January 1987 LIDS-P-1647

$\mathcal{H}^{\infty}$ Sensitivity Minimization for Delay Systems: Part II

David S. Flamm*

Sanjoy K. Mitter

Laboratory for Information and Decision Systems and the Department of Electrical Engineering and Computer Science, M.I.T., Cambridge, MA 02139

* Now with The Aerospace Corp., P.O. Box 92957, Los Angeles, CA 90009.

Acknowledgement

D. S. Flamm was supported during this work by an Advanced Study Grant from The Aerospace Corporation, El Segundo, California. The research of S. K. Mitter has been supported by the Air Force Office of Scientific Research under Grant AFOSR-85-0227 and the Army Research Office under contracts ARO DAAG29-84-K-0005 and ARO DAAL03-86-K-0171.

I. Introduction and Problem Formulation.

In Part I of this paper [Flamm and Mitter 1986] we considered the single-input/single output $\mathcal{H}^{\infty}$ weighted sensitivity minimization control problem formulated in [Zames 1981], but with transfer functions of the form

$$P(s) = e^{-s\Delta}P_0(s) \qquad (1.1)$$

where $P_0(s)$ is a minimum phase and stable rational function, and $\Delta>0$. The block diagram in Figure 1 shows the feedback system models we are considering.

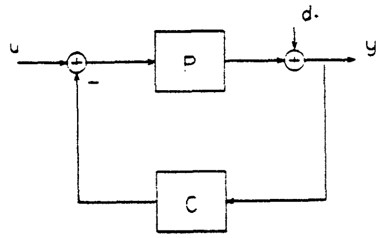


Figure 1. Feedback system considered.

The closed loop sensitivity S(s) is the transfer function from *d* to *y*. The weighted sensitivity X(s) for the weighting function W(s) is given by

$$X(s) = W(s)S(s) = W(s)[1+P(s)C(s)]^{-1} \qquad (1.2)$$

The problem is to minimize the $\mathcal{H}^{\infty}$ norm of X(s) over all stabilizing proper feedbacks C(s), that is, to solve

$$\inf_{C(s)} \|W(s)[1+P(s)C(s)]^{-1}\|_{\infty} \tag{1.3}$$

where C(s) ranges over all proper compensators for which the feedback system in Figure 1 is internally stable.

For later convenience we define the "Q-parameter" as $Q = \frac{C}{1+PC}$, and we note that $C = \frac{Q}{1-PQ}$. For stable plants P, there is a one-to-one correspondence between stable proper Q (*i.e.*, $Q \in \mathcal{H}^{\infty}$) and proper C(s) for which the system in Figure 1 is internally stable. See [Zames 1981].

The criterion of minimizing the norm of (1.2) for P(s) rational is introduced and motivated in [Zames 1981, pp. 585-586]. In this paper we continue the solution of (1.3) from Part I. As before, we assume the weighting function is given by

$$W(s) = \frac{s+1}{s+\beta} \tag{1.4}$$

with $0 < \beta$.

In Part I we restricted our attention to the modified problem

$$\inf_{H \in \mathcal{H}^{\infty}} \|W(s) - e^{-s\Delta}H(s)\|_{\infty} \tag{1.5}$$

since the infimum in (1.3) is generally not attainable when P(s) is strictly proper, and the solution to (1.5) is not greater than that of (1.3). See, for example, [Francis and Zames 1984, p. 10] for the rationale behind the transition from (1.3) to (1.5). In this part we shall use a compensator arising from a solution to (1.5) to find a sequence of compensators for which the closed loop weighted sensitivity approaches the infimum in (1.3).

II. Results from Part I and Overview of Part II.

We first repeat the main results of Part I.

Theorem 1.

For $\beta < 1$ the infimal sensitivity X(s) corresponding to problem (1.5) is given by

$$X(s) = \lambda_{max} \frac{s+1 - e^{-s\Delta}\lambda_{max}(s-\beta)}{\lambda_{max}(s+\beta) - e^{-s\Delta}(s-1)}, \tag{2.1}$$

where

$$\lambda_{max} = \left[\frac{\omega_0^2+1}{\omega_0^2+\beta^2}\right]^{1/2} \tag{2.2}$$

and ω_0 is the smallest positive solution of

$$\cot(\omega\Delta) = \frac{\omega^2-\beta}{\omega(1+\beta)}$$

∎

Theorem 2

For $\beta \geq 1$, all infimal sensitivities X(s) corresponding to problem (1.5) are given by

$$X(s) = \frac{s+1 - e^{-s\Delta}\varphi(s)(s-\beta)}{s+\beta - e^{-s\Delta}\varphi(s)(s-1)} \tag{2.3}$$

as $\varphi(s)$ ranges over $B(\mathscr{H}^{\infty})$, the unit ball in $\mathscr{H}^{\infty}$. Moreover, for $\beta > 1$

$$X(s) \text{ is inner} \Leftrightarrow \varphi \text{ is inner.}$$ ∎

Our present purpose is to use these results to solve the original problem (1.3)

Using the optimal sensitivity (2.1) for problem (1.5), we shall compute a corresponding compensator for (1.3) as if (2.1) were the solution to (1.3). The result of these calculations will show that the resulting compensator is generally improper, and therefore (2.1) cannot be obtained as a solution to (1.3). We shall see that the optimal improper compensator is also unstable, and that the resulting closed loop system has zero "delay margin."

> <u>Remark:</u> In case P is not strictly proper, the optimal compensator is proper, and there is less difficulty with realizing it. To treat the most general case, we assume that P is strictly proper.

We next show how to approximate this improper compensator by proper ones which result in stable closed loop systems having sensitivities with norms arbitrarily close to that of (2.1). Since the infimal value for (1.3) is not generally attainable, this sequence of compensators solves the problem (1.3).

Finally, since the compensators in our sequence contain delays, we show how to modify these compensators to eliminate the delays, yet preserve stability and yield sensitivities with norms approaching the infimum. Thus we obtain a sequence of proper and finite dimensional compensators which solve (1.3).

