FRACTAL ANALYSIS OF TRACK GEOMETRY DATA

by

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(Text = 4300 words)
(12 Figures)

Submission Date: July 9, 2002

ABSTRACT

A Federal Railroad Administration sponsored research project has been ongoing to explore the use of Fractal Analysis of track geometry data for indication of track geometry roughness, maintenance planning and track substructure condition assessment. Fractal analysis provides unique numerical values (fractal dimensions) that characterize railway track geometry patterns. The fractal dimensions can be used for effective maintenance planning by providing meaningful parameters for geometry deterioration modeling, and by potentially providing information about the actual condition of the track by precise quantification of the geometry patterns. The paper will present a lucid discussion of fractal theory and will demonstrate its usefulness for quantifying railroad geometry data by highlighting key aspects of the research results. This paper also discusses the relationship between track structure conditions and fractal dimensions for use in maintenance planning and condition evaluation.

INTRODUCTION

Fractal analysis is used to characterize irregular patterns and to quantify patterns that are seemingly chaotic and random (1). The fractal dimension of a pattern varies depending on the degree of "roughness" of the pattern, and will have a different value for each pattern type, with the fractal dimension being specific for that pattern (2). Railway track geometry exhibits irregular and rough characteristics. Therefore, fractal analysis can be used to precisely characterize track geometry data and provide unique numerical values that can quantify geometry signatures.

This paper presents results of a study conducted to explore the use of fractal analysis of track geometry data for an ***indication*** of track geometry condition, maintenance ***planning*** and ***evaluation*** of problem cause. The paper focuses on vertical profile geometry since it is the parameter that is related to the track substructure condition.

FRACTAL ANALYSIS

Classical geometry holds that patterns are described by integer, whole-number dimensions. For example, curves are one-dimensional, surfaces are two-dimensional, and cubes are three-dimensional. However, patterns can actually occupy space between these discrete integer values, and therefore the patterns need to be described by a fraction of a dimension. For example, a one-dimensional curve that is extremely rough, so rough that it is convoluted at even high levels of magnification, can actually fill the space in which it resides, thus being two-dimensional. Fractal dimensions are those “fractional” dimensions that reside between conventional whole-number dimensions.

Fractal analysis, i.e., the process used to determine the fractal dimension, was presented by Benoit B. Mandelbrot (1, 3) to characterize those patterns within nature that are irregular, chaotic or fragmented, and cannot be effectively quantified using classical geometry of whole-number dimensions (3). There are various techniques for determining the fractal dimension of rough patterns, including the divider, area-perimeter and box methods (1, 2, 4), as well as the parallel-line (5), power spectral density (4, 6) and distribution methods (7). The divider method will be discussed in this paper.

The divider method is based on the empirical studies of coastlines, and was used by Mandelbrot to quantify curves whose fractal dimensions were greater than one (> 1.0). The divider method is based on an equation that expresses the length of a rough line by:

$$L(\lambda) = n\lambda^{1-D_R} \quad , \qquad \text{[Eq. 1]}$$

where: λ = length of unit measurement,
$L(\lambda)$ = length of the rough line based on unit measurement length λ,
n = number of steps of length λ , and
D_R = fractal dimension of the rough line.

To better understand Equation 1 consider the measurement of the coastline of Britain. In order to determine the length, a progression of ruler lengths, λ, is used to trace the coast, as shown on Figure 1. The length of the coastline measured by each ruler length, designated $L(\lambda)$, can be calculated by simply multiplying the number of ruler lengths, n, by the length of the ruler, i.e.,

$$L(\lambda) = n\,\lambda \quad . \qquad \text{[Eq. 2]}$$

Ultimately, the true length is measured when the length of the ruler is shorter than the smallest detail of the coastline, resulting in a perfect trace of Britain. At first glance, this method seems logical. For instance, Figure 1.d certainly appears to "fit" the details of the coastline much more closely than Figure 1.a. Eventually, this technique would seem to lead to the actual length. A closer look, however, indicates that the ruler length in Figure 1.d appears to miss the details of the coastline, as apparent in Figure 1.e. At every level of magnification, the coastline continues to present more detail. The length , $L(\lambda)$, measured by a mile-long ruler, λ, would be less than the length measured by a yardstick, which would be less than the length measured by a one-inch ruler. It could be argued that the length of the coastline increases without limit as the ruler length decreases.

Empirical work with the measurement of several coastlines provided an equation that described the constant increase in measured length as the ruler length decreased:

$$L(\lambda) \sim N\lambda^{1-D} \quad , \qquad \text{[Eq. 3]}$$

where: $L(\lambda)$ = length of the coastline with respect to ruler length λ ,
N = number of ruler lengths used to measure the coast, and
D = an empirical constant.

Mandelbrot asserted that the exponent D in Equation 3 quantified the rate at which the length of the measured line increases with decreasing length of measuring ruler, and that this D was directly related to the particular "roughness" of the pattern being measured. Equation 3 thus became Equation 1.

To better understand how to quantify a pattern using the divider method consider a rough line that is measured using a "divider" or "ruler" of length, λ, as illustrated in FIGURE 2. The rough line in FIGURE 2 is measured by placing the ends of the ruler on consecutive points of intersection along the line. As the line is measured with smaller rulers (top down in FIGURE 2) the measured Total Length, $L(\lambda)$, increases since the smaller rulers have the ability to intersect more points on the line and hence better approximate the actual length. Once the pattern has been measured in such a way, Equation 1 can be used to develop the fractal dimension, D_R , by first taking the logarithm (base 10) of both sides of Equation 1, which gives:

$$\log L(\lambda) = (1-D_R)\log \lambda + \log n \ . \qquad \text{[Eq. 4]}$$

This is in the form of the equation of a line, i.e.,

$$y = mx + b \ . \qquad \text{[Eq. 5]}$$

Therefore, a plot of λ versus $L(\lambda)$ on a log-log scale (FIGURE 3) yields a linear relationship with the slope of the line defined as:

$$m = 1 - D_R \ . \qquad \text{[Eq. 6]}$$

Thus, the fractal dimension equals:

$$D_R = 1 - m \ . \qquad \text{[Eq. 7]}$$

The fractal dimension is therefore determined from the slope of $\log L(\lambda)$ versus $\log\lambda$ plot, as shown on the bottom plot in FIGURE 2. If the arbitrarily rough line in the top plot of FIGURE 2 was smoother than that shown, the difference in consecutive measured line lengths would not be as great, and the slope of the best-fit line through the data points on the fractal log-log plot would be less. This would then result in D_R being between 1.0 and 1.149.

