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Discrete Optimization

A model for dynamic chance constraints in hydro power reservoir management ☆

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ABSTRACT

In this paper, a model for (joint) dynamic chance constraints is proposed and applied to an optimization problem in water reservoir management. The model relies on discretization of the decision variables but keeps the probability distribution continuous. Our approach relies on calculating probabilities of rectangles which is particularly useful in the presence of independent random variables but works equally well in the case of correlated variables. Numerical results are provided for two and three stages.

1. Introduction

A conventional optimization problem under chance constraints is given by

$$\min\{f(x)|\mathbb{P}(g(x,\xi(\omega)) \geqslant 0) \geqslant p\}. \tag{1}$$

Here, $f:\mathbb{R}^n \to \mathbb{R}$ is some objective function, $g:\mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m$ is a constraint mapping, ξ is some s-dimensional random vector on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $p \in [0,1]$ is some specified safety level. The meaning of the chance constraint is as follows: a decision x is declared to be feasible, whenever the probability of satisfying the random inequality system

$$g_i(x,\xi(\omega)) \geqslant 0 \quad (i = 1,\ldots,m)$$

is at least p. Such constraints have importance in many engineering problems affected by random parameters the realization of which can be observed only after a (an optimal) decision has been taken. As basic references to theory, algorithms, stability and applications of chance constrained optimization problems we refer to [17,18,9].

In the setting above, no possible temporal character of randomness is taken into account. The decision is determined once and for ever, thus making (1) a *static* model. Often, however, decision making is *dynamic*, i.e., it is a process in the course of which an increasing amount of information is available due to successive realization of random variables. Then, intuitively, one could expect better solutions to problem (1) when decisions also rely on previously observed realizations of the random parameter rather than ignoring them. This is a standard viewpoint in multistage stochastic optimization problems. These, however, after discretizing the probability distribution, have the privilege of not being burdened by all the difficulties inherent to chance constraints. It is not clear, whether recent advantages in integer programming methods for chance constraints (e.g. [12]) would allow one to follow a similar way by discretizing the distribution and working with some appropriate scenario tree formulation of (1).

In this paper, we propose a semi-discrete (or semicontinuous) approach in that we suggest to discretize the dynamic decisions but not the distribution. This builds a bridge to existing methods for (continuous) static chance-constrained problems. The method will be applied to a single water reservoir management model. We keep the model as simple as possible in order to focus on aspects related to the dynamic character of chance constraints. The importance of chance constrained programming in the context of water reservoir management has been recognized a long time ago (see, e.g. the basic monograph [11] or [4,5,10,13]). We emphasize that, in contrast to most related papers,

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we consider the more appropriate and more difficult case of joint chance constraints rather than individual ones which would allow for simple quantile-based reformulations of the chance constraints via linear programming. The difference between the two models is that under joint constraints the probability is evaluated for the entire inequality system as in (1), whereas under individual constraints probabilities are computed for each inequality of the system separately. Though appealing by their easy application, individual chance constraints have the drawback, for instance when dealing with time-dependent constraints, of guaranteeing a certain probability level pointwise only, but usually one is interested in safety over a whole time horizon. So, while one could guarantee the satisfaction of some stochastic inequality with a probability of say 95% at each time step it could happen almost surely that violation occurs at least once in the considered interval.

Dealing with joint probability distributions in the context of chance constraints has been pioneered by Prékopa and corresponding models for the control of water reservoirs are found, for instance, in the early papers [15,16]. Here already, a dynamic approach is used by taking into account previously observed data via conditional distributions. On the other hand, in this model, no reaction of future decisions on future observations is anticipated, so it is quasi-static. In our paper, we adopt the viewpoint of multistage stochastic programs, where decisions at any future time stage are understood as functions of all previously observed random data (see, e.g. [14]). In contrast to the typical alternating sequence of decisions and observations in multistage programming, we let this sequence always terminate by an observation (the last one). In this way the last decision made does not have the meaning of just reacting on the last observation made but rather reacts on the observation before last and hedges against the last observation. This point of view (see also [6]) is coherent with the original philosophy of chance constraints.

2. Dynamic chance constraints

Let $\xi := (\xi_1, \ldots, \xi_T)$ be an s-dimensional discrete time stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote by

$$x = (x_1, x_2(\xi_1), x_3(\xi_1, \xi_2), \ldots, x_T(\xi_1, \ldots, \xi_{T-1})),$$

some adapted decision process, where the x_t are $\mathbb{R}^n$-valued. Furthermore, let

$$g : \mathbb{R}^{nT} \times \mathbb{R}^{sT} \to \mathbb{R}^m$$

be some constraint mapping. Then, we call the inequality

$$\mathbb{P}(g(x(\omega),\ \xi(\omega)) \geqslant 0) \geqslant p \quad (p \in [0,1]).$$

a *dynamic chance constraint*. Denoting by $\mu := \mathbb{P} \circ \xi^{-1}$ the law of ξ, we may rewrite the chance constraint as

$$\int_{A(x)} \mu(dz) \geqslant p, \tag{2}$$

where

$$A(x) = \bigcap_{i=1,\ldots,m} \{z | g_i(x_1, \ldots, x_T(z_1, \ldots, z_{T-1}), z_1, \ldots, z_T) \geqslant 0\}.$$

Here, the unknowns are the functions x_1, $x_2(z_1)$,…,$x_T(z_1,\ldots,z_{T-1})$. Choosing all these functions as (unknown) constants x_1, x_2,…,x_T, leads us back to the conventional static model of a chance constraint. Dealing with constraints like (2), where the variable x is of functional type seems to be very hard. Therefore, it is natural to discretize or parameterize x. More precisely, we put

$$x_t(z_1, \ldots, z_{t-1}) := \alpha_t(\pi_t, z_1, \ldots, z_{t-1}) \quad (t = 1, \ldots, T), \tag{3}$$

where $\pi_t \in \mathbb{R}^{P_t}$ is some finite-dimensional parameter and $\alpha_t : \mathbb{R}^{P_t} \times \mathbb{R}^{s(t-1)} \to \mathbb{R}^n$ is a given mapping. Then, (2) turns into

$$\int_{\widetilde{A}(\pi)} \mu(dz) \geqslant p, \tag{4}$$

where

$$\widetilde{A}(\pi) = \bigcap_{i=1,\ldots,m} \{z | g_i(\alpha_1(\pi_1), \alpha_2(\pi_2, z_1), \ldots, \alpha_T(\pi_T, z_1, \ldots, z_{T-1}), z_1, \ldots, z_T) \geqslant 0\} = \bigcap_{i=1,\ldots,m} \{z | \tilde{g}_i(\pi, z) \geqslant 0\}.$$

In other words, (4) has the form of a conventional (static) chance constraints with a finite dimensional decision vector $\pi = (\pi_1, \ldots, \pi_T)$ representing the parameters of the decision functions in the dynamic chance constraint. The choice of an appropriate parameterization is crucial for an efficient numerical treatment of constraints like (4).

