Exact Solution for Freezing in Cylindrically Symmetric, Porous, Moist Media

M. A. Boles[1] and M. N. Özişik[2]

Nomenclature

K_{12} = thermal conductivity ratio, $\dfrac{k_1}{k_2}$

L = dimensionless latent heat, $\dfrac{lu_0}{c_2 T_0}$

Q = dimensionless strength of the line sink, $\dfrac{q}{k_1 T_0}$

R = dimensionless radial coordinate, $\dfrac{r}{r_0}$

$S(\tau)$ = the dimensionless location of freeze-front, $\dfrac{s(t)}{r_0}$

$U(R,\tau)$ = nondimensional moisture concentration for the region $r > S(t)$, $\dfrac{u_0 - u(r,t)}{u_0}$

α = diffusivity

α_{ij} = diffusivity, $\dfrac{\alpha_i}{\alpha_j}$

Δ = nondimensional thermal gradient coefficient, $\dfrac{T_0 \delta}{u_0}$

$\theta_i(R,\tau)$ = nondimensional temperature for the regions $i = 1$ and $i = 2$, $\dfrac{T_i(r,t)}{T_0}$

θ_v = nondimensional freeze-front temperature, $\dfrac{T_v}{T_0}$

τ = nondimensional time, $\dfrac{\alpha_2 t}{r_0^2}$

Subscripts

1,2,m = refers to solid, liquid, and moisture, respectively.

Introduction

Experimental investigations [1, 2] of freezing in a porous medium, such as a wet soil, have shown that the freezing is accompanied by moisture migration towards the freezing front. The migration of moisture effects the temperature distribution in the medium and the location of the freezing front. Freezing in a wet porous medium has numerous engineering applications. For example, during the extraction of energy from the earth using a heat-pump, earth-coupled thermal system, the cooling pipe buried in the earth can be regarded as a line heat sink in an infinite medium. As the energy is extracted from the moist porous soil, the moisture is redistributed, lowered in temperature, and becomes frozen. It is of interest to know the temperature distribution and the location of the freezing front. Also, in the measurements of thermal conductivity in moist porous media, the knowledge of the moisture distribution upon freezing is important.

The standard formulation of phase-change problem without allowing for the movement of moisture within the medium is not applicable for the analysis of this type of freezing problems. More complicated phase-change problems allowing for the movement of moisture have been studied in connection with evaporation [3] and freezing [4] only for a semi-infinite medium. The studies of freezing due to a line heat sink in an infinite medium with cylindrical symmetry are also limited [5, 6] and do not allow for moisture movement.

In the present study exact solutions are developed for the temperature and moisture distribution and the location of the freezing front caused by cooling with a line heat sink in an infinite medium with cylindrical symmetry.

Formulation of the Problem

Consider a line heat sink of strength, q, (W/m) positioned at the origin of the radial coordinate, $r = 0$, in a moist, porous, infinite medium. Initially the medium is at a uniform temperature, T_o, and contains moisture of uniform distribution, u_o. At time $t = 0$, the heat sink is activated. The freezing process begins and the freezing front propagates in the radial direction. Let $r = s(t)$ be the position of the freeze front at any time, t. The dimensionless temperature distribution $\theta_1(R,\tau)$ in the freezing zone, $0 < R < S(\tau)$, where there is no moisture is governed by the equation

$$\frac{\partial \theta_1(R,\tau)}{\partial \tau} = \alpha_{12} \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial \theta_1}{\partial R}\right), \text{ in } 0<R<S(\tau), \tau>0 \quad (1)$$

In the unfrozen region $S(\tau) < R < \infty$, the dimensionless temperature distribution $\theta_2(R,\tau)$ and the moisture distribution $U(R,\tau)$ are governed by the following system obtained as a special case from the Luikov's system of equations (87) and (88) of [7]

$$\frac{\partial \theta_2(R,\tau)}{\partial \tau} = \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial \theta_2}{\partial R}\right), \text{ in } S(\tau)<R<\infty, \tau>0 \quad (2)$$

$$\frac{\partial U(R,\tau)}{\partial \tau} = \alpha_{m2} \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial U}{\partial R}\right) - \alpha_{m2} \Delta \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial \theta_2}{\partial R}\right), \text{ in } S(\tau)<R<\infty, \tau>0 \quad (3)$$