III. Computation of the Improper Compensator

Now we compute an improper compensator which would give us the computed optimal sensitivity for (1.5) as described in Theorem 1. The computation of the improper compensator involves finding the value of the free parameter-function H(s) which gives rise to the optimal weighted sensitivity, using the formula $X(s) = W(s) + e^{-s\Delta}H(s)$. Since our computation of X did not involve finding an H which attains the infimum, we need to show that H as given by the computation $H = \frac{X-W}{e^{-s\Delta}}$ is an element of $\mathscr{H}^\infty$. Suppose we have computed the optimal sensitivity X(s) as above. We know (See [Flamm and Mitter 1986, pp. 5-9]) that $\Pi_K(X|_K) = \Pi_K(W|_K)$, so $\Pi_K((X-W)|_K) = 0$. We can also see that $\Pi_K(X-W) = 0$ since $K^\perp$ is closed under multiplication by $\mathscr{H}^\infty$ functions. Therefore $(X-W)\mathscr{H}^2 \subseteq K^\perp = e^{-s\Delta}\mathscr{H}^2$. But then $e^{-s\Delta}$ divides (X-W) in $\mathscr{H}^\infty$ (this follows from the uniqueness of the inner-outer factorization), and the computation works.

We proceed to find the feedback compensator C which results in the weighted sensitivity (2.1). Using the Q-parametrization mentioned in Section I, we have $X = W(1-PQ)$ and $C = \frac{Q}{1-PQ}$, so

$$C = \frac{W-X}{PX}. \tag{3.1}$$

Recall that we are assuming $P(s) = e^{-s\Delta}\cdot P_o(s)$, with $P_o(s)$ stable and minimum phase.

Using (2.1) and (1.4) in (3.1) and simplifying, we get for the optimal compensator (with $\lambda = \lambda_{max}$)

$$\overline{C} = \frac{1}{P_o\lambda}\cdot\frac{s^2-1 - \lambda^2(s^2-\beta^2)}{-(s+1)(s+\beta) + e^{-s\Delta}\lambda(s^2-\beta^2)}$$

Substituting for λ^2 using (2.2) we get

$$\bar{C} = \frac{\lambda}{P_o} \cdot \frac{\beta^2-1}{\omega_o^2+1} \cdot \frac{s^2+\omega_o^2}{-(s+1)(s+\beta) + e^{-s\Delta}\lambda(s^2-\beta^2)}$$

Taking

$$\zeta = \frac{\lambda(1-\beta^2)}{\omega_o^2+1} = \frac{\lambda^2-1}{\lambda}$$

this is

$$\bar{C} = \zeta \cdot P_o^{-1} \cdot \frac{s^2+\omega_o^2}{s^2+(\beta+1)s+\beta} \cdot \frac{1}{1 + e^{-s\Delta}\cdot\lambda\cdot\frac{\beta-s}{s+1}},$$

which can be realized as shown in Figure 2.

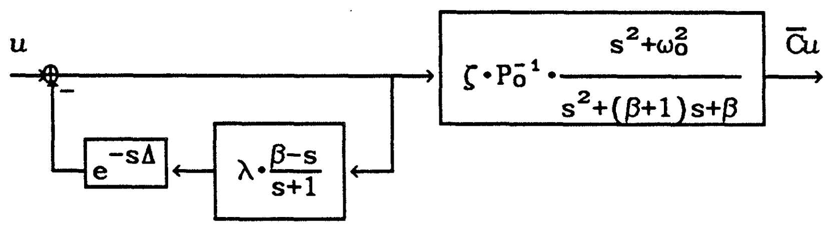


Figure 2. Realization of Optimal Compensator

Since P_o (the outer part of P) will generally be strictly proper, C and Q will in general be improper. We also note that since the compensator contains a delay it is infinite dimensional.

IV. Stability of Optimal Improper Feedback Compensator.

We present two ways to see that the optimal compensator is unstable.

The first is essentially just an extension of the situation for purely rational plants. The idea is that the optimal sensitivity is a constant times a (infinite) Blaschke product, and the numerator of the sensitivity appears in the denominator of the compensator. Therefore the compensator has right half plane poles. The details are as follows.

We know from above that $C = \frac{W-X}{PX}$, so

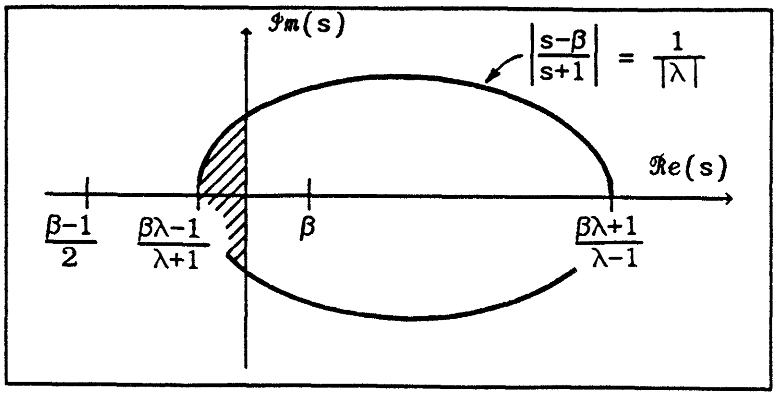
$$C = \frac{1}{P}\left[\frac{W}{X} - 1\right].$$

Since W and P have only left half plane zeros and poles by assumption, if we can show that X is a constant times a Blaschke product, we can conclude that C is unstable.

Let $\varphi = \lambda^{-1}X$. Then φ is inner. We argue (following [Sarason 1967, p. 194]) that $\varphi(s)$ is a Blaschke product: Since $\varphi(s)$ is continuous on the imaginary axis, the only singular inner functions that can divide it are of the form $e^{-s\alpha}$ with $\alpha > 0$. But $e^{s\alpha}\varphi(s)$ is unbounded on the positive real axis, so φ is purely a Blaschke product.

We can further show that φ is an <u>infinite</u> Blaschke product by applying Picard's theorem to its numerator. Thus we prove that $1-e^{-s\Delta}\lambda\frac{s-\beta}{s+1}$ has finitely many zeros in the closed left half plane, and then conclude by appealing to Picard's theorem that $1-e^{-s\Delta}\lambda\frac{s-\beta}{s+1}$ has infinitely many zeros in the right half plane.

First we note that $|e^{-s\Delta}| > 1$ for s in the left half plane, and $|e^{-s\Delta}| < 1$ in the right half plane. Now all zeros must satisfy $e^{-\Re e(s\Delta)}\cdot\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$, and therefore all closed left half plane zeros satisfy $\left|\frac{s-\beta}{s+1}\right| \leq \frac{1}{|\lambda|}$. The locus $\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$ is an ellipse, and so all closed left half plane zeros lie on or inside the intersection of the ellipse $\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$ with the closed left half plane. See Figure 3.