3. A simple single water reservoir model

In the following we consider a simple model of a single water reservoir designed for hydro power generation. By $\xi := (\xi_1, \ldots, \xi_T)$, we denote a discrete scalar random process indicating stochastic inflows (precipitation) to the reservoir at time periods $1, \ldots, T$. Associated with ξ, we consider a scalar, non-anticipative decision process

$$x = (x_1, x_2(\xi_1), x_3(\xi_1, \xi_2), \ldots, x_T(\xi_1, \ldots, \xi_{T-1})),$$

reflecting the policy of water release from the reservoir. Given an initial water level l_0 in the reservoir, we require that over the whole period of time the water level remains between some upper value l^*(flood reserve) and some lower value l_* (dead storage). Since, the current water level is the sum of the initial level and the cumulative inflows until the given time minus the cumulative release at that time, one arrives at the following system of stochastic inequalities:

$$l_* \leqslant l_0 + \sum_{i=1}^{t} \xi_i - \sum_{i=1}^{t} x_i(\xi_1, \ldots, \xi_{i-1}) \leqslant l^* \quad (t = 1, \ldots, T). \tag{5}$$

As the realizations of future observations $\xi_t(\omega), \ldots, \xi_T(\omega)$ are not known when taking a decision at time t, the constraint system (5) is required to hold at least with some probability $p \in [0,1]$. The released water is used to produce energy which is sold on the power market. The objective is to maximize the expected profit subject to satisfying the level constraints with a given probability. The energy produced E by releasing a quantity r of water is a function of the current water level l which may be approximated by the expression $r(cl + d)$ (see, e.g. [5]). Thus, the (stochastic) amount of energy produced in period t is given by

$$E_t(x, \xi) = x_t(\xi_1, \ldots, \xi_{t-1}) \left(c \left(l_0 + \sum_{i=1}^{t-1} \xi_i - \sum_{i=1}^{t-1} x_i(\xi_1, \ldots, \xi_{i-1}) \right) + d \right).$$

Now, the expected total amount of energy produced over the whole time period becomes

$$f(x) := \sum_{t=1}^{T} \mathbb{E} E_t(x, \xi), \tag{6}$$

where '$\mathbb{E}$' denotes expectation. In addition to the level constraints (to be formulated as a chance constraint), we do not allow negative releases and we require some condition on the final filling level of the reservoir (cycling constraint), in order to avoid that additional profit can be made in the considered time horizon at the expense of later horizons. There are many different ways of doing so. One possibility is the use of some valuation function for the final filling level which is subtracted from the profit in the objective of the optimization problem. A stringent alternative would consist in requiring some final filling level to be exceeded with probability 1 or – more generally in the sense of a probabilistic constraint again – with probability $p \in [0,1]$. The least demanding yet reasonable cycling constraint would insist on some final level in expectation only. All these alternatives are easily incorporated into our model. For simplicity, we content ourselves here with an expected value constraint on keeping the initial filling level at final time. Using the expression in (5) for the current filling level at $t = T$, the expected cycling constraint can be written as

$$\sum_{i=1}^{T} \mathbb{E}\xi_i = \sum_{i=1}^{T} \mathbb{E} x_i(\xi_1, \ldots, \xi_{i-1}).$$

Summarizing, we arrive at the following optimization problem under dynamic chance constraints:

$$\begin{aligned} &\max f(x) \quad \text{subject to} \\ &\mathbb{P}\left(l_* \leqslant l_0 + \sum_{i=1}^{t} \xi_i - \sum_{i=1}^{t} x_i(\xi_1, \ldots, \xi_{i-1}) \leqslant l^* \quad (t = 1, \ldots, T) \right) \geqslant p \\ &x_t(\xi_1, \ldots, \xi_{t-1}) \geqslant 0 \quad \mathbb{P}\text{-almost surely} \quad (t = 1, \ldots, T) \\ &\sum_{i=1}^{T} \mathbb{E}\xi_i = \sum_{i=1}^{T} \mathbb{E} x_i(\xi_1, \ldots, \xi_{i-1}). \end{aligned} \tag{7}$$

Evidently, the chance constraint in (7) is of type (2) with $n = s = 1$ (scalar processes) and $m = T$ (time constraints).

4. Discretization of the decision policies

In principle one could proceed as suggested in Section 2 and parameterize the decision policies according to (3), in order to arrive at a chance constraint of type (4) in a finite dimensional setting. It turns out, however, that for the convenience of numerical treatment, a simple transformation $x \to \psi$ is useful, which can easily be inverted after having calculated optimal transformed policies ψ. More precisely, we introduce the functions

$$\psi_t(z_1, \ldots, z_{t-1}) := l^* - l_0 + \sum_{i=1}^{t} x_i(z_1, \ldots, z_{i-1}) - \sum_{i=1}^{t-1} z_i \quad (t = 1, \ldots, T). \tag{8}$$

Note, that for $t = 1$, one gets the deterministic (transformed) decision variable $\psi_1 = l^* - l_0 + x_1$. With this transformation, one can rewrite the chance constraint in (7) as

$$\mathbb{P}(\psi_t(\xi_1, \ldots, \xi_{t-1}) + l_* - l^* \leqslant \xi_t \leqslant \psi_t(\xi_1, \ldots, \xi_{t-1}) \quad (t = 1, \ldots, T)) \geqslant p. \tag{9}$$

The advantage of (9) over the original form of the chance constraint in (7) is that the event of which the probability is taken is defined in terms of the very components of ξ rather than in terms of cumulated components. This allows us later to reduce the calculation of probabilities to simple rectangles. Now, we approximate the policies ψ_t by piecewise constant functions $\tilde{\psi}_t$. Accordingly we make a step function ansatz with N pieces in each dimension (of course one could also consider different numbers of pieces in each dimension but this would just increase the notational complexity here)

$$\tilde{\psi}_t(z_1, \ldots, z_{t-1}) := \sum_{i_1, \ldots, i_{t-1}=1}^{N} a(i_1, \ldots, i_{t-1}) \chi_{Q(i_1, \ldots, i_{t-1})}(z_1, \ldots, z_{t-1}) \quad (t = 1, \ldots, T), \tag{10}$$

where χ refers to the characteristic function and $Q(i_1, \ldots, i_{t-1}) \subseteq \mathbb{R}^{t-1}$ are disjoint subsets covering the whole space. For the case $t = 1$, we have $\tilde{\psi}_1 = a$ for some scalar a. Our aim is to substitute the infinite dimensional optimization problem (7) by a finite dimensional one which just depends on the parameter vector

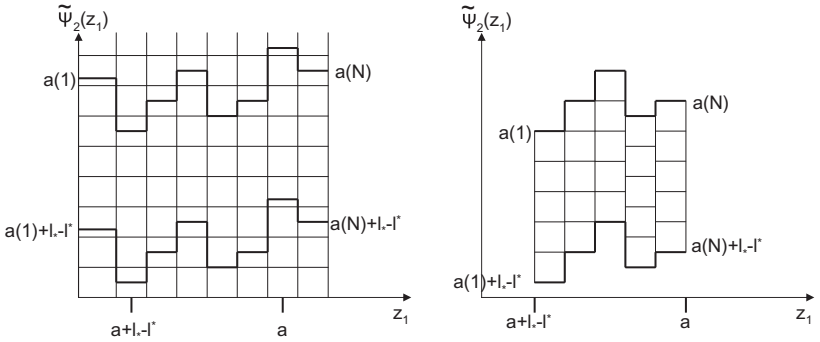

Fig. 1. Illustration of the step functions $\tilde{\psi}_2$ defined on a fixed (left part) and a variable grid (right part).