Fractal Analysis of Railway Track Geometry Data

FIGURE 3 presents a typical profile deviation pattern and the method used to employ the divider method. The divider method algorithm shown on FIGURE 3 uses constant step lengths along the x-axis (2, 4). As the x-axis increment value approaches the minimum discrete unit size, and the length of the measured line increases, an increasingly better portrayal of the pattern emerges.

FIGURE 4 presents the fractal plot that results from using the fractal dimensioning method illustrated in FIGURE 3. This fractal plot is typical of MCO profile data in that the deviation measurements along a length of track often exhibit bi-fractal aspects. The bi-fractal aspect is apparent as the two distinct linear portions of the data. This plot shows that the relationship between the step-length values and the corresponding measured length is different for the relatively large step-length values and the small step-length values. The length of the pattern measured by the large step-length values is influenced by the overall shape of the pattern, whereas the length measured by the small step-length values is predominantly influenced by the texture of the pattern. Therefore, depending upon the size of the step-length values used, different aspects or "scales" of the pattern either become apparent or no longer visible. FIGURE 4 shows that the large-scale (1^{st} order) roughness associated with the relatively large step-length values (right linear relationship) is less than the smaller-scale (2^{nd} order) roughness associated with the smaller step length values (left linear relationship).

To further illustrate the idea of different "scales" or "orders" of roughness in the data, FIGURE 5 presents continuous geometry MCO data in which two different orders of magnitude are shown. The 1^{st} order roughness is associated with the overall shape of the line and the 2^{nd} order roughness is associated with the texture of the line. The divider method applied to the pattern in FIGURE 5 would result in a bi-fractal plot similar to that in FIGURE 4.

The slope of the two linear portions of the plot, i.e., the fractal dimensions, as well as the location of the fractal breakpoint provides numerical parameters for characterization of the roughness of the pattern. The 1^{st} order and 2^{nd} order fractal dimensions (D_{R1} and D_{R2}) provide two numerical parameters to describe the geometry pattern. The D_{R1} quantifies the roughness at a relatively large scale, and the D_{R2} provides a descriptor of relatively small scale roughness. The fractal breakpoint value, expressed as a step-length value (in units of length), indicates the point of separation between the relatively small and large scale. These numerical parameters can help derive correlations between geometry measurements and track condition. Figure 6 presents four examples of vertical profile MCO data with corresponding fractal plots to illustrate how the fractal plots vary depending on the roughness of the pattern.

TRACK CONDITION EVALUATION

Fractal analysis has been applied to railway track geometry data in order to examine its use for indication of track geometry condition, maintenance planning and evaluation of the cause of substructure related problems. The fractal analysis was performed on vertical profile MCO data obtained by a high-speed inertial-based geometry recording car. Mid-chord offset data reflect an unequal emphasis of different wavelengths resulting in MCO distortion by both amplification and attenuation. The vertical profile data was analyzed in MCO format under the supposition that the MCO data, like the actual vertical space curve, reflects the track substructure condition despite the amplitude distortion.

The focus of the research was on vertical profile measurement since these are the parameters most influenced by the track substructure conditions and indicative of the cause of track condition deterioration. The fractal analysis of track geometry data was performed in this study in both discrete section analysis and in a moving-window fashion.

Indication of Roughness

Rough track geometry adversely affects the performance and longevity of track components and rolling stock, and causes passenger discomfort and an increased potential for derailments. It is for these reasons that quantifiers or "indicators" of the degree of roughness associated with track geometry are important. Fractal analysis provides good indicators of roughness in that it provides numerical values directly related to the roughness of the geometry signature. Fractal dimensions surpass other geometry data quantifiers, such as Running Roughness (8), since fractal dimensions account for the magnitude of the variations as well as their frequency structure.

Running Roughness (8), R^2, is a mean square statistical calculation that provides a magnitude analysis of the geometry measurements. As such, large values of R^2 are associated with large deviations, and small-scale roughness is not reflected in R^2 values. The first-order fractal dimension (D_{R1}) quantifies large-scale roughness, and as expected D_{R1} and R^2 correlate well with each other. This close correlation is apparent in FIGURE 7, which shows the relationship between the first-order fractal dimension (D_{R1}) and Running Roughness of geometry data for two different locations (Site A09 and A12) on Amtrak's Northeast Corridor. There are two groupings of data in FIGURE 7 that show a direct relationship between R^2 and D_{R1} . However, the relationship is not the same for the two sets of data since each set of data shows the relationship between D_{R1} and R^2 for different sections of track with different geometry patterns. Since fractal dimensions are pattern based whereas Running Roughness is strictly magnitude-based, there is not a constant relationship between D_{R1} and R^2. The relationship between D_{R1} and R^2 varies depending upon the pattern under scrutiny.

Fractal analysis provides a means to directly compare the roughness characteristics of different lengths of track since fractal analysis provides a measurement of roughness that is independent of length. This is not to say that a subsection of a larger section will have the same fractal dimensions, but that if two patterns of different lengths have different roughness characteristics, then their fractal dimensions will be different. Similarly, if two patterns of different length have similar patterns, their fractal dimensions will be equal. In this respect, fractal dimensions provide an "intesitive" measurement of pattern roughness, in the sense that R^2, for example, is an "extensitive" property, i.e., its magnitude depends on the length of the "window" used for calculation. To illustrate this, consider FIGURE 8.

FIGURE 8 (top plot) presents an approximate 17,000 ft sample of vertical profile geometry data with three different section lengths outlined. Sections A, B and C are approximately 3000 ft, 9000 ft and 1500 ft in length, respectively. Visually, Section B is less rough than both A and C, and A and C exhibit similar roughness characteristics. Fractal analysis was performed for each section (A, B & C). Fractal log-log plots for the 3 sections were developed and the results are shown in the bottom of FIGURE 8. Review of FIGURE 8 indicates how Section B is considerably less rough than Sections A and C, and that Sections A and C have similar roughness characteristics. Again, these results show that direct comparison of the roughness characteristics can be made for widely different lengths of track.

