly one edge, say (i, j), is activated independently from all earlier instants and with a positive probability $p_{ij} \in (0, 1)$ such that $\sum_{(i,j) \in \mathcal{E}} p_{ij} = 1$. In other words, every edge in $\mathcal{E}$ has a positive probability to be activated at each time, and these probabilities sum to one. Along this activated edge, node j sends information to i, while node i receives the information and makes an update accordingly.

In the first part of this paper, we consider the general consensus problem as described below. Let a subset $\mathscr{C}$ of $\mathbb{Z}^n$ be the set of consensus states:

$$\mathscr{C} := \{x : x_1 = \cdots = x_n\}. \tag{1}$$

Definition 1: The agents are said to achieve *quantized consensus almost surely* if for every initial condition $x(0)$, $x(k) \to \mathscr{C}$ as $k \to \infty$ with probability one.

Problem 1: Design distributed algorithms and find graphical connectivity such that the agents achieve quantized consensus almost surely.

For this problem, in Section III we will propose a class of algorithms, and derive a necessary and sufficient graphical condition that ensures almost sure quantized consensus.

In the second part, we extend the above problem to average consensus by further requiring that the consensus value be the average of the initial state sum. Formally, let $S := x(0)^T \mathbf{1}$, where $\mathbf{1} = [1 \cdots 1]^T$ is the vector of 1s. Hence the average of the initial states is S/n, a number that need not be an integer in general. We can, however, always write $S = nL + R$, where L and R are both integers with $0 \le R < n$. Thus, either L or $L+1$ (the latter if $R > 0$) may be viewed as an integer approximation of the average S/n. Henceforth we refer to $x^* := L\mathbf{1}$ or $(L+1)\mathbf{1}$ as the *true (quantized) average*.

To ensure converging to the average, the algorithms reported in the literature (e.g., [10], [15]) rely on a key property that the state sum $x^T \mathbf{1}$ remains invariant at each iteration. Unfortunately, this property in general fails in our gossip digraph setup where only one agent is allowed to update its state at each time. To overcome this difficulty, we propose associating to each agent an additional variable to record the changes in individual states; then the agents communicate these 'records' to their neighbors such that this important information can be utilized for state updates. We call these additional variables *surpluses*, and view them as augmented state components. The rules of how to use these surpluses mark the distinctive feature of our averaging algorithm compared to those in the literature; the concrete description is deferred to Section IV.

Formally, let the surplus of agent i at time k be $s_i(k) \in \mathbb{Z}$; thus the aggregate surplus is $s(k) = [s_1(k) \cdots s_n(k)]^T$, the initial value of which is set to be $s(0) = [0 \cdots 0]^T$. As described, the surplus is introduced so as to make the quantity $(x+s)^T \mathbf{1}$ invariant during iterations, i.e., for $k \ge 0$,

$$(x(k) + s(k))^T \mathbf{1} = (x(0) + s(0))^T \mathbf{1} = nL + R. \tag{2}$$

Consequently, $s^T \mathbf{1} = R$ (≥ 0) if $x = L\mathbf{1}$, and $R - n$ (< 0) if $x = (L+1)\mathbf{1}$. Now define the set of average consensus states, which is a subset $\mathscr{A}$ of $\mathbb{Z}^n \times \mathbb{Z}^n$, by

$$\mathscr{A} := \begin{cases} \mathscr{A}_L, & \text{if } R = 0; \\ \mathscr{A}_L \cup \mathscr{A}_{L+1}, & \text{if } 0 < R < n, \end{cases} \tag{3}$$

where

$$\mathscr{A}_L := \{(x, s) : x_i = L \ \&\ s_i \ge 0,\ i = 1, ..., n\},$$
$$\mathscr{A}_{L+1} := \{(x, s) : x_i = L+1 \ \&\ s_i \le 0,\ i = 1, ..., n\}.$$

Definition 2: The agents are said to achieve *quantized average almost surely* if for every initial condition $(x(0), 0)$, $(x(k), s(k)) \to \mathscr{A}$ as $k \to \infty$ with probability one.