$$\vec{a} = (a, a(1), \ldots, a(N), a(1,1), \ldots, a(N,N), \ldots, a(1, \ldots, 1), \ldots, a(N, \ldots, N)), \tag{11}$$

defining the functions $\tilde{\psi}_t$ via (10). To start with, we replace the chance constraint (9) by the approximate chance constraint

$$\mathbb{P}(\tilde{\psi}_t(\xi_1, \ldots, \xi_{t-1}) + l_* - l^* \leqslant \xi_t \leqslant \tilde{\psi}_t(\xi_1, \ldots, \xi_{t-1}) \quad (t = 1, \ldots, T)) \geqslant p. \tag{12}$$

Fig. 1 (left part) illustrates the second ansatz function $\tilde{\psi}_2$ ($\tilde{\psi}_1$ reducing to the scalar a as mentioned before) defined by the step coefficients $a(1), \ldots, a(N)$ on some fixed grid. If the problem were just a two-stage one (i.e., $T = 2$), then (12) would mean that the probability of the set between the step function and its downwards shifted copy as well as between the limits $a + l_* - l^*$ and a be not smaller than p. Considering a fixed grid leads to certain difficulties due to the fact that the components of $\vec{a}$ will not lie on a grid point in the course of iterations which complicates the calculation of the desired probability. Moreover, for checking (12), it is not necessary that the step functions $\tilde{\psi}_t$ be determined on their whole domain spaces. It is sufficient to know them on the domain defined by the previous relations

$$\tilde{\psi}_\tau(\xi_1, \ldots, \xi_{\tau-1}) + l_* - l^* \leqslant \xi_\tau \leqslant \tilde{\psi}_\tau(\xi_1, \ldots, \xi_{\tau-1})$$

for $\tau < t$ (see right part of Fig. 1). This leads us to propose a kind of dynamic grid adapted to the current values of the variable $\vec{a}$. To be more precise, given a discretization parameter $N \in \mathbb{N}$, let for $t = 1, \ldots, T$

$$\begin{aligned} &Q(i_1, \ldots, i_t) := [r_{i_1}, r_{i_1+1}) \times [r_{i_1,i_2}, r_{i_1,i_2+1}) \times \cdots \times [r_{i_1,i_2,\ldots,i_t}, r_{i_1,i_2,\ldots,i_t+1}), \\ &r_{i_1,i_2,\ldots,i_t} := a(i_1, \ldots, i_{t-1}) + l_* - l^* + \Delta_{i_t}, \\ &\Delta_i := (i-1)(l^* - l_*)/N \quad (i = 1, \ldots, N+1). \end{aligned} \tag{13}$$

Observe that $\Delta_1 = 0$ and $\Delta_{N+1} = l^* - l_*$, whence $r_{i_1,i_2,\ldots,i_{t-1},1} = a(i_1, \ldots, i_{t-2}) + l_* - l^*$ and $r_{i_1,i_2,\ldots,i_{t-1},N+1} = a(i_1, \ldots, i_{t-2})$. The two-stage grid defined by the half open rectangles

$$Q(i_1, i_2) := [r_{i_1}, r_{i_1+1}) \times [r_{i_1,i_2}, r_{i_1,i_2+1})$$

is illustrated in the right part of Fig. 1. Of course, one might object that choosing a grid size depending on the (arbitrary) variable sizes of the intervals

$$[a(i_1, \ldots, i_{t-1}) + l_* - l^*, a(i_1, \ldots, i_{t-1})]$$

may not be the best idea from the numerical point of view, but we are not concerned with this detail in the present paper yet. Rather, we benefit from the conceptual advantage of the chosen variable grid in order to facilitate the calculation of (12). As a preparatory result we have the following proposition whose proof is straightforward and therefore omitted.

Proposition 1. *A vector $z \in \mathbb{R}^T$ satisfies the inequality system*

$$\tilde{\psi}_t(z_1, \ldots, z_{t-1}) + l_* - l^* \leqslant z_t < \tilde{\psi}_t(z_1, \ldots, z_{t-1}) \quad (t = 1, \ldots, T) \tag{14}$$

with $\tilde{\psi}_t$ as in (10) and $Q(i_1, \ldots, i_t)$ as in (13) if and only if

$$z \in \bigcup_{i_1=1}^{N} \cdots \bigcup_{i_T=1}^{N} Q(i_1, \ldots, i_T). \tag{15}$$

Now, we are in a position to make the probability (12) explicit in terms of the coefficients contained in $\vec{a}$:

Theorem 2. *Let the random vector ξ have an absolutely continuous distribution. Then,*

$$\begin{aligned} &\mathbb{P}(\tilde{\psi}_t(\xi_1, \ldots, \xi_{t-1}) + l_* - l^* \leqslant \xi_t \leqslant \tilde{\psi}_t(\xi_1, \ldots, \xi_{t-1}) \quad (t = 1, \ldots, T)) \\ &\qquad = \sum_{i_1=1}^{N} \cdots \sum_{i_{T-1}=1}^{N} \mathbb{P}(\xi \in A_{i_1}(\vec{a}) \times A_{i_1,i_2}(\vec{a}) \cdots \times A_{i_1,i_2,\ldots,i_{T-1}}(\vec{a}) \times B_{i_1,i_2,\ldots,i_{T-1}}(\vec{a})), \end{aligned}$$

where $\vec{a}$ refers to (11) and the one-dimensional intervals $A_{i_1,i_2,\ldots,i_t}(\vec{a})$ and $B_{i_1,i_2,\ldots,i_{T-1}}(\vec{a})$ are defined as

$$A_{i_1,i_2,\ldots,i_t}(\vec{a}) := [a(i_1, \ldots, i_{t-1}) + l_* - l^* + \Delta_{i_t}, a(i_1, \ldots, i_{t-1}) + l_* - l^* + \Delta_{i_t+1}] (t = 1, \ldots, T-1).$$

$$B_{i_1,i_2,\ldots,i_{T-1}}(\vec{a}) := [a(i_1,\ldots,i_{T-1}) + l_* - l^*, a(i_1,\ldots,i_{T-1})].$$

Proof. We first observe that

$$\begin{aligned}
\bigcup_{i_1=1}^{N}\cdots\bigcup_{i_T=1}^{N} Q(i_1,\ldots,i_T) &= \bigcup_{i_1=1}^{N}\cdots\bigcup_{i_{T-1}=1}^{N}\bigcup_{i_T=1}^{N}[r_{i_1},r_{i_1+1})\times\cdots\times[r_{i_1,i_2,\ldots,i_{T-1}},r_{i_1,i_2,\ldots,i_{T-1}+1})\times[r_{i_1,i_2,\ldots,i_T},r_{i_1,i_2,\ldots,i_T+1})\\
&= \bigcup_{i_1=1}^{N}\cdots\bigcup_{i_{T-1}=1}^{N}[r_{i_1},r_{i_1+1})\times\cdots\times[r_{i_1,\ldots,i_{T-1}},r_{i_1,i_2,\ldots,i_{T-1}+1})\times\left\{\bigcup_{i_T=1}^{N}[r_{i_1,i_2,\ldots,i_T},r_{i_1,i_2,\ldots,i_T+1})\right\}\\
&= \bigcup_{i_1=1}^{N}\cdots\bigcup_{i_{T-1}=1}^{N}[r_{i_1},r_{i_1+1})\times\cdots\times[r_{i_1,i_2,\ldots,i_{T-1}},r_{i_1,i_2,\ldots,i_{T-1}+1})\times[r_{i_1,i_2,\ldots,i_{T-1},1},r_{i_1,i_2,\ldots,i_{T-1},N+1}).
\end{aligned}$$