The ability to directly compare different-length sections of track is useful for ranking and categorizing purposes, and is also useful for assessing "equivalent utility". Equivalent Utility refers to the determination of the dynamic response of certain pieces of equipment on a segment of track that has the same geometry characteristics as another segment on which the equipment performed satisfactorily during testing. Two sections of track will have equivalent utility if their roughness characteristics are similar with respect to dynamic ride performance. That is, once a particular piece of equipment (e.g., a new high-speed train-set) has qualified to run at a certain speed over a certain piece of track with a particular roughness condition, it can then be certified to run on other sections of like-roughness track without making a complete series of qualifying runs. Sections of track with "like-roughness" with respect to train dynamics are said to have "equivalent utility".

Planning

By quantifying track geometry data with fractal analysis and developing the trends of fractal parameters over time, predictions can be made regarding the future condition of the track. Maintenance and/or remedial measures, including allocation of capital resources, can then be planned based on these predictions. Also, comparing fractal dimensions of geometry data for different sections of track can be used to rank the track sections for maintenance prioritization.

FIGURE 9 presents an example of trend-analysis using fractal dimensions. This figure was developed by performing fractal analysis and running roughness (R^2)

calculations for vertical profile MCO data obtained over time for a discrete section of revenue service track. Surfacing and undercutting maintenance input information is also shown on FIGURE 9.

As shown in FIGURE 9, there is a clear reflection of the deterioration in geometry (increase in roughness) by D_{R1}, D_{R2} and R^2 between September 1994 and March 1995. During this time period no maintenance was performed. Also apparent in FIGURE 9 is the improvement in geometry (decrease in roughness) due to maintenance intervention in August 1994, April 1995 and late June 1995. The relationship between D_{R1} and R^2 is consistent (note parallel trendlines of R^2 and D_{R1}) throughout the time period until September 1995 when undercutting of the track was performed and the geometry pattern was thereby changed.

Fractal analysis of individual discrete sections of track is useful for section ranking and also for looking at the change of a section over time. FIGURE 10 presents a one-mile section of running D_{R1} derived from the moving-window approach. By aligning the running D_{R1} plots to a layout of track features the influence of track features such as undergrade bridges, turnouts (switches), signal locations and other physical features that can influence geometry measurements can be determined. For instance, in FIGURE 10 it is known that the roughness occurring between Station 0 and Station 2250 ft is due to a long undergrade river bridge (UGBR). The roughness from Station 2250 to Station 4550 ft, on the other hand, is not associated with any track feature and is likely caused by the track substructure.

To view and evaluate the behavior of the entire rough section over a period of time, a single fractal log-log plot was calculated for the entire 2300 ft dataset for each geometry measurement date that was taken during the time period of interest. The results are plotted in the bottom of FIGURE 10, and this provides an indication of the geometry performance for the entire section of track. The rate of deterioration for this section of track can be compared and prioritized with that of other track sections.

The trends of D_{R1} shown on FIGURE 10 show that the vertical profile track geometry was relatively stable until November 1999 at which point it began to deteriorate until March 2000 when the track was smoothed by track surfacing. D_{R2} similarly begins to deteriorate in November 1999. The high D_{R2} degradation values of January through March 2000 are due to a high level of noise generated by a failing sensor on the geometry car. The second-order fractal dimension is good at spotting the eminent failures before they show up in the calibration process.

Evaluation

Track geometry patterns are influenced by the track's structural conditions. In particular, vertical profile space curve and MCO measurements are influenced by the track's substructure condition. By meaningfully quantifying the vertical profile geometry pattern it should be possible to obtain useful information on the substructure condition of the track. Fractal analysis provides precise numerical quantification of geometry patterns, and therefore has the potential to evaluate track substructure condition by characterizing the geometry signature.

The *functional* condition of railway track, i.e. the loaded position of the rails, depends on both the *unloaded profile* and the *elastic deflection* of the track under load, and is directly indicated by the geometry data obtained by track geometry recording cars. The elastic deflection component of the geometry measurement is related to the structural behavior of railway track. The structural behavior of railway track is defined by the strength and stiffness properties of the track superstructure (rail, fasteners, ties) and substructure (ballast, subballast, subgrade), as well as the dynamic load that the train traffic imparts to the track.

To use geometry data to evaluate the structural condition of the track an understanding is required of the complex interrelationship between the load imparted by the track geometry car, the actual geometry car measurement (top of rail response), and the structural behavior of the track. Understanding the contributions of the many variables is difficult, to say the least. However, a new field of science called *Chaos Theory* describes a new way to examine such a complex system.

Chaos theory is based on the concept that complex systems actually have a fundamentally simple structure or behavior and that the system can be understood by studying the "dynamical" behavior of the system. The key to the science of chaos is its ability to account for every detail involved. Rather than dismantle a system into its fundamental elements, the chaos perspective observes the behavior of the system as a whole (9).

The approach taken in this study to examine the potential for fractal analysis of geometry data to evaluate track substructure condition relies on the fundamental assumption of *chaos* theory that a complex system can be understood by studying its "dynamical" behavior. Therefore, the approach taken in this study uses fractal analysis to characterize the complex pattern of geometry data, which are a function of a complex interrelationship, to gain insight into the structural condition of the track. This project has begun to explore the application of fractal analysis of geometry car data for substructure condition assessment. This is being done by performing empirical studies to see if a correlation exists between the patterns that appear in the geometry car measurements and the condition of the substructure.

A relatively common substructure problem is excessively fouled ballast. Ballast with a large amount of fouling material will lose the ability to support the track and will tend to deform more under repeated loading. Fouled ballast reduces the effectiveness of tamping by preventing ballast interlocking, and also results in high rates of differential settlement within the ballast. Fouled ballast results in loss of elasticity and greater plastic deformation. The amount of ballast fouling can vary along the track over short distances. This, combined with varying drainage and subgrade conditions, results in varying support of the ties over relatively short distances. This translates into rough track. Conceptually, the rough track that is caused by fouled ballast would be roughness of small scale since rapid variations in tie support would translate into a relatively high frequency. Therefore, the difference in geometry pattern from a clean ballast section should be reflected in the D_{R2} measurement.