It is to be noted that in the Luikov system [7] the zones 1 and 2 refer, respectively, to evaporation and moisture zones, whereas in the present system they refer to solid and liquid zones, respectively. The boundary and initial conditions become

$$\mathcal{L}im_{R \to 0} \left(2\pi R \frac{\partial \theta_1}{\partial R}\right) = Q \quad (4)$$

$$\theta_1(S,\tau) = \theta_2(S,\tau) = \theta_v \quad (5)$$

$$K_{12} \frac{\partial \theta_1(S,\tau)}{\partial R} - \frac{\partial \theta_2(S,\tau)}{\partial R} = L[1 - U(S,\tau)] \frac{dS}{d\tau} \quad (6)$$

$$\frac{\partial U(S,\tau)}{\partial R} - \Delta \frac{\partial \theta_2(S,\tau)}{\partial R} = 0 \quad (7)$$

$$\theta_2(R,0) = 1 \quad (8a)$$

$$U(R,0) = 0 \quad (8b)$$

$$\theta_2(\infty,\tau) = 1 \quad (9a)$$

$$U(\infty,\tau) = 0 \quad (9b)$$

[1] Mechanical and Aerospace Engineering Department, North Carolina State University, Raleigh, N. C. 27650, Mem. ASME

[2] Mechanical and Aerospace Engineering Department, North Carolina State University, Mem. ASME

Contributed by the Heat Transfer Division for publication in the Journal of Heat Transfer. Manuscript received by the Heat Transfer Division May 13, 1982.

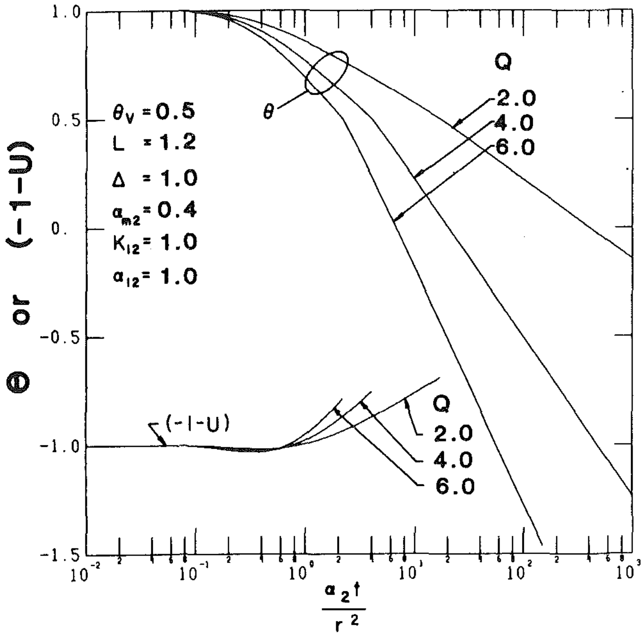

Fig. 1 Effect of nondimensional line heat strength, Q, on temperature, θ, and moisture, U