Then, the absolute continuity of the distribution of ξ along with Proposition 1 yield:

$$\begin{aligned}
&\mathbb{P}(\tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) + l_* - l^* \leqslant \xi_t < \tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) \quad (t=1,\ldots,T))\\
&\qquad = \mathbb{P}(\xi\in\bigcup_{i_1=1}^{N}\cdots\bigcup_{i_T=1}^{N} Q(i_1,\ldots,i_T)) = \mathbb{P}(\xi\in\bigcup_{i_1=1}^{N}\cdots\bigcup_{i_{T-1}=1}^{N} A_{i_1}(\vec{a})\times A_{i_1,i_2}(\vec{a})\cdots\times A_{i_1,i_2,\ldots,i_{T-1}}(\vec{a})\times B_{i_1,i_2,\ldots,i_{T-1}}(\vec{a}))\\
&\qquad = \sum_{i_1=1}^{N}\cdots\sum_{i_{T-1}=1}^{N}\mathbb{P}(\xi\in A_{i_1}(\vec{a})\times A_{i_1,i_2}(\vec{a})\cdots\times A_{i_1,i_2,\ldots,i_{T-1}}(\vec{a})\times B_{i_1,i_2,\ldots,i_{T-1}}(\vec{a})).
\end{aligned}$$

Here, the last equality relies on the fact that for different $(T-1)$- tuples $(i_1,i_2,\ldots,i_{T-1})$, the sets

$$A_{i_1}(\vec{a})\times A_{i_1,i_2}(\vec{a})\cdots\times A_{i_1,i_2,\ldots,i_{T-1}}(\vec{a})\times B_{i_1,i_2,\ldots,i_{T-1}}(\vec{a})$$

are almost disjoint in the sense that they overlap only at sets of Lebesgue measure zero. □

The chosen approach turns out to be particularly useful under the assumption of ξ having independent components because this allows us to reduce the calculation of probabilities to one-dimensional distribution functions. As an immediate consequence of Theorem 2 we have:

Corollary 3. *Let the random vector ξ have an absolutely continuous distribution with independent components. Then,*

$$\begin{aligned}
&\mathbb{P}(\tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) + l_* - l^* \leqslant \xi_t \leqslant \tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) \quad (t=1,\ldots,T))\\
&\qquad = \sum_{i_1=1}^{N}\cdots\sum_{i_{T-1}=1}^{N}(\Phi_T(a(i_1,\ldots,i_{T-1})) - \Phi_T(a(i_1,\ldots,i_{T-1}) + l_* - l^*))\cdot\prod_{t=1}^{T-1}(\Phi_t(a(i_1,\ldots,i_{t-1}) + l_* - l^* + \Delta_{i_t+1})\\
&\qquad\quad - \Phi_t(a(i_1,\ldots,i_{t-1}) + l_* - l^* + \Delta_{i_t})),
\end{aligned}$$

where Φ_t refers to the one-dimensional distribution function of the component ξ_t. In the two-stage situation (T = 2) one gets

$$\begin{aligned}
&\mathbb{P}(\tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) + l_* - l^* \leqslant \xi_t \leqslant \tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) \quad (t=1,2))\\
&\qquad = \sum_{i=1}^{N}(\Phi_2(a(i)) - \Phi_2(a(i) + l_* - l^*))\cdot(\Phi_1(a + l_* - l^* + \Delta_{i+1}) - \Phi_1(a + l_* - l^* + \Delta_i)).
\end{aligned}$$

But also in the more realistic case of correlated components of ξ one may benefit from the obtained structure by the possibility to reduce rectangle probabilities to (now: multidimensional) distribution functions. At least for a small number T of stages this seems to be a viable procedure. We present a corresponding result for the special case of two stages:

Corollary 4. *Let the random vector ξ have an absolutely continuous distribution. In the two-stage situation (T = 2), one gets*

$$\begin{aligned}
&\mathbb{P}(\tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) + l_* - l^* \leqslant \xi_t \leqslant \tilde{\psi}_t(\xi_1,\ldots,\xi_{t-1}) \quad (t=1,2))\\
&\qquad = \sum_{i=1}^{N}\Phi(a + l_* - l^* + \Delta_{i+1}, a(i)) - \Phi(a + l_* - l^* + \Delta_i, a(i)) + \Phi(a + l_* - l^* + \Delta_i, a(i) + l_* - l^*)\\
&\qquad\quad - \Phi(a + l_* - l^* + \Delta_{i+1}, a(i) + l_* - l^*),
\end{aligned}$$

where Φ denotes the two-dimensional distribution function of $\xi = (\xi_1, \xi_2)$.

Proof. The proof follows immediately from Theorem 2 upon observing that, for any rectangle $[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]$ one has

$$\mathbb{P}(\xi\in[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]) = \Phi(\alpha_2,\beta_2) - \Phi(\alpha_1,\beta_2) + \Phi(\alpha_1,\beta_1) - \Phi(\alpha_2,\beta_1). \quad \square$$

5. The finite dimensional optimization problem

In the last section, we were able via Theorem 2 to represent a finite dimensional approximation (in terms of the finite dimensional variable $\vec{a}$) of the infinite dimensional chance constraint in problem (7). To do so with the whole optimization problem (7), we have to express

first the original unknown x in terms of $\vec{a}$ and then to substitute the corresponding expressions in the remaining constraints as well as in the objective of (7). At the same time, this will show us how to recover x from a solution $\vec{a}^*$ of the finite dimensional problem. Note that (8) is easily inverted in terms of x:

$$x_1 = \psi_1 + l_0 - l^*,$$
$$x_t(z_1,\dots,z_{t-1}) = \psi_t(z_1,\dots,z_{t-1}) - \psi_{t-1}(z_1,\dots,z_{t-2}) + z_{t-1} \quad (t = 2,\dots,T).$$

Now, using the approximations $\tilde{\psi}_t$ of ψ_t, we get corresponding approximations $\hat{x}_t$ of x_t. Using (10), the explicit dependence of $\hat{x}_t$ on the unknown $\vec{a}$ is revealed:

$$\hat{x}_1 = \tilde{\psi}_1 + l_0 - l^* = a + l_0 - l^*,$$

$$\hat{x}_t(z_1,\dots,z_{t-1}) = \tilde{\psi}_t(z_1,\dots,z_{t-1}) - \tilde{\psi}_{t-1}(z_1,\dots,z_{t-2}) + z_{t-1}$$
$$= \sum_{i_1,\dots,i_{t-1}=1}^{N} a(i_1,\dots,i_{t-1})\chi_{Q(i_1,\dots,i_{t-1})}(z_1,\dots,z_{t-1}) - \sum_{i_1,\dots,i_{t-2}=1}^{N} a(i_1,\dots,i_{t-2})\chi_{Q(i_1,\dots,i_{t-2})}(z_1,\dots,z_{t-2}) + z_{t-1} \quad (t = 2,\dots,T).$$