Problem 2: Design distributed algorithms and find graphical connectivity such that the agents achieve quantized average almost surely.

To solve this problem, in Section IV we will propose a novel class of algorithms, under which we derive a necessary and sufficient graphical condition that guarantees almost sure quantized average.

III. Quantized Consensus

In this section we solve Problem 1, the almost sure quantized consensus. We start by presenting a class of algorithms, called *quantized consensus* (**QC**) algorithm. Then we provide the convergence result under a certain graphical condition.

A. **QC** *Algorithm*

Here we present **QC** algorithm. Suppose the edge $(i,j) \in \mathcal{E}$, $i,j \in \mathcal{V}$, is activated at time k. Along the edge node j sends to i its state information, $x_j(k)$, but does not perform any update, i.e., $x_j(k+1) = x_j(k)$. On the other hand, node i receives j's state $x_j(k)$ and updates its own as follows:

(R1) If $x_i(k) = x_j(k)$, then $x_i(k+1) = x_i(k)$;
(R2) if $x_i(k) < x_j(k)$, then $x_i(k+1) \in (x_i(k), x_j(k)]$;
(R3) if $x_i(k) > x_j(k)$, then $x_i(k+1) \in [x_j(k), x_i(k))$.

In words, node i stays put if its own state is the same as the received one; otherwise, it updates the state in the direction of diminishing the difference.

B. *Convergence Result*

First, we need to review some notions from standard graph theory (e.g., [1]). In a digraph a node i is *reachable* from a node j if there exists a path from j to i which respects the direction of the edges. In particular, a node is reachable from itself. A digraph is *strongly connected* if every node is reachable from every other node. Now let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph, and $\mathcal{U}$ a nonempty subset of $\mathcal{V}$. The subset $\mathcal{U}$ is said to be *closed* if every node v in $\mathcal{V} - \mathcal{U}$ is not reachable from any node u in $\mathcal{U}$. Intuitively, there is no edge pointing out from the subset $\mathcal{U}$. Also, the digraph $\mathcal{G}_\mathcal{U} = (\mathcal{U}, \mathcal{E} \cap (\mathcal{U} \times \mathcal{U}))$ is called the *induced subdigraph* by $\mathcal{U}$. A *strong component* of $\mathcal{G}$ is a maximal induced subdigraph of $\mathcal{G}$ which is strongly connected. Note that a maximal induced subdigraph need not be unique in general. Lastly, a node $v \in \mathcal{V}$ is called a *globally reachable node* if v is reachable from every other node [12, p.15]. Clearly the digraph $\mathcal{G}$ is strongly connected if and only if every node is globally reachable.

We present the main result of this section.

Theorem 1: Using **QC** algorithm, the agents achieve quantized consensus almost surely if and only if their communication digraph $\mathcal{G}$ has a globally reachable node.

It has been known (e.g., [12], [17]) that the existence of a globally reachable node is a necessary and sufficient graphical condition which ensures consensus in the case of real-valued states. In this respect, Theorem 1 extends this result to the setting where both stored and communicated states are quantized.

Our analysis technique is a blend of graph-theoretic and probabilistic arguments, which differs from the typical one (e.g., [15], [17]) that exploits the spectral properties of stochastic matrices associated to the graph structure. Indeed, owning to our integer state setup, the overall system does not enjoy a linear representation, and consequently the matrix approach cannot be applied.

Also, notice that the rules **(R2)** and **(R3)** of **QC** algorithm can be chosen so that the algorithm is similar to those for the real-valued case. Hence, we conjecture that the convergence rate of **QC** algorithm may be close to that of real-valued algorithms. The rigorous analysis of this conjecture is left open for future work.

To prove Theorem 1 we need a key lemma from [12, Theorem 2.1], which establishes an important relation between digraph connectivity and its structure.