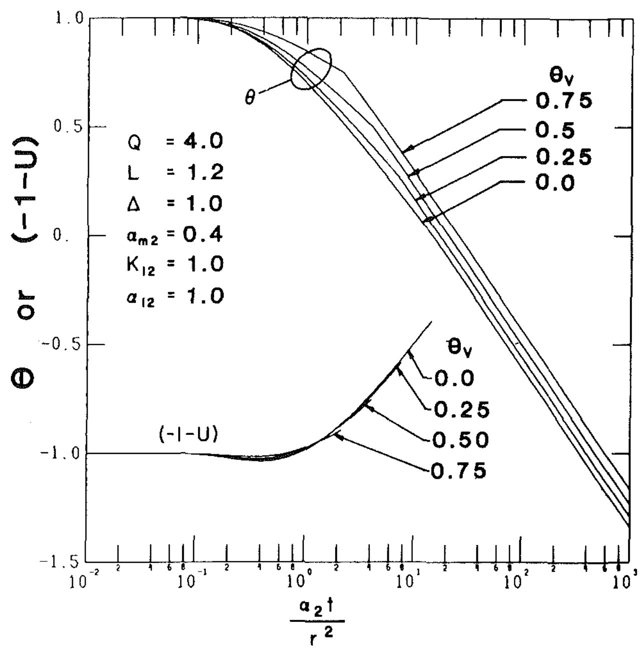

Fig. 2 Effect of nondimensional freeze-front temperature, θ_V, on temperature, θ, and moisture, U

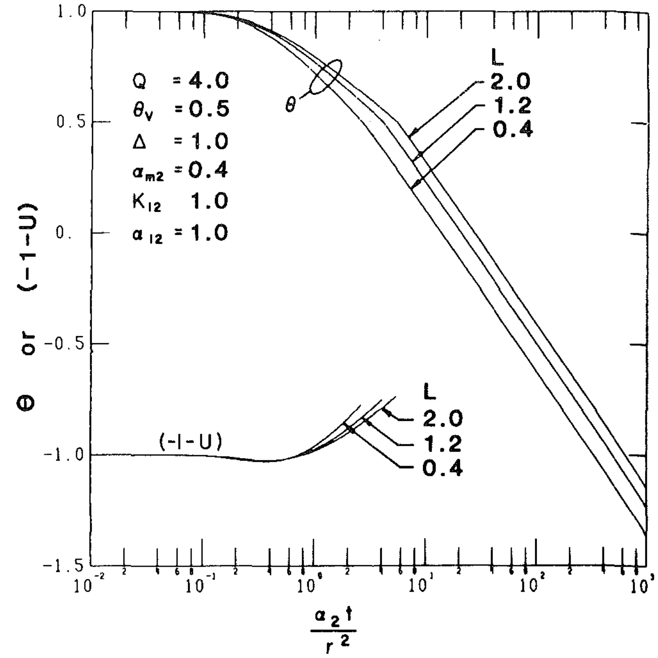

Fig. 3 Effect of nondimensional latent heat, L, on temperature, θ, and moisture, U

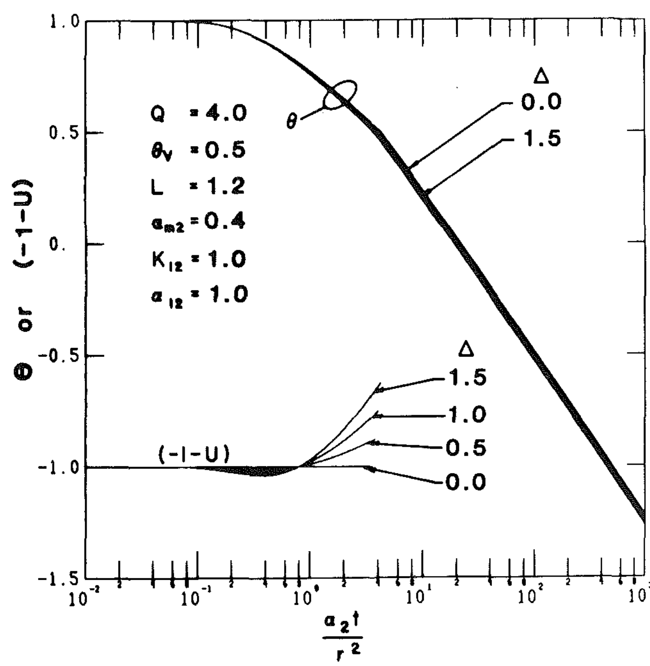

Fig. 4 Effect of nondimensional thermal gradient coefficient, Δ, on temperature, θ, and moisture, U