<u>Figure 3. Region for possible left half plane zeros.</u>

Thus all closed left half plane zeros lie in a compact region, and we conclude from analyticity that there are only finitely many in that region. Picard's theorem tells us that there are infinitely many zeros, so that we must conclude that there are infinitely many in the open right half plane. A similar argument shows that the denominator of X(s) in (2.1) has only finitely many right half plane zeros. Therefore the numerator and denominator of X(s) have only finitely many common zeros, and X(s) is an infinite Blaschke product.

(The same conclusion might be reached by looking at the Nyquist plot for the feedback loop in Figure 2, although we know of no version of the Nyquist stability criterion which applies to the case of infinitely many right half plane poles.)

A second instability proof gives us more detailed information about the distribution of the right half plane zeros without much more trouble. As stated above, the right half plane zeros must satisfy $\left|\frac{s-\beta}{s+1}\right| > \frac{1}{|\lambda|}$. As $|s| \to \infty$, $\left|\frac{s-\beta}{s+1}\right| \to 1$. Therefore as $|s| \to \infty$, the zero set $\{z_i\}$ approaches the line $|e^{-s\Delta}| = \frac{1}{|\lambda|}$, which is the same as $\mathcal{Re}(s) = \frac{\ln|\lambda|}{\Delta}$, and $\mathcal{Im}(z_i) \to (2n+1)\pi$. Also, since right half plane zeros must satisfy $\left|\frac{s-\beta}{s+1}\right| < 1$ (by comparing distances from β and from the point -1), we have $|\lambda|\cdot\left|\frac{s-\beta}{s+1}\right| < |\lambda|$. Then since $|e^{-s\Delta}|\cdot\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$, $|e^{s\Delta}| < |\lambda|$, and we conclude $\mathcal{Re}(s) < \frac{\ln|\lambda|}{\Delta}$ for these zeros.

IV. Motivation for a Sequence of Proper Compensators.

We consider the problem of approximating the sensitivity resulting from the improper ideal compensator using a sequence of realizable compensators.

There are two problems with the ideal compensator: First, it is generally improper. The physical interpretation of this is that it would have to contain differentiators, which can only be approximated with real systems. That is why our problem (1.3) requires C(s) to be proper. The second problem is that the optimal compensator contains a ideal delay, and again this cannot be constructed exactly.

The best we can hope for is that we can approximate the ideal compensator over a finite bandwidth, and design the system so that the behavior outside this band does not significantly affect performance. We would like to describe an approximation procedure such that we can pick whatever finite bandwidth we want, and the performance will approach the optimum as the bandwidth grows.

We are not attempting to approximate the optimal compensator, but rather describe a sequence of compensators for which the weighted $\mathcal{H}^\infty$ norm of the closed loop sensitivity approaches the infimal value. It does turn out, however, that our sequence of compensators approaches the optimal compensator uniformly on compact sets. For this point see [Fagnani 1986, pp. 45-46].

Since the infimal weighted sensitivity is unique (when it has norm greater than 1), and the corresponding compensator is improper when the plant is strictly proper, there can be no proper compensator which achieves the infimum.

For the case of a purely rational plant it is also the case that the optimal compensator is generally improper, and there are procedures in the literature to compute proper compensators for this case ([Zames and Francis 1983, p. 591] and [Vidyasagar 1985, p. 178]). The procedure in [Zames and Francis 1983] requires the evaluation of the term $B_z(\infty)$, where $B_z(s)$ is the Blaschke product formed from plant zeros. In our case there is no Blaschke product involved, but rather a singular inner function. If we interpret $B_z(s)$ to be this inner function, $B_z(\infty)$ is not defined. There is no apparent way to fix this problem for our case.

The procedure in [Vidyasagar 1985] does not work for our case either. The essence of the difficulty is the same as in the Zames-Francis procedure — the inner factor of the plant is not continuous at infinity. In the case of a stable plant, the Vidyasagar procedure consists of multiplying the optimal "Q-parameter" by a rational function, the magnitude of which decreases with increasing frequency at a sufficiently high rate. As the breakpoint of this "roll-off" function increases, the $\mathcal{H}^\infty$-norm of the rolled-off sensitivity function approaches the minimum. The idea is then to compute the compensator which yields this suboptimal Q-parameter, and in the rational plant case one will have a satisfactory sequence of approximating compensators.

Additional details on why that approach does not work for our delay case are given in [Flamm 1986, pp. 64-67]. The essential point is that in the rational plant case, at high frequency the product $P\overline{Q}$ approaches a real constant. Roll-off can therefore be introduced into the feedback loop at high frequency without regard for the phase, whereas in the delay case $P\overline{Q}$ has unbounded phase.

V. Approximation by Proper Finite Dimensional Compensators

We now describe how to construct an approximating sequence of compensators which *does* produce sensitivities which approach the optimum. We summarize this result as

Theorem 3. There exists a sequence of rational proper feedback compensators which result in weighted sensitivities of norm approaching the optimal value λ_{max} of Theorem 1. ∎

Remark: Theorem 3 solves (Problem 1.3) for $\beta < 1$. A similar results holds for $\beta \geq 1$. Since the infimal weighted sensitivity is unique (when it has norm greater than 1), and the corresponding compensator is improper when the plant is strictly proper, there can be no proper compensator which achieves the infimum. (For $\beta \geq 1$, the solution is not unique, and there is a proper compensator which achieves the infimum, namely the zero compensator $C(s) = 0$. See Theorem 2 above.)

The essential idea is to roll-off the ideal Q-parameter by multiplying it by a stable transfer function for which the Bode magnitude plot has slope less than 1 (such as (5.1) below), so as to limit the phase deviation due to the roll-off, until sufficient attenuation has been obtained. This this can be accomplished, for example, with a lead-lag network which approximates such compensation by having average slope magnitude less than 1.

We note that any stable Q-parameter results in a stable closed loop system, so that this roll-off technique preserves stability just as do the procedures for the case of a purely rational plant transfer function.

Assume $\bar{Q}$ is the optimal (improper) Q parameter resulting in the optimal weighted sensitivity $\bar{X}(s) = W(s)[s-e^{-s\Delta}\bar{Q}(s)]$. Define $\lambda = \|\bar{X}\|_\infty$, and note that $\lambda \geq 1$. We can write $\bar{X}(j\omega) = \lambda e^{j\alpha(\omega)}$, where $\alpha(\omega)$ is real.