Lemma 1: A digraph has a globally reachable node if and only if it has a unique closed strong component. Furthermore, this unique closed strong component is the induced subdigraph by the set of all globally reachable nodes.

Proof of Theorem 1: (Necessity) Suppose $\mathcal{G}$ does not have a globally reachable node. By Lemma 1, $\mathcal{G}$ has at least two distinct closed strong components, say $\mathcal{V}_1$ and $\mathcal{V}_2$. Consider some initial condition $x(0)$ such that all nodes in $\mathcal{V}_1$ have the same state $a \in \mathbb{Z}$ and all nodes in $\mathcal{V}_2$ have $b \in \mathbb{Z}$, but $a \neq b$. Then the quantized consensus is achieved *almost never* (with probability 0), for both $\mathcal{V}_1$ and $\mathcal{V}_2$ are closed.

(Sufficiency) Due to space limitation, we refer to [4]. ■

IV. Quantized Average

We move on to solve Problem 2, the quantized average consensus, by appropriately extending **QC** algorithm. A direct application of **QC** algorithm in general fails to ensure convergence to the true average, because the state sum need not be invariant at each iteration, hence causing the shift of the average. To handle this average shift, we propose associating to each agent an additional variable, termed surplus. These surpluses are used to keep track of the state changes of individual agents, so that the information of the amount of average shift is not lost but kept *locally* in these variables. Then the agents communicate the surpluses to their neighbors for state updates in such a way that the average of the initial states may be recovered. Furthermore, to assist the use of surpluses, two more auxiliaries are needed, which we call threshold and local extrema. We use these three augmented elements to make the extension of **QC** algorithm.

In the sequel, we first present the extended algorithm, called *quantized average* (**QA**) algorithm, and then provide the convergence result under a certain graphical condition. Further, we discuss suitable values for the threshold, and demonstrate the result using two numerical examples.

A. **QA** *Algorithm*

First, we introduce the three augmented elements.

1. *Surplus.* Every agent is associated with a surplus variable to record its state changes. Recall from Section II that the surplus of agent i is denoted by $s_i \in \mathbb{Z}$. Thus the aggregate surplus is $s = [s_1 \cdots s_n]^T \in \mathbb{Z}^n$, whose initial value is set to be $s(0) = [0 \cdots 0]^T$. The rules of specifying how these surpluses are updated locally and communicated over the network form the core of **QA** algorithm.

2. *Threshold.* All agents have a common threshold number, denoted by $\delta \in \mathbb{Z}_+$. This (constant) number is involved in deciding whether or not to update a state using available surpluses. A proper value for the threshold will be found crucial to ensure that the set $\mathscr{A}$ (defined in (3)) is the unique equilibria set under **QA** algorithm. We shall determine the range of such threshold values in Section IV-C. To keep the presentation clear, in this section we fix $\delta = n$, the total number of agents in the network.

3. *Local extrema.* Each agent i is further assigned two variables, m_i and $M_i \in \mathbb{Z}$, to record respectively the minimal and maximal states among itself and its neighbors. These *local* extrema will be used to prevent a state, when updated by available surpluses, from exceeding the interval of all initial states (i.e., $[m(0), M(0)]$). As to the initial values of local extrema, we set $m_i(0) = M_i(0) = x_i(0)$, for each $i \in \mathcal{V}$. We will design updating rules for m_i and M_i as part of **QA** algorithm.

Thus, we have augmented the state of each agent i from a single x_i to a tuple of four elements (x_i, s_i, m_i, M_i). In addition, a common threshold δ needs to be stored. It is also worth noting that only x_i and s_i will be involved in communication.