Observe that $\hat{x}_t$ above is not globally defined as a consequence of the same fact holding true for the functions $\tilde{\psi}_t$. The latter were defined only on a subdomain corresponding to the region of interest of the chance constraint. This raises the question of how to extend them globally (which is important for instance, in order to calculate their expectations as needed in (7)). One could agree that the $\hat{x}_t$ remain free outside the mentioned subdomain thus serving the aim of maximizing the objective in (7). However, one could also adopt the viewpoint that arguments outside the subdomain represent violations of the filling level constraint, and in such cases no benefit in terms of profit from energy production should be allowed. This idea, which we shall follow from now on, reflects some additional stimulus for not violating the filling level constraints which, in addition to the imposed chance constraint, acts in the same direction by not rewarding profit under constraint violation. Therefore, we put $\hat{x}_t \equiv 0$ outside the domain

$$\bigcup_{i_1=1}^{N} \cdots \bigcup_{i_{t-1}=1}^{N} Q(i_1,\dots,i_{t-1}).$$

The computation of the expectation $\mathbb{E}\hat{x}_t$ is simple as far as the contributions of step functions to the definition of $\hat{x}_t$ are concerned. However, the remaining contribution coming from z_{t-1} is less obvious because it amounts to the expectation of ξ_t restricted to the mentioned subdomain. Calculating this expectation exactly may be complicated or not even possible. To overcome this difficulty, we propose to approximate the functions $f(z_1,\dots,z_{t-1}) = z_{t-1}$ as step functions defined on the same grid as the $\hat{x}_t$:

$$\tilde{z}_t(z_1,\dots,z_t) := \sum_{i_1,\dots,i_t=1}^{N} \bar{a}(i_1,\dots,i_t)\chi_{Q(i_1,\dots,i_t)}(z_1,\dots,z_t) \quad (t = 1,\dots,T-1).$$

Here, the coefficients $\bar{a}(i_1,\dots,i_t)$ are chosen as the midpoints of the intervals

$$[r_{i_1,i_2,\dots,i_t}, r_{i_1,i_2,\dots,i_t+1})$$

or stated explicitly (see (13)):

$$\bar{a}(i_1,\dots,i_t) := (r_{i_1,i_2,\dots,i_t} + r_{i_1,i_2,\dots,i_t+1})/2 = a(i_1,\dots,i_{t-1}) + l_* - l^* + \left(i_t - \frac{1}{2}\right)\frac{l^* - l_*}{N}. \tag{16}$$

Doing so, $\bar{a}(i_1,\dots,i_t)$ approximates the value of z_t on the rectangle $Q(i_1,\dots,i_t)$ by its average. With this approximation $\tilde{z}_t$ of z_t, we arrive, starting from the previously obtained expressions for $\hat{x}_t$, at the following second approximation $\tilde{x}_t$ of x_t which finally is fully amenable to the desired formulation of our finite dimensional optimization problem:

$$\tilde{x}_1 = a + l_0 - l^*,$$

$$\tilde{x}_t(z_1,\dots,z_{t-1}) = \sum_{i_1,\dots,i_{t-1}=1}^{N} (a(i_1,\dots,i_{t-1}) + \bar{a}(i_1,\dots,i_{t-1}))\chi_{Q(i_1,\dots,i_{t-1})}(z_1,\dots,z_{t-1}) - \sum_{i_1,\dots,i_{t-2}=1}^{N} a(i_1,\dots,i_{t-2})\chi_{Q(i_1,\dots,i_{t-2})}(z_1,\dots,z_{t-2}) \quad (t=2,\dots,T).$$

Note that the characteristic functions $\chi_{Q(i_1,\dots,i_{t-2})}$ themselves can be made explicit in terms of the unknown $\vec{a}$ via (13). Now, for instance, the nonnegativity constraint in (7) can be written as

$$a \geqslant l^* - l_0 \sum_{i_1,\dots,i_{t-1}=1}^{N} (a(i_1,\dots,i_{t-1}) + \bar{a}(i_1,\dots,i_{t-1}))\chi_{Q(i_1,\dots,i_{t-1})}(z_1,\dots,z_{t-1})$$
$$\geqslant \sum_{i_1,\dots,i_{t-2}=1}^{N} a(i_1,\dots,i_{t-2})\chi_{Q(i_1,\dots,i_{t-2})}(z_1,\dots,z_{t-2}) \quad (\forall (z_1,\dots,z_{t-1}); \quad \forall t \in \{2,\dots,T\}).$$

Evaluating pointwise the last system of inequalities, we may invoke the definition of the rectangles $Q(i_1,\dots,i_t)$ to derive the identity

$$\chi_{Q(i_1,\dots,i_{t-1})}(z_1,\dots,z_{t-1}) = \chi_{Q(i_1,\dots,i_{t-2})}(z_1,\dots,z_{t-2}) = 1 \quad \forall (z_1,\dots,z_{t-1}) \in Q(i_1,\dots,i_{t-1}).$$

Since the $Q(i_1,\dots,i_{t-1})$ form a partition of the domain of $\tilde{x}_t$, we may further exploit (16) in order to reduce the relations above to the following explicit inequality system in the components of the unknown $\vec{a}$:

$$a \geqslant l^* - l_0,$$

$$a(i_1,\ldots,i_{t-1}) \geqslant l^* - l_* + \left(i_{t-1} - \frac{1}{2}\right)\frac{l_* - l^*}{N} \quad \forall (i_1,\ldots,i_{t-1}) \in \{1,\ldots,N\}^{t-1} \quad \forall t \in \{2,\ldots,T\}.$$

Next, we obtain the expectations

$$\mathbb{E}\tilde{x}_1 = a + l_0 - l^*,$$

$$\mathbb{E}\tilde{x}_t = \sum_{i_1,\ldots,i_{t-1}=1}^{N} (a(i_1,\ldots,i_{t-1}) + \bar{a}(i_1,\ldots,i_{t-1}))\mathbb{P}((\xi_1,\ldots,\xi_{t-1}) \in Q(i_1,\ldots,i_{t-1})) - \sum_{i_1,\ldots,i_{t-2}=1}^{N} a(i_1,\ldots,i_{t-2})\mathbb{P}((\xi_1,\ldots,\xi_{t-2}) \in Q(i_1,\ldots,i_{t-2})) \quad (t = 2,\ldots,T).$$

Similarly to the calculation of the chance constraints, we are again faced with the need of computing probabilities of rectangles, this time, however, in one dimension less. If, for instance, $T = 2$, then the computation of $\mathbb{E}\tilde{x}_2$ requires just the probability of intervals, etc. Again, as for the chance constraints, one might benefit from the assumption of independent components for larger values of T.

Now, that we are able to calculate expectations $\mathbb{E}\tilde{x}_t$, we can immediately rephrase the last constraint (cycling constraint) in (7) in terms of the unknown $\vec{a}$.