To examine the D_{R2} response to different ballast fouling conditions, fractal analysis was performed on vertical profile geometry data for two different sections of track with excessively fouled ballast conditions. The approximately 8 to 10 in. below the bottom of tie of the ballast layers of both of the sections of track were then cleaned with a high-production track ballast cleaner (track undercutter). Figure 11 shows the vertical profile MCO data of a 3500 ft section of track before and after undercutting, and the fractal log-log plot for the entire 3500 ft section. The large-scale roughness (D_{R1}) does reduce from before to after undercutting likely due to the overall smoothing of the track.

However, the small-scale roughness (D_{R2}), which is hypothesized (but not yet proven) to be affected by the ballast condition, reduces dramatically from before to after undercutting.

Furthermore, the geometry patterns used to develop FIGURE 9 reflect the excessively fouled ballast condition at that particular site (Site A12). Undercutting and replacement of fouled ballast with clean ballast affected the geometry patterns as reflected in the change in D_{R1} and D_{R2} values. The relationship between R^2 and D_{R1}, and the behavior of D_{R2}, change significantly for sections of track with fouled ballast layers from before ballast cleaning to after; indicating the potential to determine the ballast fouling condition from the geometry car measurements.

The effects of the track superstructure may also be seen in the geometry data. To examine this effect, fractal analysis was performed on two different sections of Class 4 mainline freight track and one section of Class 7 mainline track. The Class 7 and "premium" Class 4 track were comprised of the premium track components: concrete ties, elastic fasteners, and high quality continuous welded rail (CWR). The other "conventional" Class 4 mainline freight track was comprised of wood ties, cut-spike fasteners and CWR. The substructure condition for all sections were relatively similar, i.e., good drainage, clean ballast layer, decent subgrade. FIGURE 12 shows the results of fractal analysis on these three sections of track.

The geometry of the premium Class 4 track is maintained to a much tighter tolerance than the conventional Class 4 track, which is the reason for the lower D_{R1} of the premium Class 4 track. The D_{R1} of the Class 7 track is even lower due to the even tighter maintenance tolerances than both of the Class 4 tracks. The D_{R2} of the two premium tracks (Class 4 and Class 7) are similar, and much lower than the conventional Class 4 track. This result suggests that D_{R2} reflects the condition of the track components (i.e., the ties, rails and fasteners).

CONCLUSIONS

The following are major conclusions derived from the study thus far:

A. Fractal analysis is a good indicator of the roughness of the geometry data and is able to provide unique numerical values that characterize railway track geometry patterns.

B. Fractal dimensions vary depending upon the degree of pattern roughness within the track geometry, and can discern different orders (scales) of roughness within track geometry data.

C. Fractal analysis is effective for comparing geometry between sections of track with different lengths since it provides numerical quantifiers that are independent of the length of the pattern being analyzed.

D. Fractal analysis has been shown to be effective for maintenance planning by providing parameters directly related to geometry roughness that can be used for trend analysis (degradation modeling).

E. Although the study of relationships between fractal parameters and track substructure condition was constrained by the limitations of available information, the fractal parameters show some indication of representing field trends. There appears to be a correlation between the 2nd order fractal dimension (D_{R2}) of vertical profile MCO data and the D_{R2} of continuous track stiffness measurements. Also, there is some indication that fractal dimensions can discern track with fouled ballast conditions from those with clean ballast, and possible a distinction between poorly draining cuts and fill embankments. Field verification is essential to confirm or deny these preliminary findings.

Acknowledgment

The research discussed in this paper was performed at the University of Massachusetts and was funded through the Federal Railroad Administration's University Cooperative Research Grant Program. The authors wish to thank Mahmood Fateh at the FRA for his support throughout this study.

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Figure Captions

FIGURE 1 Coastline of Britain.

FIGURE 2 Example of divider fractal dimensioning technique and fractal log-log plot.

FIGURE 3 Example of fractal dimensioning of geometry deviation pattern.

FIGURE 4 Fractal plot of example pattern of FIGURE 3.

FIGURE 5 Orders of roughness of geometry car deviation (MCO) data.

FIGURE 6 Example patterns and fractal plots.

FIGURE 7 Relationship between D_{R1} and R^2 for Site A09 and A21.

FIGURE 8 Example Fractal Analysis of different section lengths.

FIGURE 9 Example trend analysis for Site A.

FIGURE 10 Example Fractal Analysis of discrete section of track.

FIGURE 11 Example of Fractal Analysis results for before and after undercutting.

FIGURE 12 Comparison of premium and conventional track.

FIGURE 1 Coastline of Britain.