The proof proceeds by developing approximations to $\bar{C}(s)$, first to make C(s)P(s) strictly proper, next to make C(s) proper, and finally to make C(s) finite dimensional. Again, our goal is to describe a sequence of compensators for which the weighted $\mathcal{H}^\infty$ norm of the closed loop sensitivity approaches the infimal value.

The proof consists of three propositions. The first step is the following:

<u>Proposition 1.</u> Let $\bar{X}(s)$ be the infimal sensitivity given in (2.1), with norm $\|\bar{X}(s)\|_\infty = \lambda \geq 1$. Let

$$h_n(s) = [n/(s+n)]^{1/n} \tag{5.1}$$

Then the compensator given by

$$C_n(s) = \frac{h_n(s)[W(s) - \bar{X}(s)]}{P(s)[(1-h_n(s))W(s) + h_n(s)\bar{X}(s)]}$$

results in a stable closed loop for which the sensitivity approaches λ as $n \to \infty$. Furthermore, the loop gain $|P(j\omega)C_n(j\omega)| \to 0$ as $|\omega| \to \infty$. ∎

<u>Remark:</u> This formula amounts to using a different roll-off function in the Vidyasagar approach. The reason for the given form of the roll-off function $h_n(s)$ is that it is necessary to control the phase of $h_n(s)$ until the loop gain has decreased sufficiently. Otherwise the sensitivity can be bounded away from the infimal value as the breakpoint frequency increases, even if the roll-off does not start until high frequency.

Proof: We examine the effect of multiplying the parameter $\overline{Q}$ by the roll-off function $h_n(s)$. The critical feature of h_n is that we can make the magnitude of $\arg(h_n)$ as small as we want by taking n sufficiently large.

We first note that using the sequence of Q parameters $Q_n(s) = h_n(s)\overline{Q}(s)$ preserves a stable closed loop since this is a stable Q parameter, and that the resulting loop gains $P(j\omega)C_n(j\omega)$ go to zero because $P(j\omega)C_n(j\omega)$ is strictly proper.

We now consider the magnitude squared of the sensitivity,

$$|X_n(j\omega)|^2 = \left|W(j\omega) + h_n(j\omega)[\lambda e^{j\alpha(\omega)} - W(j\omega)]\right|^2.$$

Suppose ω_n is the frequency at which $|X_n(j\omega_n)| = \|X_n\|_\infty$ (ω_n is finite for any given n since the sensitivity function is 1 at ∞), and define

$$h = |h_n(j\omega_n)|,\ \delta = \arg[h_n(j\omega_n)],\ W = W(j\omega_n),\ \text{and}\ \alpha = \alpha(\omega_n).$$

Note that h, δ and α are functions of n, as is ω_n. We also note for later use that δ satisfies $0 \le \delta \le \frac{2\pi}{n}$, *i.e*, $\delta \sim O(1/n)$.

Now we show that the norm of this sensitivity approaches the infimal value λ as n increases.

$$\begin{aligned}
|X_n(j\omega)|^2 &\le |W + h \cdot e^{j\delta}(\lambda e^{j\alpha} - W)|^2 \\
&= |W - W \cdot h \cdot e^{j\delta} + h \cdot e^{j\delta}\lambda e^{j\alpha}|^2 \\
&= |W|^2 - 2h|W|^2\cos\delta + h^2|W|^2 + h^2\lambda^2 + 2h\lambda\,\mathcal{Re}[We^{-j\alpha}(e^{-j\delta} - h)] \\
&= |W|^2 - 2h|W|^2\cos\delta + h^2|W|^2 + h^2\lambda^2 + 2h\lambda\Big\{\mathcal{Re}(W)\Big[\cos(\alpha+\delta) - \\
&\qquad h\cdot\cos(\alpha)]\Big] + \mathcal{Im}(W)\Big[\sin(\alpha+\delta) - h\cdot\sin(\alpha)\Big]\Big\}
\end{aligned}$$

Assume we have taken n large enough so that $\cos(\delta) \approx 1$ and $\sin(\delta) \approx \delta$. Then we write

$$|X_n(j\omega)|^2 \le |W|^2 - 2h|W|^2 + h^2|W|^2 + h^2\lambda^2 + 2h\lambda\Big\{\mathcal{R}e(W)\Big[\cos(\alpha) - h\cos(\alpha) - \delta\sin(\alpha)\Big] + \mathcal{I}m(W)\Big[\sin(\alpha)[1-h] + \delta\cdot\cos(\alpha)\Big]\Big\} + O(\delta^2)$$

$$= |W|^2 - 2h|W|^2 + h^2|W|^2 + h^2\lambda^2 + 2h\lambda\Big\{(1-h)\Big[\mathcal{R}e(W)\cos(\alpha) + \mathcal{I}m(W)\sin(\alpha)\Big] + \delta\Big[-\mathcal{R}e(W)\sin(\alpha) + \mathcal{I}m(W)\cos(\alpha)\Big]\Big\} + O(1/n^2)$$

$$\le |W|^2 - 2h|W|^2 + h^2|W|^2 + h^2\lambda^2 + 2h\lambda|W|[1-h+\delta] + O(1/n^2)$$

$$= [\,|W|(1-h) + h\lambda]^2 + 2h\lambda|W|\delta + O(1/n^2)$$

Given n, there are two possibilities: either (i) $\omega_n > n$, or else (ii) $\omega_n \le n$. We examine both cases.

case (i). ($\omega_n > n$) In this case we shall use the fact that $|W| \to 1$ as $\omega_n \to \infty$. $|W(j\omega_n)|^2 < \dfrac{n^2+1}{n^2+\beta^2} = 1 + \dfrac{1-\beta^2}{n^2+\beta^2} < 1 + \dfrac{1}{n^2}$, since $W(j\omega) = \dfrac{j\omega+1}{j\omega+\beta}$ and $\beta < 1$. Therefore

$$|X_n(j\omega)|^2 \le [(1+\tfrac{1}{n})(1-h) + h\lambda]^2 + 2h\lambda\delta(1+\tfrac{1}{n}) + O(1/n^2)$$

$$= [(1+\tfrac{1}{n})-\tfrac{h}{n}+h(\lambda-1)]^2 + 2h\lambda\delta(1+\tfrac{1}{n}) + O(1/n^2)$$

$$= [1+\tfrac{1-h}{n}+h(\lambda-1)]^2 + 2h\lambda\delta(1+\tfrac{1}{n}) + O(1/n^2)$$

$$\le [1+\tfrac{1-h}{n}+\lambda-1]^2 + 2h\lambda\delta(1+\tfrac{1}{n}) + O(1/n^2)$$

$$= [\tfrac{1-h}{n}+\lambda]^2 + 2\lambda\delta(1+\tfrac{1}{n}) + O(1/n^2)$$

$$= \lambda^2 + O(1/n)$$

We conclude that $(\|\bar{X}\|-\|X_n\|) \to 0$ as $n \to \infty$.