As far as the objective(6) is concerned, we approximate the terms $E_t(x)$ in the vein of the previous derivations as

$$E_t(x,z) \approx E_t(\tilde{x},z) = \tilde{x}_t(z_1,\ldots,z_{t-1})\left(c\left(l_0 + \sum_{i=1}^{t-1}\tilde{z}_i(z_1,\ldots,z_i) - \sum_{i=1}^{t-1}\tilde{x}_i(z_1,\ldots,z_{i-1})\right) + d\right).$$

Using the already developed step function representations for $\tilde{x}_t$ and $\tilde{z}_i$ one may again explicitly calculate the expectations $\mathbb{E}E_t(\tilde{x},\xi)$. Rather than deriving the involved general formula here, we present its expression for the two-stage case $T = 2$. Trivially,

$$\mathbb{E}E_1(\tilde{x},\xi) = \tilde{x}_1(cl_0 + d) = (a + l_0 - l^*)(cl_0 + d).$$

Next, using the step function representations for $\tilde{x}_2$ and $\tilde{z}_1$, we obtain

$$E_2(\tilde{x},z) = \tilde{x}_2(z)(c(l_0 + \tilde{z}_1(z) - \tilde{x}_1) + d) = \left(\sum_{i=1}^{N}(a(i) + \bar{a}(i))\chi_{Q(i)}(z) - a\right)\left(c\left(l_0 + \sum_{j=1}^{N}\bar{a}(j)\chi_{Q(j)}(z) - \tilde{x}_1\right) + d\right).$$

The further development of this expression relies on the fact that the product of characteristic functions of two sets equals the characteristic function of the intersection of these sets and on the sets $Q(i)$ and $Q(j)$ being disjoint for $i \neq j$ (see (13)). One then finally arrives at

$$E_2(\tilde{x},z) = \sum_{i=1}^{N}(a(i) + \bar{a}(i) - a)((c(l^* - a + \bar{a}(i)) + d))\chi_{Q(i)}(z).$$

Recalling from (13) that

$$Q(i) = [a + l_* - l^* + \Delta_i, a + l_* - l^* + \Delta_{i+1}),$$

we get

$$\mathbb{E}E_2(\tilde{x},\xi) = \sum_{i=1}^{N}(a(i) + \bar{a}(i) - a)((c(l^* - a + \bar{a}(i)) + d)) \cdot (\Phi_1(a + l_* - l^* + \Delta_{i+1}) - \Phi_1(a + l_* - l^* + \Delta_i)),$$

where Φ_1 refers to the one-dimensional distribution function of ξ_1.

In order to give an impression about the discretized counterpart to the original problem (7) we write it out explicitly for the two-stage case ($T = 2$):

maximize

$$(a + l_0 - l^*)(cl_0 + d) + \sum_{i=1}^{N}(a(i) + \bar{a}(i) - a)((c(l^* - a + \bar{a}(i)) + d)) \cdot \mathbb{P}\left(a + \frac{N+1-i}{N}(l_* - l^*) \leqslant \xi_1 \leqslant a + \frac{N-i}{N}(l_* - l^*)\right)$$

subject to

$$\sum_{i=1}^{N}\mathbb{P}\begin{pmatrix} a + \frac{N+1-i}{N}(l_* - l^*) & \leqslant \xi_1 \leqslant & a + \frac{N-i}{N}(l_* - l^*) \\ a(i) + l_* - l^* & \leqslant \xi_2 \leqslant & a(i) \end{pmatrix} \geqslant p$$

$$a \geqslant l^* - l_0, a(i) \geqslant l^* - l_* + \left(i - \frac{1}{2}\right)\frac{l_* - l^*}{N} (i = 1,\ldots,N)$$

$$\mathbb{E}\xi_1 + \mathbb{E}\xi_2 = l_0 - l^* + \sum_{i=1}^{N}(a(i) + \left(\frac{i - N - 1/2}{N}\right)(l^* - l_*) + a) \cdot \mathbb{P}\left(a + \frac{N+1-i}{N}(l_* - l^*) \leqslant \xi_1 \leqslant a + \frac{N-i}{N}(l_* - l^*)\right).$$

The unknowns in this problem are the variables a, $a(1),\ldots,a(N)$, whereas the $\bar{a}(i)$ are calculated via (16). Once, the optimal values for these unknowns have been determined, one can easily identify the approximated release policies $\tilde{x}_t$ as described above in this section. As the optimization problem is nonconvex, one may not expect to find global solutions to it. Local solutions can be determined by means of any appropriate algorithm for constrained nonlinear optimization. The basic requirement to do so is to be able to calculate probabilities of rectangular sets and gradients thereof (with respect to upper and lower limits). This is not a big challenge in the two-stage setting presented above because there all occurring probabilities are taken in one or two dimensions. If more stages are included, then again, no problem arises as long as the components of the random vector are assumed to be independent because everything can be reduced then to the calculus of probabilities in one dimension. If, however, correlations between components are admitted in the general T-stage case, then things become much more difficult. Fortunately, at least for multivariate normal distributions, efficient codes for calculating rectangular probabilities exist (see [7,8]). Moreover, it is possible to reduce the determination of gradients for rectangular probabilities to the calculus of probabilities themselves in reduced dimension (see [1]), so the same code can be employed to provide functional values and gradients of probabilistic constraints at a time.

6. Numerical results

In this section, we illustrate our approach to dynamic chance constraints by numerical examples in a two-stage and three-stage setting. We consider the water reservoir problem presented in Section 3 with the following data:

$$T = 2, l_* = 1, l^* = 3, p = 0.9, c = 2, d = 1, \xi \sim \mathcal{N}\left((1,1), \begin{pmatrix} 0.09 & 0 \\ 0 & 0.09 \end{pmatrix}\right),$$

the latter expression meaning that the random vector $\xi = (\xi_1, \xi_2)$ has a bivariate normal distribution with independent components each of them having mean 1 and variance 0.09 (or standard deviation 0.3). We take the initial (and expected final) filling level l_0 as a parameter moving between l_* and l^*. We solve the finite dimensional optimization problem formulated in Section 5 with a discretization parameter $N = 160$. For dealing with the dynamic chance constraint, we make use of the second statement in Corollary 3. Similarly, we make use of the explicit representation of the objective developed for the two-stage case in Section 5. For the numerical solution we employed the nonlinear optimization solver of *Mathematica* which was sufficient for our purposes and did the job in the range of seconds or few minutes.