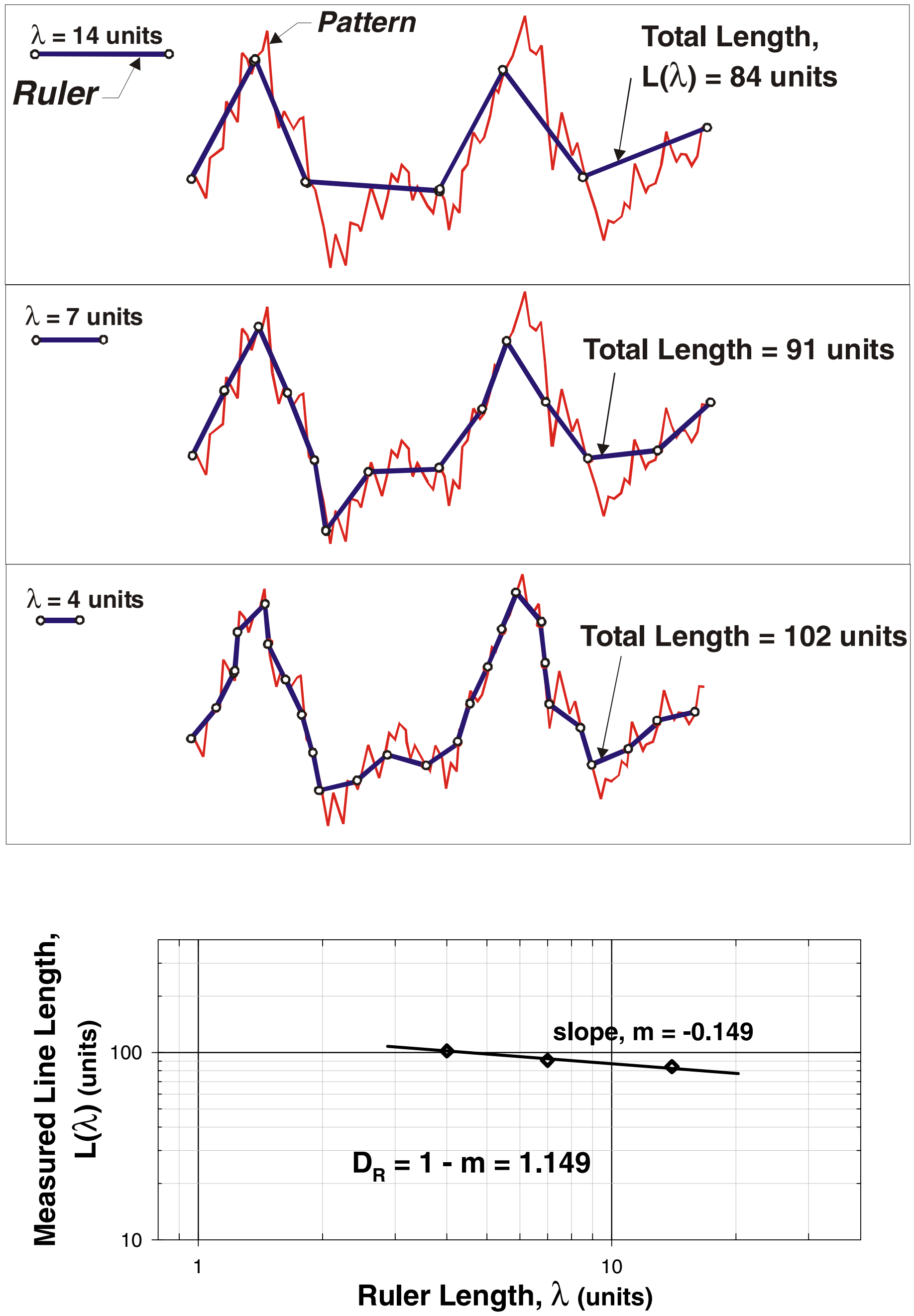

FIGURE 2 Example of Divider fractal dimensioning technique and fratal log-log plot.

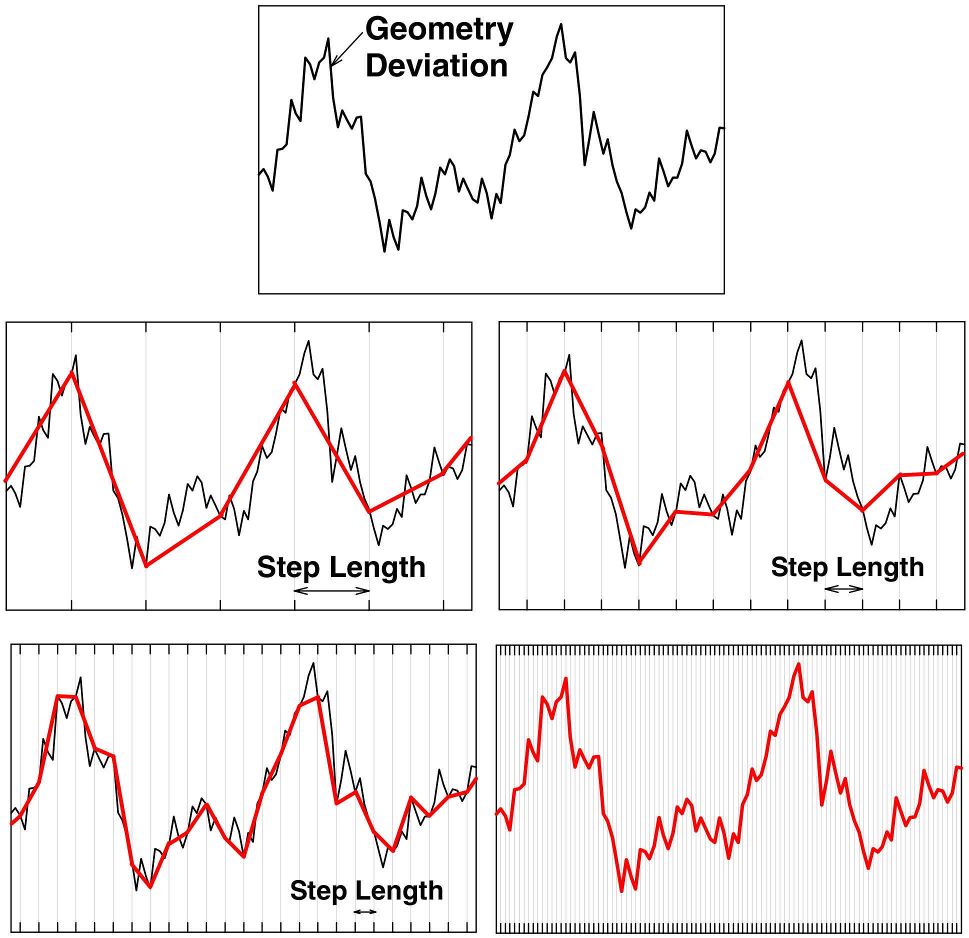

FIGURE 3 Example of fractal dimensioning of geometry deviation pattern.