case (ii). **($\omega n \leq n$)** The idea in this case is that $h \to 1$ since $\omega n \leq n$. Since $|W| \geq 1$ and $h \leq 1$,

$$|X(j\omega n)|^2 \leq [|W|(1-h) + h\lambda]^2 + 2h\lambda\delta|W| + O(1/n^2)$$

$$= [|W|-(|W|-1)h+h(\lambda-1)]^2 + 2h\lambda\delta|W| + O(1/n^2)$$

$$= [(|W|-1)(1-h)+1+h(\lambda-1)]^2 + 2h\lambda\delta|W| + O(1/n^2)$$

$$\leq [(|W|-1)(1-h)+\lambda]^2 + 2\lambda\delta|W| + O(1/n^2)$$

$$= \lambda^2 + O(1-h) + O(1/n)$$

Now since $\omega_n \leq n$ in this case, $h^2 = |h(j\omega n)|^2 = \left[\frac{n^2}{n^2+\omega_n^2}\right]^{1/n}$, and $(0.5)^{1/n} \leq h^2 < 1$. Therefore $1-h \leq 1-(0.5)^{1/2n} < \frac{3}{4n}$ (using a Taylor series for $x^{1/n}$). This gives us $|X(j\omega n)|^2 \leq \lambda^2 + O(1/n)$, and once again we conclude that $(\|\bar{X}\|-\|X_n\|) \to 0$ as $n \to \infty$. ∎

The compensator resulting from Proposition 1 is given by

$$C_n' = \frac{W-X}{PX} = \frac{h_n\bar{Q}}{1-Ph_n\bar{Q}}.$$

Recall that we are assuming $W(s) = \frac{s+1}{s+\beta}$. Take $n(s) = s+1$ and $d(s) = s+\beta$. Let $\check{f}(s)$ denote $f(-s)$. Then some calculations give

$$C_n' = \frac{h_n(\lambda^2 d\check{d} - n\check{n})P_o^{-1}}{\lambda n d + e^{-s\Delta}[(1-h_n)n\check{n} - \lambda^2 d\check{d}]}.$$

In Proposition 1 the roll-off could be fast after the loop gain has decreased sufficiently: As ω increases and after $|h_n(j\omega)|$ is sufficiently small for $\omega > \omega_n$ for some ω_n, since $|h_n(j\omega)| \to 0$ as $\omega \to \infty$ and $|e^{-s\Delta}P_o\overline{Q}|$ is bounded, we can modify $h_n(j\omega)$ (call the modification $g_n(j\omega)$) so that $|g_n(j\omega)|$ decreases arbitrarily fast without increasing the $\mathscr{H}^\infty$-norm of the resulting sensitivity. It suffices to take ω_n large enough so that for $\omega > \omega_n$ we have both

$$|h_n(j\omega)| < \frac{\lambda-1}{\mu(\lambda+|W(j\omega)|)} \tag{5.2}$$

and

$$|W(j\omega)| < \frac{\lambda+1}{2} \tag{5.3}$$

These conditions are motivated by graphical considerations: They ensure that the graphs of the sensitivity $X_n(j\omega)$ and the weighting $W(j\omega)$ are close enough to the point 1 so that rotation of the graph of $X_n(j\omega)$about any point in the locus $W(j\omega)$ will not cause it to leave the disk $|s| = \lambda$. Such rotation will be an effect of increased phase due to rolling off $h_n(j\omega)$ more rapidly. $\mu \geq 2$ will ensure that h_n can safely be rolled off, but we will want to take $\mu > 2$ to leave "room" for the approximation of h_n by a finite dimensional function.

In order to demonstrate a specific $g_n(j\omega)$, we must must estimate the smallest value of ω_n which allows (5.2) to hold. Then we must choose $g_n(j\omega)$ so that for $\omega < \omega_n$ it differs very little from $h_n(j\omega)$ both in magnitude and phase.

Some simple calculations give a sufficient condition of

$$\omega_n \geq n\left[\frac{\mu(\lambda+1)}{\lambda-1}\right]^n \tag{5.4}$$

for (5.2) to hold. (5.3) will hold for ω_n sufficiently large independently of n. Now we pick $g_n(j\omega) = f_n(j\omega) \cdot h_n(j\omega)$ where $f_n(s)$ is a stable rational function of s which satisfies $|1-f_n(j\omega)| < 1/n^2$ for $\omega \leq \omega_n$, $|f_n(j\omega)|$ is strictly decreasing for $|\omega| > \omega_n$, and which eventually rolls off at least as fast as P(s). One such function when $m = 1$ (m is the differenc ebetween the degree of the denominator of $P_o(s)$ and the degree of its numerator) is $f_n(s) = \dfrac{n^2\omega_n}{s+n^2\omega_n}$.

The argument works with a stable finite dimensional approximation to h_n. We also note that roll-off of the parameter $\overline{Q}$ can ensure a proper compensator: Let $Q_n = \overline{Q}h_n$, the rolled-off Q parameter. Then if C_n is the resulting compensator, $C_n = \frac{Q_n}{1-PQ_n}$. Thus if Q_n is proper, so is C_n since P is proper. These points form the basis for the next proposition.