Fig. 2 illustrates the obtained results. The left upper part provides a plot of optimal first stage release decisions x_1 as a function of the initial filling level l_0. The right upper part plots the optimal second stage release policies x_2 as functions of the first inflow realization ξ_1 for initial filling levels l_0 reaching from 1.5 to 2.6 (curves from left to right). As can be seen, the optimal release policies become linear on their domain for extreme initial levels whereas they are strongly nonlinear in between. The reason is that for extreme initial filling levels of 1.5 and 2.6, respectively, the feasible domain shrinks in a way that only trivial release (release of second stage equals inflow of first stage) is left as an option. In between, however, the feasible set becomes larger and allows for adjusting release policies in a way to maximize profit. The resulting optimal policies are nonlinear then (in contradiction to the assumption in the application of linear decision rules). The lower left

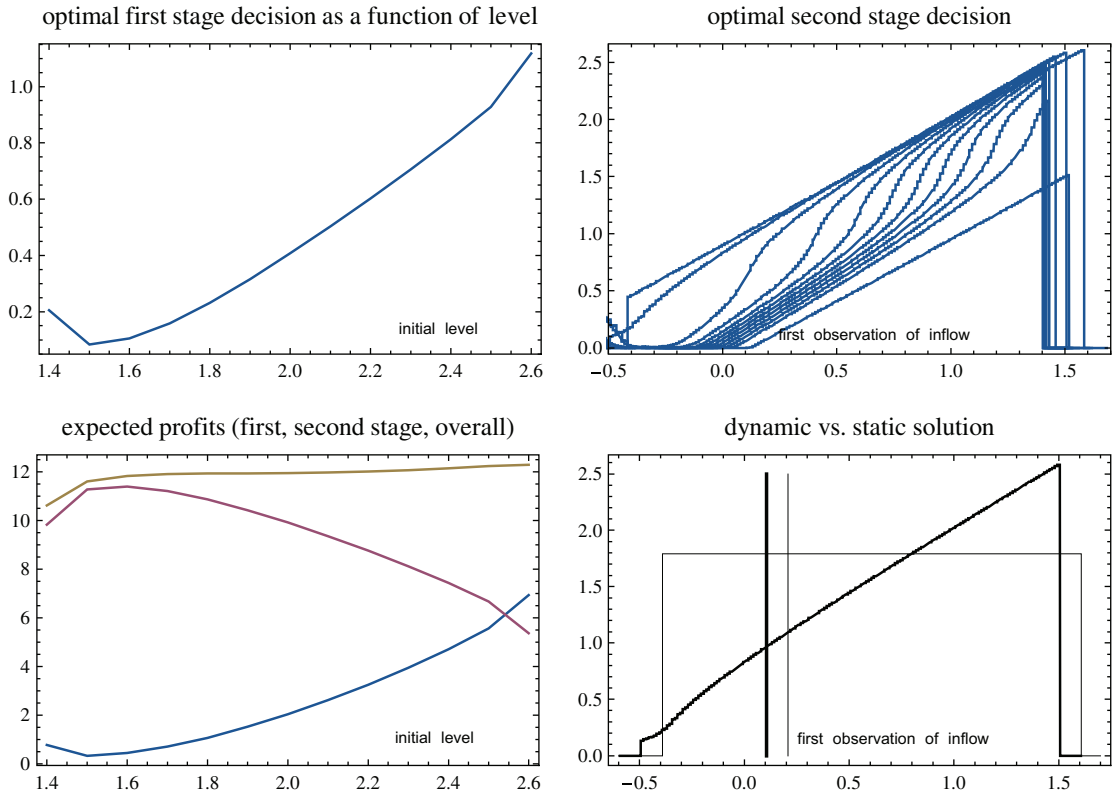

Fig. 2. Illustration of numerical results for a two stage instance of the water reservoir problem assuming independent components of the random inflow. For details, see text.

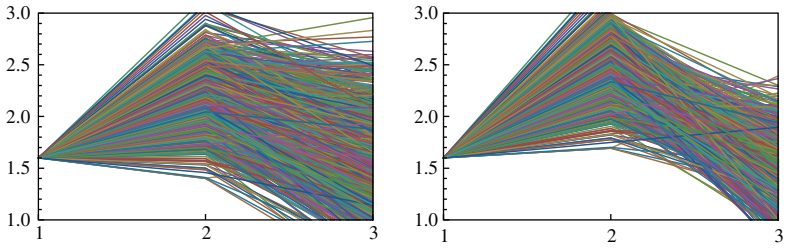

Fig. 3. Comparison of 1000 filling level scenarios for a static and a dynamic solution. Filling levels are plotted over the current stage.

diagram plots the expected profits (from bottom to top) of the optimal first and second stage decisions as well as the overall expected profit as functions of l_0. Roughly speaking, the same overall expected profit can be realized over the whole interval of initial filling levels but with strongly varying contributions by the first and second order release decisions. Finally, the lower right diagram contrasts the optimal static (thin lines) and dynamic (thick lines) solutions of the problem for a fixed $l_0 = 1.6$. The (scalar) first stage decisions are represented by vertical lines identifying the corresponding value on the first axis. In contrast to the dynamic release policy, the optimal second stage decisions in the static case reduces to a constant (on the domain of feasible operation).

It is obvious that a static solution cannot realize the same value of the objective function as the dynamic one, but has to be worse due to a lack of flexibility in its reaction on observations of first stage inflow. Indeed, the expected profit of the static solution amounts to 9.89 as opposed to an expected profit of 11.83 for the dynamic solution. The difference $11.83 - 9.89 = 1.94$ can be interpreted as the *Value of the Dynamic Solution* (VDS). Taken as a relative figure, the improvement of the objective reaches approximately 16% in this example.

The difference between the static and the dynamic solution can also be made out from Fig. 3, where the two different solutions mentioned above are applied to 1000 precipitation scenarios simulated for the two stages according to the assumed normal distribution. In both cases the cycling constraint (expected value of final level equals that of the initial level) as well as the probabilistic constraints for the upper and lower level at both stages are satisfied. We emphasize that due to our model considering the joint distribution of precipitations over the time horizon, approximately 900 out of the 1000 profiles stay completely inside the frame of the figure. With the usually employed simpler individual chance constraints, one could guarantee only for each of the two stages separately, that approximately 900 of these profiles remain within the feasable limits. This generally results in much less than 900 profiles staying inside the whole frame. The difference between the static and the dynamic solution is revealed by a reduced variance for the final filling level in case of the more flexible dynamic solution.

If correlations between the random inflow components are to be taken into account, then only the computation of the chance constraint is affected. One can no longer rely on Corollary 3. Nevertheless, one may employ Corollary 4 which comes at the price of calculating bivariate distribution functions. This, however, is the only change as compared to the case of independent components. Fig. 4 illustrates the optimal second stage decision policies in the previous example for different correlation coefficients between the components of the inflow variables.

Another issue of interest in optimization problems with chance constraints is the right choice of the probability level p. Of course, first of all, this is a matter of experience, insight into the problem and of the right interpretation of results. It has to be taken into account however, that increasing p corresponds to shrinking the feasible set of the problem. Accordingly, there may exist some maximum probability level p^* which can be realized at most under the remaining constraints of the problem. In other words, further increasing p leads to empty feasible sets then, which should be avoided in the numerical treatment of the problem. Moreover, even below this maximum value, increasing p will decrease the optimal value of the objective function, i.e., the expected profit, in the optimization problem. Both aspects are illustrated in Fig. 5 for the two-stage case with uncorrelated components. It can be seen that, over a reasonably large range of probability levels, the expected profit decreases only moderately whereas it quickly collapses for excessively high probabilities. Such diagrams can help the decision maker to find a good compromise between safe operation and profit making.