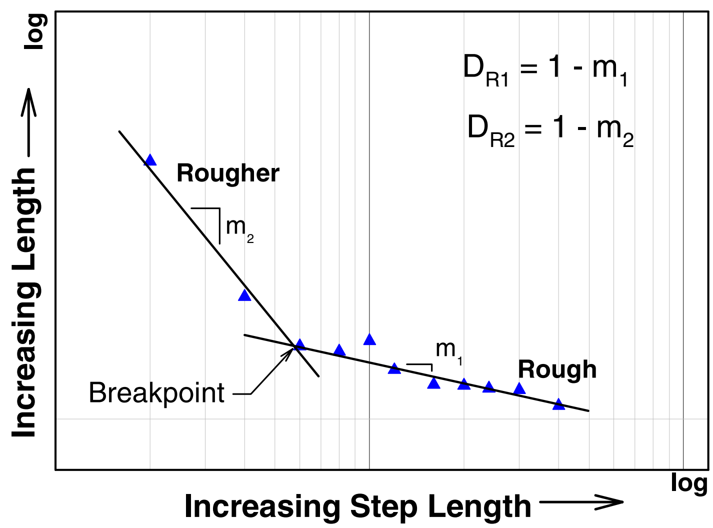

FIGURE 4 Fractal plot of example pattern of FIGURE 3.

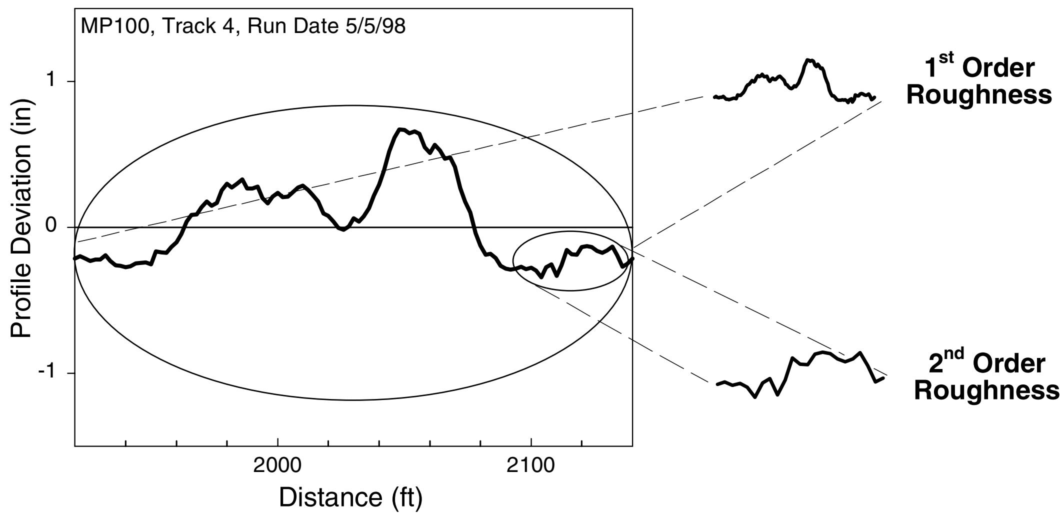

FIGURE 5 Orders of roughness of geometry car deviation (MCO) data.

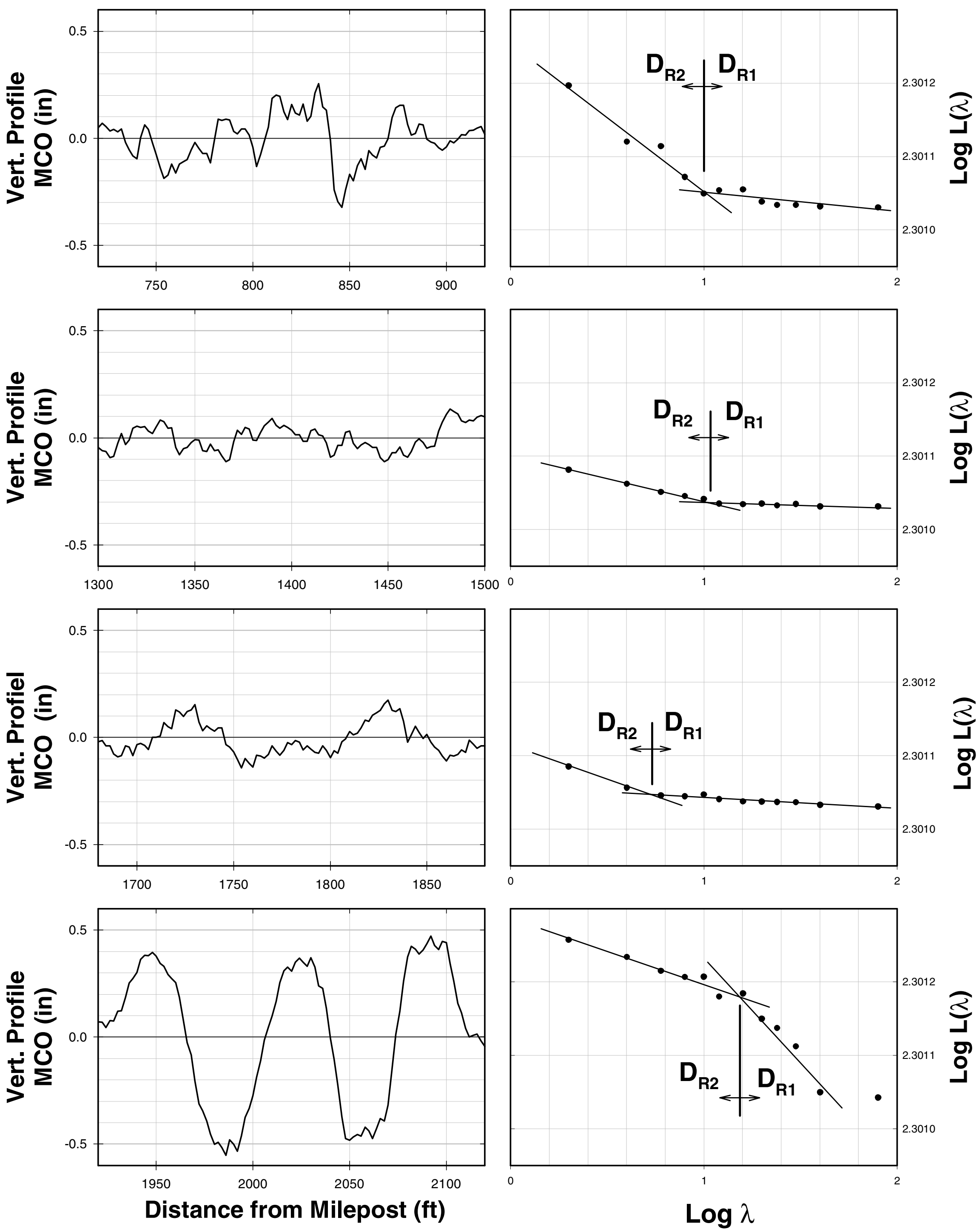

FIGURE 6 Example patterns and fractal plots.