<u>Proposition 2.</u> Let $h_n(s)$ be as in Proposition 1. For each n take ω_n to satisfy (5.4) with $\mu = 4$. Take ω_r to be the least frequency above ω_n at which $|f_n(j\omega)| \leq |h_n(j\omega)|$ for $|\omega| \geq \omega_r$. Let $\tilde{h}_n(s)$ be any stable rational minimum phase function which satisfies

$$\|\tilde{h}_n\|_\infty \leq 1 \tag{5.6}$$

and

$$|h_n(j\omega)-\tilde{h}_n(j\omega)| < \frac{1}{n^2} \text{ for } |\omega| \leq \omega_r \tag{5.7}$$

Take $f_n(s)$ to be a stable minimum phase rational function of s which satisfies

$$\|f_n\|_\infty \leq 1 \tag{5.8}$$

and

$$|1-f_n(j\omega)| < 1/n^2 \text{ for } \omega \leq \omega_n \tag{5.9}$$

and which eventually rolls off at least as fast as P(s). Define

$$g_n(s) = f_n(j\omega)\cdot\tilde{h}_n(s) \tag{5.10}$$

Define

$$C_n(s) = \frac{g_n(s)[W(s) - \overline{X}(s)]}{P(s)[(1-g_n(s))W(s) + g_n(s)\overline{X}(s)]} \tag{5.11}$$

Then the closed loop feedback system using $C_n(s)$ as the compensator is stable, and the closed loop weighted sensitivity

$$X_n(s) = W(s)[1+P(s)C_n(s)]^{-1}$$

has ∞-norm which approaches the infimal value λ as $n \to \infty$. Furthermore $C_n(s)$ is a proper function. ∎

Proof: The expression (5.6) results from using the Q-parameter $g_n(j\omega)\overline{Q}(j\omega)$ in the formula $C(s) = \frac{Q(s)}{1-P(s)Q(s)}$. The closed loop is stable because $g_n(j\omega)\overline{Q}(j\omega)$ is a stable Q-parameter.

We show that the ∞-norm of the sensitivity approaches the optimal value by showing first that the magnitude of the sensitivity function is bounded above by λ on the range $|\omega| \in (\omega_n,\infty)$, and second that on the range $[0,\omega_n]$ it can be made arbitrarily close to λ by increasing n.

Remark: From (5.4) we see $\omega_n \to \infty$ as $n \to \infty$.

For $\omega > \omega_n$ (5.2) implies

$$\begin{aligned} \frac{\lambda-1}{4} &> |h_n(j\omega)|\left[\lambda + |W(j\omega)|\right] \\ &\geq |h_n(j\omega)|\cdot|\lambda e^{j\alpha(j\omega)} - W(j\omega)| \\ &= |h_n(j\omega)\cdot(\lambda e^{j\alpha(j\omega)} - W(j\omega))| \end{aligned}$$

On $[\omega_r,\infty)$, we have from (5.5) that

$$|f_n(j\omega)| \leq |h_n(j\omega)| \text{ for } \omega > \omega_r,$$

and

$$|\tilde{h}_n(j\omega)| \leq 1$$

from (5.9), so

$$|gn(j\omega)\cdot(\lambda e^{j\alpha(j\omega)} - W(j\omega))| \le \frac{\lambda-1}{4}.$$

But

$$Xn(j\omega) = W(j\omega) + gn(j\omega)\cdot(\lambda e^{j\alpha(j\omega)} - W(j\omega)),$$

so using (5.3) we conclude that

$$|Xn(j\omega)| < \frac{3\lambda+1}{4} < \lambda \text{ for } \omega \ge \omega_r.$$

On $(\omega n, \omega_r)$,

$$|gn(j\omega)| = |fn(j\omega)\tilde{h}n(j\omega)|$$
$$\le |hn(j\omega)| + \frac{1}{2n^2}$$

using (5.2), (5.7) and (5.8). Further calculation using (5.2) shows that the condition

$$n^2 > \frac{2(\lambda+\|W\|)}{\lambda-1}$$

will ensure that

$$|gn(j\omega)(\lambda e^{j\alpha(j\omega)} - W(j\omega))| \le \frac{\lambda-1}{2}$$

and it follows from (5.3) that

$$|Xn(j\omega)| < \lambda \text{ on } (\omega n, \omega_r).$$

For $\omega \le \omega n$, we repeat the argument in Proposition 1. Take

$$|Xn|^2 = |W(j\omega) + gn(j\omega)[\lambda e^{j\alpha(\omega)} - W(j\omega)]|^2$$

and set ω_m to the frequency at which $|Xn(j\omega_m)| = \sup_{\omega\in[0,\omega n]} |Xn(j\omega)|$. Now define

$$g = |gn(j\omega_m)|,\ \delta = \arg[gn(j\omega_m)],\ W = W(j\omega_m), \text{ and } \alpha = \alpha(\omega_m).$$

g, δ and α are functions of n, as is ω_m. One can see from (5.4), (5.7), (5.9) and the definition of $hn(s)$ that $\delta \sim O(1/n)$.

Now we show that the norm of this sensitivity approaches the infimal value λ as n increases. As in the proof of Proposition 1,

$$|X_n(j\omega)|^2 \le [|W|(1-g) + g\lambda]^2 + 2g\lambda|W|\delta + O(1/n^2).$$

Given n, there are two possibilities: either (i) $\omega_m > n$, or else (ii) $\omega_m \le n$. We examine both cases.

case (i). ($\omega_m > n$) Exactly as in the proof of Proposition 1 we find

$$|X_n(j\omega)|^2 \le \lambda^2 + O(1/n)$$

We conclude that $(\|\bar{X}\| - \|X_n\|) \to 0$ as $n \to \infty$.

case (ii). ($\omega_m \le n$) Let $\eta = \|W(j\omega_m)\|_\infty$. Then since $\eta \ge 1$ and $g \le 1$, as before

$$|X_n(j\omega_m)|^2 \le \lambda^2 + O(1-g) + O(1/n)$$

$$\begin{aligned}1-g &\le |1-g_n(j\omega_m)| \\ &= |1-f_n(j\omega_m)\tilde{h}_n(j\omega_m)| \\ &\le |1-h_n(j\omega_m)| + |h_n(j\omega_m)-\tilde{h}_n(j\omega_m)| + |1-f_n(j\omega_m)||\tilde{h}_n(j\omega_m)|\end{aligned}$$

Using the estimate of $(1-h)$ from the proof of Proposition 1, along with (5.6), (5.7) and (5.9), we find

$$1-g \le \frac{3}{4n} + \frac{1}{n^2} + \frac{1}{n^2}.$$

Once again we get $|X_n(j\omega_m)|^2 \le \lambda^2 + O(1/n)$, and we conclude that $(\|\bar{X}\| - \|X_n\|) \to 0$ as $n \to \infty$.