We are now ready to present **QA** algorithm. Suppose that edge $(i, j) \in \mathcal{E}$, $i, j \in \mathcal{V}$, is activated at time k. Along the edge, node j sends to i its state information, $x_j(k)$, as well as its surplus, $s_j(k)$. While it does not perform any update on its state (nor on its local minimum and maximum), node j does always set its surplus to be 0 after transmission, meaning that the surpluses, if any, are entirely passed onto its neighbor i; that is,

$$m_j(k+1) = m_j(k), \quad M_j(k+1) = M_j(k),$$
$$x_j(k+1) = x_j(k), \quad s_j(k+1) = 0.$$

On the other hand, node i receives the information from j, namely $x_j(k)$ and $s_j(k)$, and performs the following updates.
1. For local minimum and maximum,

$$m_i(k+1) = \min\{m_i(k), x_j(k)\},$$
$$M_i(k+1) = \max\{M_i(k), x_j(k)\}.$$

2. State and surplus are updated as follows:
(R1) If $x_i(k) = x_j(k)$, then there are three cases:
(i) If $s_i(k) + s_j(k) \geq \delta$ and $x_i(k) \neq M_i(k)$, then

$$x_i(k+1) = x_i(k) + 1,$$
$$s_i(k+1) = s_i(k) + s_j(k) - 1.$$

(ii) If $s_i(k) + s_j(k) \leq -\delta$ and $x_i(k) \neq m_i(k)$, then

$$x_i(k+1) = x_i(k) - 1,$$
$$s_i(k+1) = s_i(k) + s_j(k) + 1.$$

(iii) Otherwise (i.e., $|s_i(k) + s_j(k)| < \delta$ or $s_i(k) + s_j(k) \geq \delta$ & $x_i(k) = M_i(k)$ or $s_i(k) + s_j(k) \leq -\delta$ & $x_i(k) = m_i(k)$),

$$x_i(k+1) = x_i(k),$$
$$s_i(k+1) = s_i(k) + s_j(k).$$

(R2) If $x_i(k) < x_j(k)$, then

$$x_i(k+1) \in (x_i(k), x_j(k)],$$
$$s_i(k+1) = s_i(k) + s_j(k) - \big(x_i(k+1) - x_i(k)\big).$$

(R3) If $x_i(k) > x_j(k)$, then

$$x_i(k+1) \in [x_j(k), x_i(k)),$$
$$s_i(k+1) = s_i(k) + s_j(k) - \big(x_i(k+1) - x_i(k)\big).$$

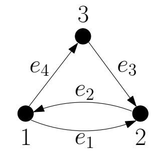

Fig. 1. Illustration of features of **QA** algorithm

In the algorithm, first observe that the surplus is updated such that for every $k \geq 0$, $(x(k+1) + s(k+1))^T\mathbf{1} = (x(k) + s(k))^T\mathbf{1} = x(0)^T\mathbf{1}$. That is, the quantity $(x + s)^T\mathbf{1}$ stays invariant at each iteration, and thus equals the initial state sum. Also, notice that the updates of state x_i in **(R2)** and **(R3)** are exactly the same as those in **QC** algorithm. The difference, however, lies in **(R1)**: Even when the state x_i coincides with x_j, it is still updated if the sum of surpluses, $s_i + s_j$, exceeds the interval $(-\delta, \delta)$; here this interval is $(-n, n)$. This is because, when the surpluses are more than n (resp., less than $-n$), the true average must be at least x_i+1 (resp., $x_i - 1$). Indeed, these surpluses should be distributed over the network such that every agent's state increases by at least 1 (resp., decreases by 1). An exception, however, is when the state x_i equals its local maximum (resp., local minimum), since in that case, x_i might undesirably exceed the interval $[m(0), M(0)]$. We illustrate these features of **QA** algorithm in the following example.