Solution of the Problem

Equations (2) and (3) can be decoupled by defining a new variable $Z(R,\tau)$ as [8]

$$Z(R,\tau)=\theta_2(R,\tau)+CU(R,\tau) \tag{10a}$$

where

$$C=\frac{1-\alpha_{m2}}{\alpha_{m2}\Delta} \tag{10b}$$

After some manipulation the equation for $Z(R,\tau)$ is determined as

$$\frac{\partial Z(R,\tau)}{\partial \tau}=\alpha_{m2}\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial Z}{\partial R}\right), \text{ in } S(\tau)<R<\infty, \tau>0 \tag{11}$$

Now the problem has been reduced to the solution of three diffusion type homogeneous equations, (1), (2), and (11), subject to the boundary and initial conditions given above. These solutions are determined as

$$\theta_1(R,\tau)=\theta_v+\frac{Q}{4\pi}\left[Ei\left(-\frac{R^2}{4\alpha_{12}\tau}\right)-Ei(-\lambda^2)\right], \text{ in } 0<R<S(\tau) \tag{12}$$

$$\theta_2(R,\tau)=1+(\theta_v-1)\frac{Ei\left(-\frac{R^2}{4\tau}\right)}{Ei(-\lambda^2\alpha_{12})}, \text{ in } R>S(\tau) \tag{13}$$

and

$$U(R,\tau)=\frac{\alpha_{m2}\Delta}{1-\alpha_{m2}}\ \frac{\theta_v-1}{Ei(-\lambda^2\alpha_{12})}\left[\frac{1}{\alpha_{m2}}e^{\frac{1-\alpha_{m2}}{\alpha_{m1}}\lambda^2}\right.$$

$$\left.\bullet Ei\left(-\frac{R^2}{4\alpha_{m2}\tau}\right)-Ei\left(-\frac{R^2}{4\tau}\right)\right],R>S(\tau) \qquad (14)$$

where

$$\lambda=\frac{S(\tau)}{2(\alpha_{12}\tau)^{1/2}} \qquad (15)$$

Finally, the transcendental equation for the determination of λ is obtained by introducing the solutions given by equations (12), (13), (14) together with the definition (15) into the interface energy balance equation (6). We find

$$K_{12}\frac{Q}{4\pi}e^{-\lambda^2}-\frac{\theta_v-1}{Ei(-\lambda^2\alpha_{12})}e^{-\lambda^2\alpha_{12}}=\lambda^2\alpha_{12}L\left\{1\right.$$

$$\left.-\frac{\Delta}{1-\alpha_{m2}}(\theta_v-1)\left[\frac{e^{\frac{1-\alpha_{m2}}{\alpha_{m1}}\lambda^2}}{Ei(-\lambda^2\alpha_{12})}Ei(-\lambda^2\alpha_{1m})-\alpha_{m2}\right]\right\}$$

$$(16)$$

Results and Discussion

Once the constant λ is computed from the solution of the transcendental equation (16) for a given set of system parameters, the location of the freeze-front $S(\tau)$ at any time τ is determined according to equation (15), the temperature distribution $\theta_1(R,\tau)$ in the freeze-zone is calculated from equation (12), and the temperature distribution $\theta_2(R,\tau)$ and the moisture distribution $U(R,\tau)$ outside the freeze-zone are calculated from equations (13) and (14), respectively.

Sample calculations are performed to illustrate the effects of various system parameters on the temperature and moisture distribution in the region. These results are presented in Figs. 1–4 by plotting the temperature and the moisture content as a function of the dimensionless parameter $\tau/R^2 \equiv \alpha_2 t/r^2$. The physical significance of such figures can be interpreted in two different ways:

1 For a fixed value of time, the curves represent the variation of temperature or moisture as a function of the radial position, r, in the medium. Then, on these figures the origin $r = 0$ corresponds to a location $\alpha_2 t/r^2 \rightarrow \infty$, and the right-hand side of the curves correspond to the frozen zone. The curves for the moisture content show that the moisture is highest at the freeze-front; moving away from the freeze-front with increasing r, there exists a region where moisture content falls below the initial value, followed by the zone where the initial value of the moisture content remains unchanged.