The properness of C_n follows from the definitions (5.10) and (5.11), and the assumed roll-off of $f_n(s)$. ∎

The resulting compensator is given by

$$C_n = \frac{g_n(\lambda^2 d\check{d} - n\check{n})P_o^{-1}}{\lambda nd + e^{-s\Delta}[(1-g_n)n\check{n} - \lambda^2 d\check{d}]}.$$

A realization of this clearly contains a pure delay. We now examine the effect of approximations to this delay on the closed loop sensitivity, in order to further approximate the ideal compensator with one which is finite dimensional. In approximating $e^{-s\Delta}$ with a rational function, we must be concerned with two things: First, we must preserve the stability of the closed loop, and second, we must preserve the approximation of the closed loop sensitivity to the optimal sensitivity. The restrictions these impose on rational approximation of the delay amount to (1) the delay must be approximated closely enough until g_n and W are sufficiently small, and (2) after that the delay approximation must not exceed 1 in magnitude.

We approximate the delay by replacing $e^{-s\Delta}$ with the rational function $\rho(s)$. (We repeat that approximation of the delay in our sense means only that our closed loop sensitivity approximates the infimal norm of the sensitivity.) The following gives one set of criteria for selecting $\rho(s)$.

Proposition 3. Let ω_r and $g_n(j\omega)$ be as in Proposition 2. Take $\omega_c \geq \omega_r$ such that,
if

$$\mathcal{R}e(s) \geq 0 \text{ and } \quad |s| \geq \omega_c, \text{ then } |W(s)| < \lambda,$$

and if

$$|\omega| > \omega_c, \text{ then } |gn(j\omega)| < \frac{\lambda - |W(j\omega_c)|}{\lambda + |W(j\omega_c)|} \tag{5.12}$$

Let $\gamma = \inf_{\Re e(s)\geq 0} |\lambda + e^{-s\Delta}\cdot\frac{\check{n}(s)}{d(s)}|$. Let $\rho(s)$ be a stable rational approximation to $e^{-s\Delta}$ such that

$$\|\rho\| \leq 1,$$

and for $\Re e(s) \geq 0$ and $|s| < \omega_c$

$$|\rho(s)-e^{-s\Delta}| < \epsilon < \frac{\gamma}{2\cdot\|W\|_\infty}.$$

Define $\tilde{C}n(s)$ as $Cn(s)$ with $\rho(s)$ substituted for $e^{-s\Delta}$. Under these conditions, as $n \to \infty$ and $\epsilon \to 0$, the closed loop system with compensator $\tilde{C}n(s)$ and weighted sensitivity $\tilde{X}n(s)$ is stable and satisfies $\|\tilde{X}n(s)\| \to \lambda$. ∎

<u>proof:</u> The Q-parameter corresponding to Cn is

$$Qn(s) = gn(s)\cdot P_o^{-1}(s)\cdot\frac{(\lambda^2 d(s)\check{d}(s) - n(s)\check{n}(s))}{n(s)(\lambda d(s) + e^{-s\Delta}\check{n}(s))}.$$

Let $\tilde{Q}n$ be Qn with $\rho(s)$ substituted for $e^{-s\Delta}$, and let $\tilde{C}n$ be the resulting compensator. Let $\tilde{Q}$ represent the optimal Q-parameter $\overline{Q}$ with $\rho(s)$ substituted for $e^{-s\Delta}$.

Since stability of the closed loop is equivalent to stability of the Q-parameter, $n(s)$ and $d(s)$ have no zeros in the right half plane, and Qn is stable, we can show that $|\lambda + e^{-s\Delta}\cdot\frac{\check{n}(s)}{d(s)}| > \gamma$ for some $\gamma > 0$, when $\Re e(s) \geq 0$. For stability of $\tilde{Q}n$ it suffices to show that $\left[\lambda + \rho(s)\cdot\frac{\check{n}(s)}{d(s)}\right]$ has no zeros in the right half plane. This is equivalent to showing that $|\lambda + \rho(s)\cdot\frac{\check{n}(s)}{d(s)}| > 0$. The condition

$|\rho(s)-e^{-s\Delta}| < \frac{\gamma}{2\cdot\|W\|_\infty}$ for $\Re e(s) \geq 0$ and $|s| < \omega_c$ suffices to show this.

A simple calculation gives, for $\Re e(s) \geq 0$ and $|s| < \omega_c$,

$$|\lambda + \rho(s)\cdot\frac{\check{n}(s)}{\check{d}(s)}| \geq |\lambda + e^{-s\Delta}\cdot\frac{\check{n}(s)}{\check{d}(s)}| - |(\rho(s)-e^{-s\Delta})\cdot\frac{\check{n}(s)}{\check{d}(s)}|$$

$$\geq \gamma - |\rho(s)-e^{-s\Delta}|\cdot|W(s)| \geq \gamma - \frac{\gamma|W(s)|}{2\|W\|} \geq \frac{\gamma}{2}$$

For $\Re e(s) \geq 0$ and $|s| \geq \omega_c$, $|W(s)| < \lambda$ and $|\rho(s)| \leq 1$, so $|\lambda + \rho(s)\cdot\frac{\check{n}(s)}{\check{d}(s)}| > 0$ there as well.

Using the condition (5.12), we now show that $\|\tilde{X}_n\| \to \|X\|$ as $n \to \infty$ and $\epsilon \to 0$. The first step is to note that we need only show that $|WP\tilde{Q}_n| < \|X\| - |W|$, since then $|W(1-P\tilde{Q}_n)| < \|X\|$, and therefore $\|\tilde{X}_n\| \leq \|X\|$. So we calculate as follows.

$$|WP\tilde{Q}_n| = \left|W(s)P(s)gn(s)\cdot P_o^{-1}(s)\cdot\frac{(\lambda^2 d(s)\check{d}(s) - n(s)\check{n}(s))}{n(s)(\lambda d(s) + \rho(s)\check{n}(s))}\right|$$

$$= \left|W(s)gn(s)\cdot\frac{(\lambda^2 d(s)\check{d}(s) - n(s)\check{n}(s))}{n(s)(\lambda d(s) + \rho(s)\check{n}(s))}\right|$$

$$= \left|W(s)gn(s)\cdot\frac{\lambda^2 - |W(s)|^2}{W(s)\left[\lambda\cdot\frac{d(s)}{\check{d}(s)} + \rho(s)\check{W}(s)\right]}\right|$$

$$= \left|gn(s)\cdot\frac{\lambda^2 - |W(s)|^2}{\left[\lambda\cdot\frac{d(s)}{\check{d}(s)} + \rho(s)\check{W}(s)\right]}\right|$$