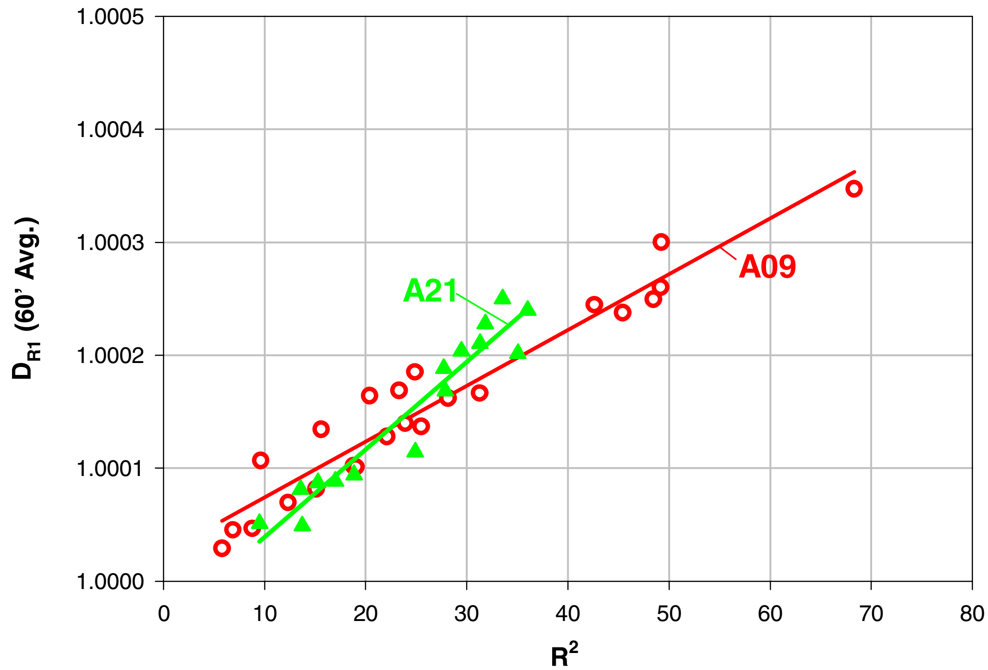

FIGURE 7 Relationship between D_{R1} and R^2 for Site A09 and A21.