Example 1: Consider three agents with communication network in Fig. 1. Let the initial condition be as follows:

agent i	$x_i(0)$	$s_i(0)$	$m_i(0)$	$M_i(0)$
1	0	0	0	0
2	3	0	3	3
3	3	0	3	3

Hence the true average is $x^* = 2\mathbf{1}$. Suppose that at $k = 0$, edge e_1 is activated with a positive probability; then **(R2)** of **QA** algorithm applies since $x_1(0) < x_2(0)$. For the possible update values $(x_1(0), x_2(0)]$ we let $x_1(1) = x_2(0)$; the corresponding state change, $x_1(1) - x_1(0)$, is recorded in the surplus $s_1(1)$. Thus we obtain that

agent i	$x_i(1)$	$s_i(1)$	$m_i(1)$	$M_i(1)$
1	3	−3	0	3
2	3	0	3	3
3	3	0	3	3

Now the agents reach consensus at value 3. If **QC** algorithm is used, then no further update will take place, and consequently the true average cannot be achieved. However, that agent 1 has surplus -3 $(= -n)$ indicates that this amount should be distributed among the three agents, thereby each decreasing its state by 1. One way to distribute the surplus is to select the edges e_4, e_2, and e_3 sequentially; the probability of this selection is positive. It can then be readily verified that **(R1)(ii)**, **(R3)**, and again **(R3)** of **QA** algorithm will sequentially apply, and that at $k = 4$ we have

agent i	$x_i(4)$	$s_i(4)$	$m_i(4)$	$M_i(4)$
1	2	0	0	3
2	2	0	2	3
3	2	0	2	3

Therefore, the true average is achieved, and there is no further update because only **(R1)(iii)** will apply.

B. Convergence Result

We present the main result of this section.

Theorem 2: Using **QA** algorithm, the agents achieve quantized average almost surely if and only if their communication digraph $\mathcal{G}$ is strongly connected.

First of all, Theorem 2 can be seen as an extension of the main result in [10] from undirected to directed graphs. The problem of achieving quantized average with directed graphs is, however, more difficult than its undirected counterpart in that the state sum need not be invariant at each iteration. Our proposed **QA** algorithm handles this difficulty, by an essential augment of surplus variables.

Second, without augmenting extra elements, it is well known (e.g., [15], [17]) that a necessary and sufficient graphical condition for average consensus is that the communication digraph is both strongly connected and *balanced* (or, equivalently, the system matrix is doubly stochastic). A balanced digraph is one where every node has the same number of incoming and outgoing edges. However, this condition can be difficult to be maintained when the communication is asynchronous. By contrast, our condition on digraphs does not require the balanced property, since only one edge is activated at a time. An exemplification was given in Example 1, where the digraph that is strongly connected but not balanced achieves average consensus.

Finally, we note that in some quantized consensus algorithms (e.g., [6], [7], [14]), the agents converge to the average with an error which could undesirably get large as the number of agents increases. To address this *unscalable* situation, several approaches are proposed using special graph topologies [7], finer quantizers [14], and probabilistic quantizers [6]. In contrast, our result ensures, for a general (strongly connected) graph and a fixed (deterministic) quantizer, that the quantized average is always achieved regardless of the number of agents.

The foregoing merits, however, come with some costs which are twofold: For one, the convergence rate of **QA** algorithm is in general slower than **QC** algorithm due to averaging. This requires additional processing based on surpluses even after the agents achieve consensus (not at the average). For the other, as to local memories each agent needs to update, in addition to its state, another three variables — surplus, local minimum, and local maximum — and needs to store a constant threshold. The corresponding updating computations are, however, purely local and fairly simple. Moreover, each agent has to transmit surpluses, along with its state, through communication channels, which could double the network loads. Nevertheless, we can show that Theorem 2 holds even if the surpluses are transmitted one unit at each time; namely, the transmitted surpluses may take values only from the set $\{-1, 0, 1\}$. Therefore, the additional transmission of surpluses requires merely two bits increase in communication.

We now provide the proof of Theorem 2.