Since $W(s) = \frac{s+1}{s+\beta}$, $|\omega| > \omega_c$ implies that $|W(j\omega)| < |W(j\omega_c)|$. Now we have for $|\omega| > \omega_c$

$$|g_n(j\omega)| < \frac{\lambda - |W(j\omega_c)|}{\lambda + |W(j\omega_c)|} < \frac{\lambda - |W(j\omega)|}{\lambda + |W(j\omega)|}$$

$$\leq \left[\lambda \cdot \left(\frac{d(j\omega)}{\check{d}(j\omega)}\right) + \rho(j\omega) \cdot \check{W}(j\omega)\right] \Big/ \left[\lambda + |W(j\omega)|\right]$$

This implies

$$|g_n(j\omega)| \cdot \left[\lambda^2 - |W(j\omega)|^2\right] \Big/ \left[\lambda \cdot \left(\frac{d(j\omega)}{\check{d}(j\omega)}\right) + \rho(j\omega) \cdot \check{W}(j\omega)\right] < \lambda - |W(j\omega)|$$

These arguments apply on the imaginary axis for $|\omega| > \omega_c$. For $|\omega| \leq \omega_c$ taking ϵ sufficiently small and n sufficiently large make $\left| |\check{X}_n(j\omega)| - |\overline{X}(j\omega)| \right|$ as small as desired. Since $\check{X}_n \in \mathcal{H}^\infty$, this behavior on the imaginary axis guarantees that $\|\check{X}_n\| \rightarrow \|\overline{X}\|$. ∎

This proposition completes the proof of Theorem 3, since we have now shown a way to construct a sequence of rational proper feedback compensators $\tilde{C}_n(s)$ for which the weighted sensitivities $\tilde{X}_n(s)$ have norms approaching the optimal value λ. To summarize, we have three ranges of frequency over which the approximation of the optimal compensator takes effect. For $|\omega| < \omega_n$ the approximating compensator is very close in magnitude and phase to the optimal compensator. Over $\omega_n \leq |\omega| \leq \omega_c$ the compensator starts to roll off while maintaining a close approximation to the delay, until by ω_c (5.12) is satisfied. From then on, $|\omega| \geq \omega_c$, so long as $|\rho(j\omega)| \leq 1$, $\rho(j\omega)$ need not be close to $e^{-j\omega\Delta}$.

V. Conclusion

We have presented the solution to the simplest meaningful $\mathcal{H}^{\infty}$ minimal weighted sensitivity problem for the case of a plant having a delay at the input.

Parallel solutions are detailed in [Flamm 1986] for more general rational weighting functions, and for plants having right half plane poles and zeros in addition to the input delay. The basic techniques are essentially generalizations of those presented here with modifications made for right half plane poles and zeros in the plant. However, in the most general case, when W(s) results in a non-compact operator on $\mathcal{H}^2$, we are not necessarily able to construct an optimal sensitivity, although we know from the theory of Sarason that one exists. We are able to "slightly" modify any given W(s) so as to make the corresponding operator compact, and thus obtain a solution to a "close" problem. See [Flamm 1986] for details.

Areas for future work include completing the picture for the non-compact case, computational issues for the case of general weighting functions, and extensions to plants with other non-rational transfer functions.

REFERENCES

Fagnani, F. (1986), "Problemi di minimizzesione in $\mathcal{H}^{\infty}$ per sistemi in dimenzione infinita," Universitá degli Studi di Pisa, Dipartimento di Matematica, October 1986 (Bachelor's thesis).

Flamm, D. S. (1985), "$\mathcal{H}^{\infty}$-Optimal Sensitivity for Delay Systems," Ph. D. thesis proposal, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, July 1985.

Flamm, D. S. (1986), "Control of Delay Systems for Minimax Sensitivity," Ph.D. Thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, June 1986.

Flamm, D. S. and S. K. Mitter (1985), "Progress on $\mathcal{H}^{\infty}$ Optimal Sensitivity for Delay Systems. Part I: Minimum phase plant with input delay, 1 pole/zero weighting function," Report No. LIDS-P-1513, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, November 1985.

Flamm, D. S. and S. K. Mitter (1986), "$\mathcal{H}^{\infty}$ Sensitivity Minimization for Delay Systems: Part I," Report No. LIDS-P-1605, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, September 1986.

Foias, C., A. Tannenbaum and G. Zames (1986a), "On Decoupling the $\mathcal{H}^{\infty}$-Optimal Sensitivity Problem for Products of Plants," *Systems and Control Letters*, 7(4), July 1986, pp. 239-245.

Foias, C., A. Tannenbaum and G. Zames (1986b), "Weighted Sensitivity Minimization for Delay Systems," *IEEE Trans. Automatic Control*, **AC-31**(8), August 1986, pp. 763-766.

Francis, B. A. and J. C. Doyle (1985), "Linear Control Theory with an $\mathcal{H}^{\infty}$ Optimality Criterion," Control Science and Engineering Systems Control Group Report #8501, University of Toronto.

Francis, B. A. and G. Zames (1984), "On $\mathcal{H}^{\infty}$-Optimal Sensitivity Theory for SISO Feedback Systems," *IEEE Trans. Automatic Control*, **AC-29**(1), pp. 9-16.

Halmos, P. R. (1967), A *Hilbert Space Problem Book*, American Book Co., New York.

Hoffman, K. (1962), *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ.

Rudin, W. (1973), *Functional Analysis*, McGraw-Hill, New York.

Sarason, D. (1967), "Generalized Interpolation in $\mathcal{H}^\infty$," *Trans. AMS*, **127**(2), pp. 179-203.

Sarason, D. (1985), "Operator-theoretic Aspects of the Nevanlinna-Pick Interpolation Problem," pp. 279-314, in *Operators and Function Theory*, S. C. Power, ed., Reidel, Dordrecht, Holland.

Tadmor, G. (1986a), "$\mathcal{H}^\infty$ Weighted Sensitivity Minimization for Systems with Commensurate Input Lags," Report No. LIDS-P-1546, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, March 1986.

Tadmor, G. (1986b), "An Interpolation Problem Associated with $\mathcal{H}^\infty$ Optimal Design in Systems with Distributed Input Lags," Report No. LIDS-P-1545, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, April 1986.

Vidyasagar, M. (1985), *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA.

Zames, G. (1981), "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses," *IEEE Trans. Automatic Control*, **AC-26**(2), pp. 301-320.

Zames, G., and B. A. Francis (1983), "Feedback, Minimax Sensitivity, and Optimal Robustness," *IEEE Trans. Automatic Control*, **AC-28**(5), pp. 585-600