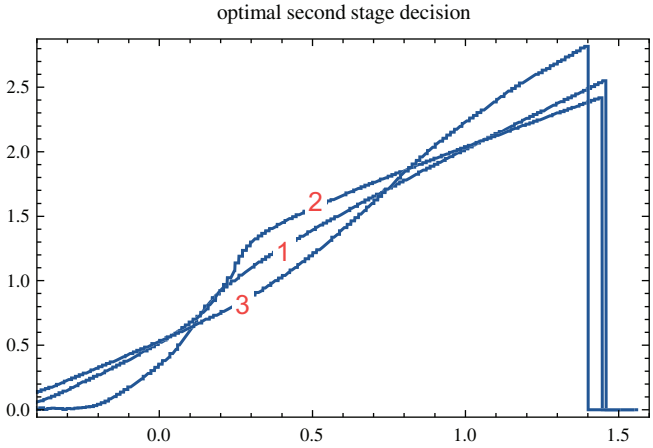

Fig. 4. Second stage release policies for the previous example (at $l_0 = 1.7$) but with an assumed correlation between inflow components of $\rho = -0.3$ (1), $\rho = 0$ (2) and $\rho = 0.9$ (3), respectively.

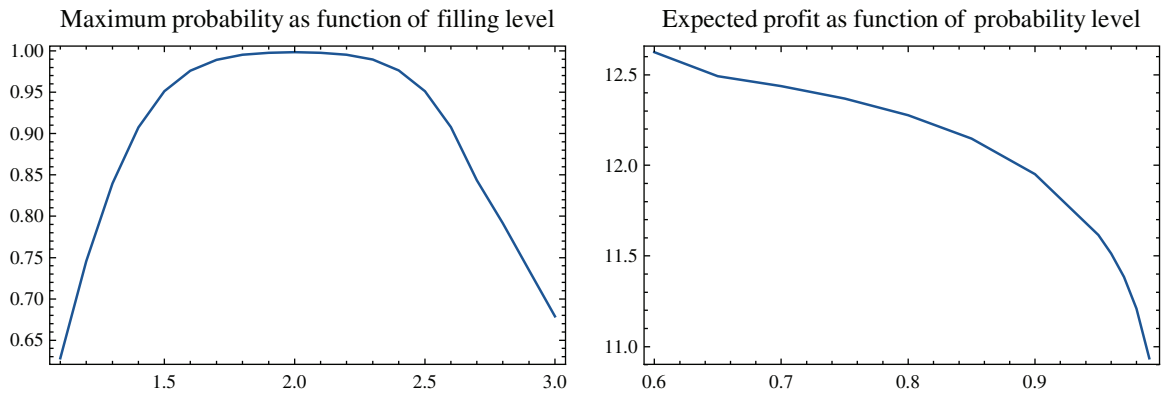

Fig. 5. Maximum probability level p as a function of the initial filling level (left part) and expected profit as a function of the chosen probability level (right part).

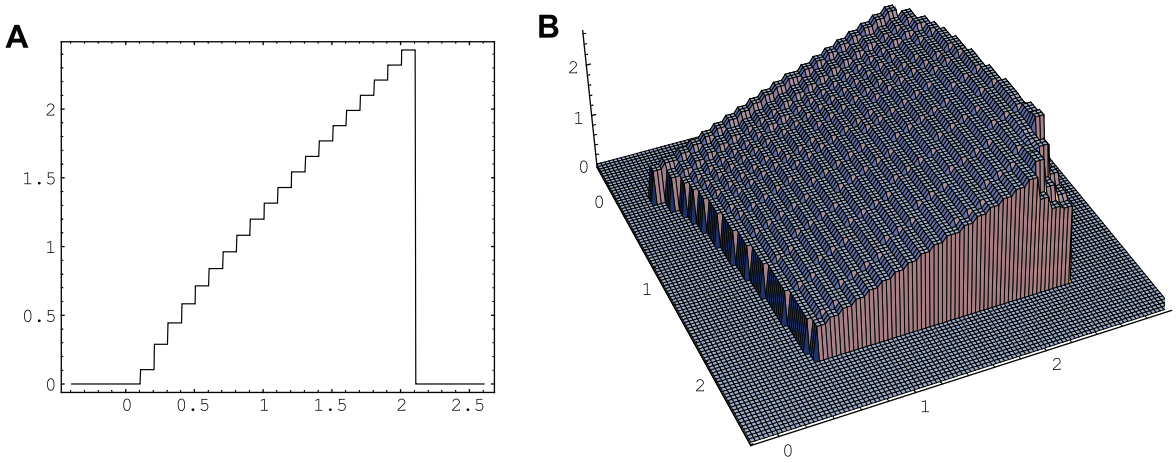

Fig. 6. Optimal second stage and third stage release policies for a three-stage example with independent components.

Finally, we want to illustrate the three-stage case. To do so we consider again independent components, now of the three-dimensional random vector $\xi = (\xi_1, \xi_2, \xi_3)$. These components are supposed to have the same distribution parameters as before (normal distribution with expectation = 1 and standard deviation = 0.3). To simplify the representation and numerical computations, we restrict the objective of the problem to the first two stages which means, that the expected profit is calculated until the second stage only, whereas all constraints have to be satisfied for the whole of three stages. Due to the increasing complexity of computations, we restrict the discretization parameter to the value $N = 20$. The resulting plots of the optimal second stage (as a function of ξ_1) and third stage (as a function of ξ_1, ξ_2) release policies are provided in Fig. 6.

7. Conclusion

The main purpose of this paper was to present a conceptual approach to the treatment of **joint dynamic** chance constraints in the case of continuous distribution. This is an attempt to adapt the ideas of multistage stochastic programming to the situation of chance constraints which is more involved due to nonlinearity. Consequently, none of the difficulties already present in the former (e.g., complexity issues) can disappear in the latter. Therefore, already passing to a two-stage model, constitutes an important step forward providing better results and new insight. Our numerical results have illustrated the presented dynamic model for a small number ($\leqslant 3$) of stages. Though it cannot be the aim of this paper, fully to explore the limits in the number of stages one could afford to deal with, we are convinced that going beyond three has a good chance to be successful. A few remarks supporting this conviction are in order here:

- The reason not to consider larger numbers of stages had mainly to do with graphical aspects. Restricting to a small number of stages did not only allow us to plot at all the release policies as functions of one or two previous random observations, but the plots were also made at high precision because comparatively large discretization parameters N could be afforded. On the other hand, such precision, which was comfortable for the purpose of illustration, is not needed in real life applications. We observed, for instance, that in the two-stage case the precision of the optimal value already reached 0.8% (relative deviation from true value) for $N = 2$ and 0.08% for $N = 5$ (recall that $N = 160$ in Fig. 2).
- Employing a professional NLP code for the solution of the discretized optimization problem has the potential of reducing the computing times needed so far by magnitudes and thus should also allow one to increase the number of stages considered.
- A further increase of efficiency could be obtained by adapting the idea of linear decision rules (LDRs) to our setting. This is a challenge by itself because LDRs are used so far in the context of individual chance constraints (e.g. [3]) which are not appropriate in general (see discussion in the introduction). Making LDRs accessible in the context of joint chance constraints is not straightforward and subject to future research. Moreover, one has to check the loss in optimality caused by such restriction of the model.

- A future adaptation of sparse grid ideas (see, e.g. [2]) or tensorial decomposition to our setting shall open a new perspective to overcome present limitations to the number of stages.

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