Proof of Theorem 2: (Necessity) Suppose that $\mathcal{G}$ is not strongly connected. Then at least one node of $\mathcal{G}$ is not globally reachable. Let $\mathcal{V}_g^*$ denote the set of non-globally reachable nodes; thus $\mathcal{V}_g^* \neq \emptyset$, and write its cardinality $|\mathcal{V}_g^*| = r,\ r \in \mathcal{V}$. If $r = n$, then $\mathcal{G}$ does not have a globally reachable node. Thus similar to the proof of Theorem 1, quantized average is achieved almost never. Now consider the case $r \in [1, n-1]$. Let $\mathcal{V}_g := \mathcal{V} - \mathcal{V}_g^*$ denote the set of all globally reachable nodes, and thus $|\mathcal{V}_g| = n - r$. By Lemma 1, $\mathcal{V}_g$ is the unique closed strong component in $\mathcal{G}$. Consider some initial condition $(x(0), 0)$ such that all nodes in $\mathcal{V}_g$ have the same state $a \in \mathbb{Z}$ and the state sum of the nodes in $\mathcal{V}_g^*$ is $n + ar$. It can be readily checked that the total sum of all initial states is $(a+1)n$; hence, the quantized average is $a+1$. However, no state or surplus update is possible for the nodes in $\mathcal{V}_g$ for it is closed. Hence, the quantized average is achieved almost never.

(Sufficiency) Due to space limitation, we refer to [4]. ■

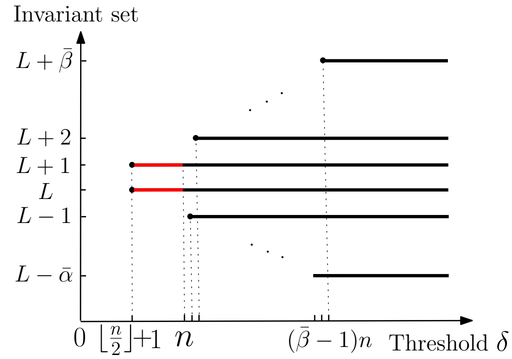

Fig. 2. The relation between threshold values and the invariant set ($\bar{\beta} > \bar{\alpha}$).

C. Threshold Range

So far, we have assumed the threshold δ to be the total number n of agents in the network. If the agents' communication digraph $\mathcal{G}$ is strongly connected, then Theorem 2 suggests that $\mathscr{A}$ (defined in (3)) is the unique invariant set where all trajectories converge. Now we proceed to investigate the systemic behavior when $\delta \neq n$. In particular, we aim at finding the range of threshold values necessary and sufficient to ensure that $\mathscr{A}$ is the unique invariant set to which all trajectories converge. This investigation is important because if the threshold δ has to be exactly n in order to guarantee average consensus, then **QA** algorithm may not be *robust* in applications where some agents could fail and/or new agents could join.

For this investigation we have the following result: The range of suitable threshold values turns out to be $\left[\lfloor \frac{n}{2} \rfloor + 1, n\right]$, which is fairly large in practice.

Theorem 3: Suppose that **QA** algorithm is used and the digraph $\mathcal{G}$ is strongly connected. Then $\mathscr{A}$ is the unique invariant set to which all trajectories converge if and only if the threshold $\delta \in \left[\lfloor \frac{n}{2} \rfloor + 1, n\right]$.

For the proof, see [4].

Now let $\bar{\alpha} := L - m(0)$ and $\bar{\beta} := M(0) - L$, where $m(0), M(0)$ are respectively the minimum and maximum initial states. For $\alpha \in [1, \bar{\alpha}]$, $\beta \in [2, \bar{\beta}]$, define the following subsets of $\mathbb{Z}^n \times \mathbb{Z}^n$