2 For a fixed value of the radial position, r, the curves represent the variation of temperature or moisture with time. Then, $\alpha_2 t/r^2 \rightarrow 0$ corresponds to the beginning of the cooling process. As time increases, the temperature at a given location continuously decreases; on the other hand, the moisture content falls below the initial value for a short period, then begins to increase continuously until the freezing front reaches to that location.

Having discussed the general behavior of the variation of temperature and moisture as a function of time and position in the region, we now focus our attention to the effects of various system parameters.

Figure 1 illustrates the effects of the nondimensional line heat strength, Q, on temperature and moisture. As the sink strength is increased, the freeze-front moves deeper into the medium at a given time; or the freeze-front arrives much quicker at a given location. Increasing Q, lowers the values of temperature and moisture.

Figure 2 illustrates the effects of the nondimensional freeze-front temperature, θ_v, on the temperature and moisture distribution in the medium. Decreasing θ_v, increases the maximum moisture content and moves the freeze-front towards the origin, $r = 0$.

Figure 3 illustrates the effects of nondimensional latent heat, L, on the temperature and moisture distribution. Lowering L forces the freeze-front further away from the origin, $r = 0$. The minimum and maximum moisture contents do not seem to depend on L.

Figure 4 shows the effects of the nondimensional thermal gradient coefficient, Δ, on the temperature and moisture distribution. The maximum moisture content is strongly dependent upon Δ; increasing Δ increases the maximum moisture content. The temperature level and the freeze-front location, however, are very little affected by Δ.

Acknowledgment

This work was supported in part by the National Science Foundation through the Grant MEA-8110 705.

References

1 Aguirre-Puente, J., and Philippe, A., "Quelques recherches effectuées en France sur le probléme de la congélation des sols," *Rev. Gén. Thermique*, Vol. 96, 1969, pp. 1123–1141.

2 Aguirre-Puente, J., Dupas, A., and Philippe, A., "Quelques résultats concernant le gonflement au gel et leur application au classement des sols selon leur degré de gélivité," *Bull. liason Labo. P. et Ch.*, Vol. 68, 1973, pp. 23–29.

3 Mikhailov, M. D., "Exact Solution of Temperature and Moisture Distributions in a Porous Half-Space With Moving Evaporation Front," *International Journal of Heat and Mass Transfer*, Vol. 18, 1975, pp. 797–804.

4 Mikhailov, M. D., "Exact Solution for Freezing of Humid Porous Half-Space," *International Journal of Heat and Mass Transfer*, Vol. 19, 1976, pp. 651–655.

5 Patterson, S., "Propagation of a Boundary of Fusion," *Proceedings of the Glasgow Mathematics Association*, Vol. 1, 1952, pp. 42–47.

6 Özişik, M. N., and Uzzel, J. C., "Exact Solution for Freezing in Cylindrical Symmetry With Extended Freezing Temperature," JOURNAL OF HEAT TRANSFER, Vol. 101, 1979, pp. 331–334.

7 Luikov, A. V., "Systems of Differential Equations of Heat and Mass Transfer in Capillary-Porous Bodies (Review)," *International Journal of Heat and Mass Transfer*, Vol. 18, 1975, pp. 1–14.

8 Mikhailov, M. D., "General Solution of the Diffusion Equations Coupled at the Boundary," *International Journal of Heat and Mass Transfer*, Vol. 16, 1973, pp. 2155–2164.

9 Lebeolev, N. N., *Special Functions and Their Applications*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

Natural Convection From Needles With Variable Wall Heat Flux

J. L. S. Chen[1]

Nomenclature

- a = dimensionless needle size
- b = needle shape parameter, $2(m-1)/5$
- f = dimensionless stream function
- g = gravitational acceleration
- Gr = modified Grashof number, $g\beta q_o L^4/(k\nu^2)$
- h = convection heat transfer coefficient

[1] Associate Professor, Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, Pa. 15261, Mem. ASME

Contributed by the Heat Transfer Division for publication in the JOURNAL OF HEAT TRANSFER. Manuscript received by the Heat Transfer Division July 23, 1982.