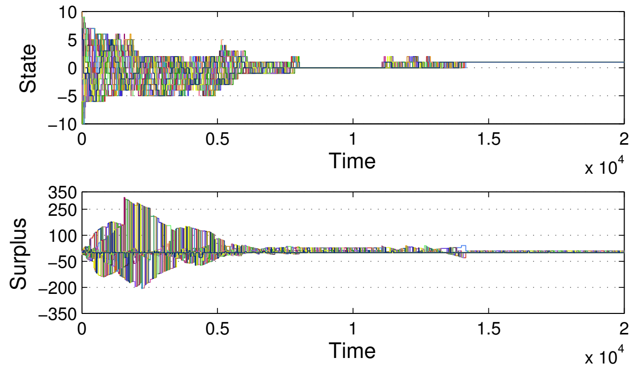

Fig. 3. Cyclic digraph of 30 agents

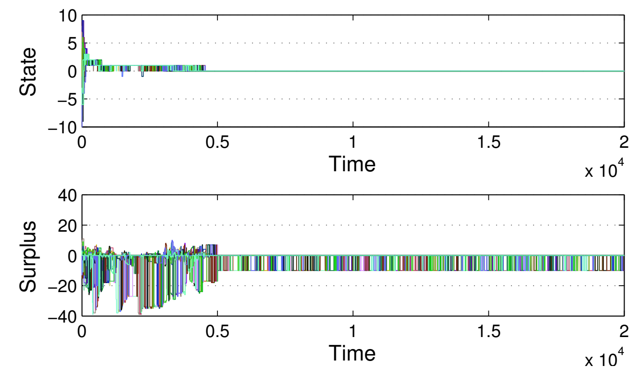

Fig. 4. Complete digraph of 30 agents

$$\mathscr{A}_{L-\alpha} := \{(x, s) : x_i = L - \alpha \ \& \ s_i \geq 0, \ i = 1, ..., n\},$$
$$\mathscr{A}_{L+\beta} := \{(x, s) : x_i = L + \beta \ \& \ s_i \leq 0, \ i = 1, ..., n\}.$$

Then we obtain the general relation between threshold values and the invariant set (as displayed in Fig. 2): For all pairs $(x(k), s(k))$, (i) when the threshold is $\delta \in [0, \lfloor \frac{n}{2} \rfloor]$, there is no invariant set; (ii) when $\delta \in [\lfloor \frac{n}{2} \rfloor + 1, n]$, $\mathscr{A}$ is the unique invariant set; (iii) when $\delta \in [n+1, \infty)$, the invariant set expands as δ increases, but lower bounded by $L - \bar{\alpha}$ and upper bounded by $L + \bar{\beta}$. The justification of this result can be found in [4].

D. Numerical Examples

We present two numerical examples to demonstrate **QA** algorithm. First, we consider a *cyclic digraph* of 30 agents, whose states are randomly initialized from a uniform distribution on the interval $[-10, 10]$; we do the same for the second example. In Fig. 3 we display the case where the initial state sum is $\sum_{i=1}^{30} x_i(0) = 40$, hence the true average being either 1 or 2. The trajectories show that all states converge to 1, and the corresponding total surplus settles at 10. Two immediate observations are respectively that it takes rather long time to converge to the true average, and that in the transient rather large surpluses are generated. These phenomena may be together due to the limited choices for inter-agent communications caused by the cyclic topology.

To obtain better performance in terms of convergence time and transient surplus amount, we consider a *complete digraph* of 30 agents. Fig. 4 exhibits the case where $\sum_{i=1}^{30} x_i(0) = -10$, thus the true average being either 0 or -1. The trajectories show that all states converge to 0, and the corresponding total surplus settles at -10. Compared to the cyclic case, the performance contrasts in convergence time and surplus peak are noticeable, which indeed match the intuition on the tradeoffs between communication costs and achievable performances.

V. Conclusion

We have studied distributed consensus problems in the setup where the states are quantized and the networks are directed and randomized. Specifically, we have derived necessary and sufficient graphical conditions that ensure general and average consensus. In future work, we are interested in analyzing the convergence rate of the proposed algorithms. In addition, the issue of finding other faster consensus algorithms deserves further investigation.

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