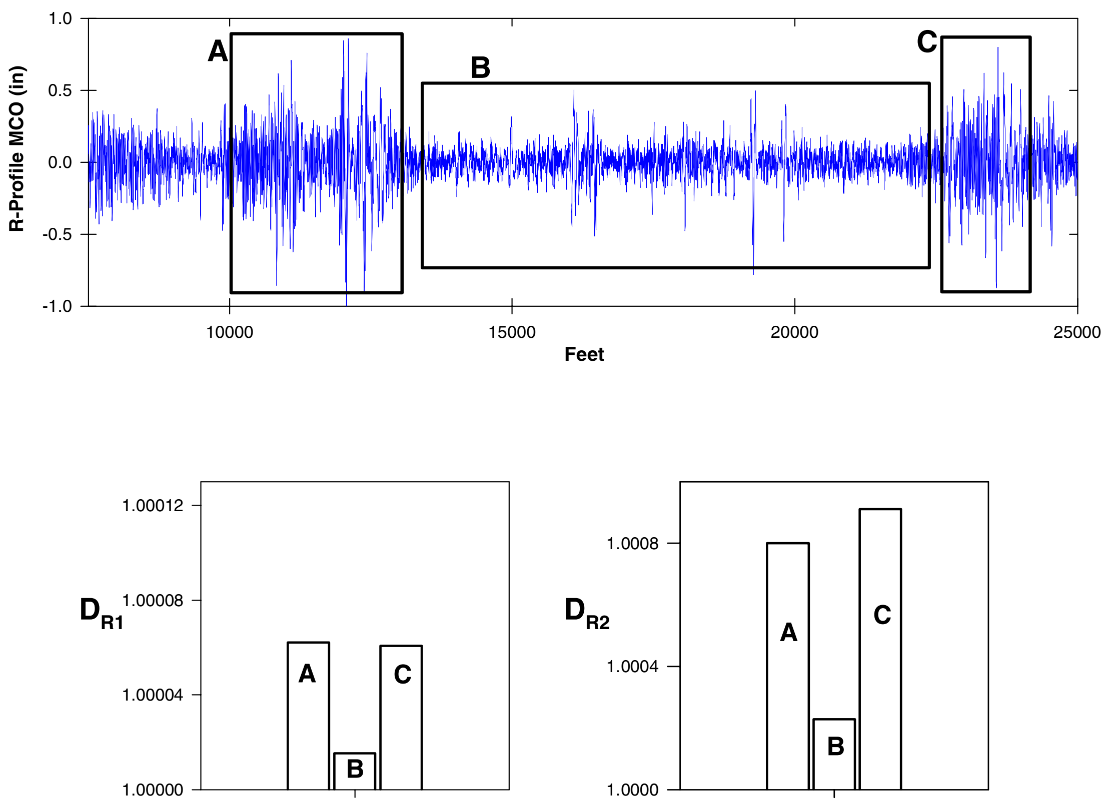

FIGURE 8 Example Fractal Analysis of Different Section Lengths

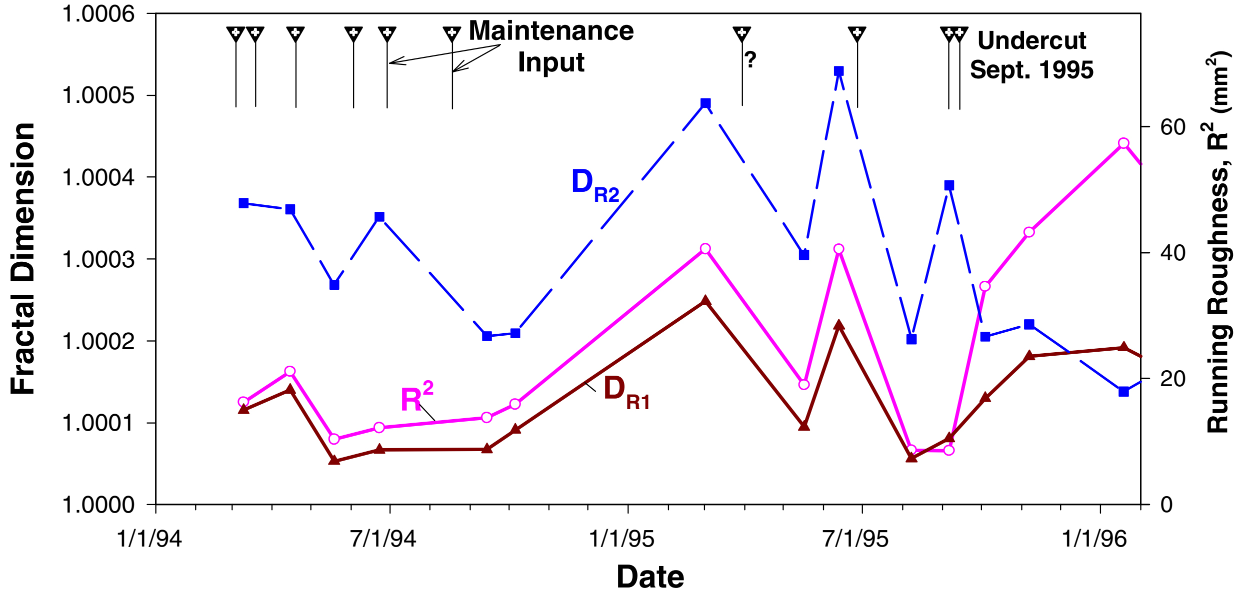

FIGURE 9 Example trend analysis for Site A.

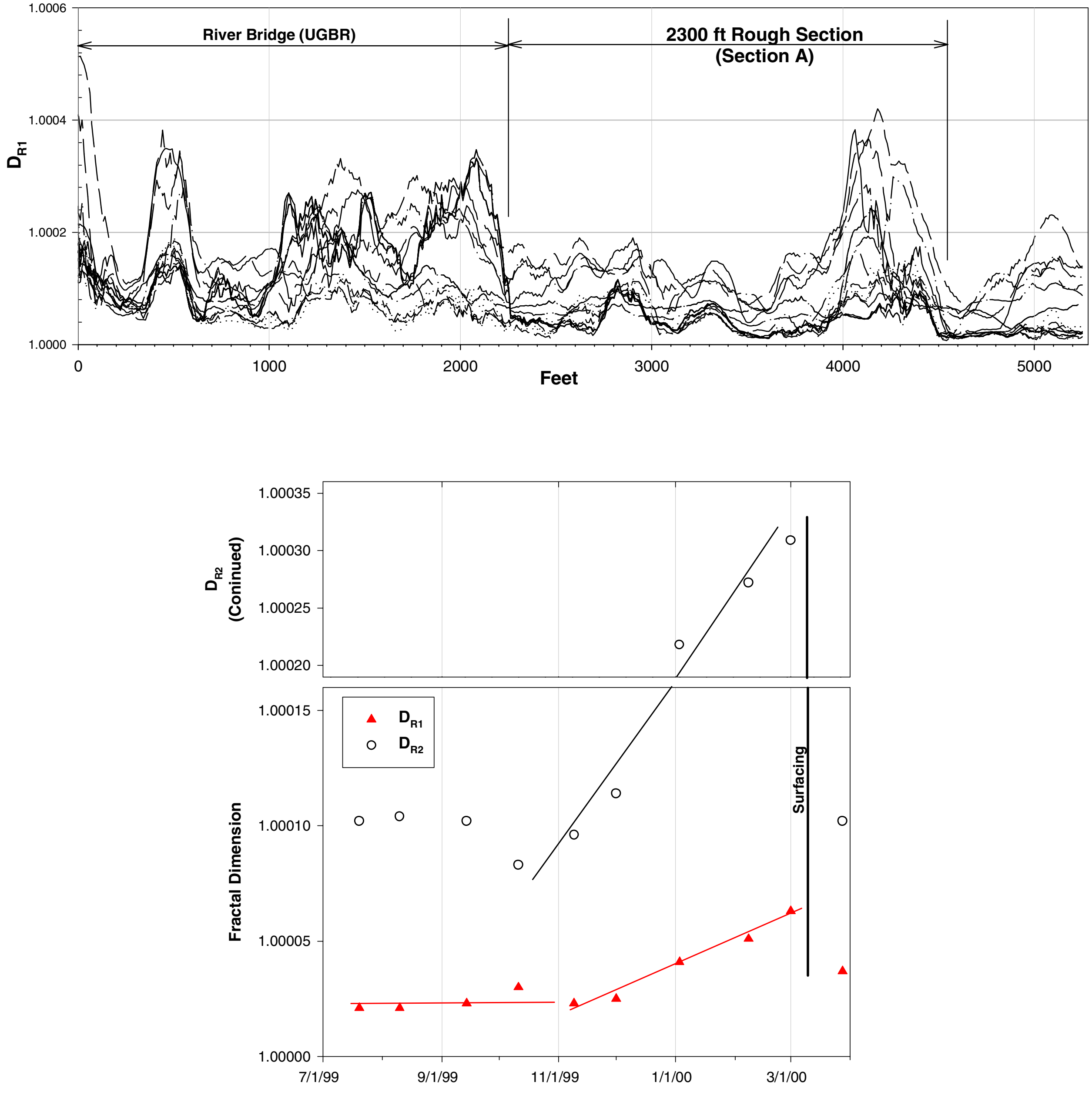

FIGURE 10 Example Fractal Analysis of discrete section of track.

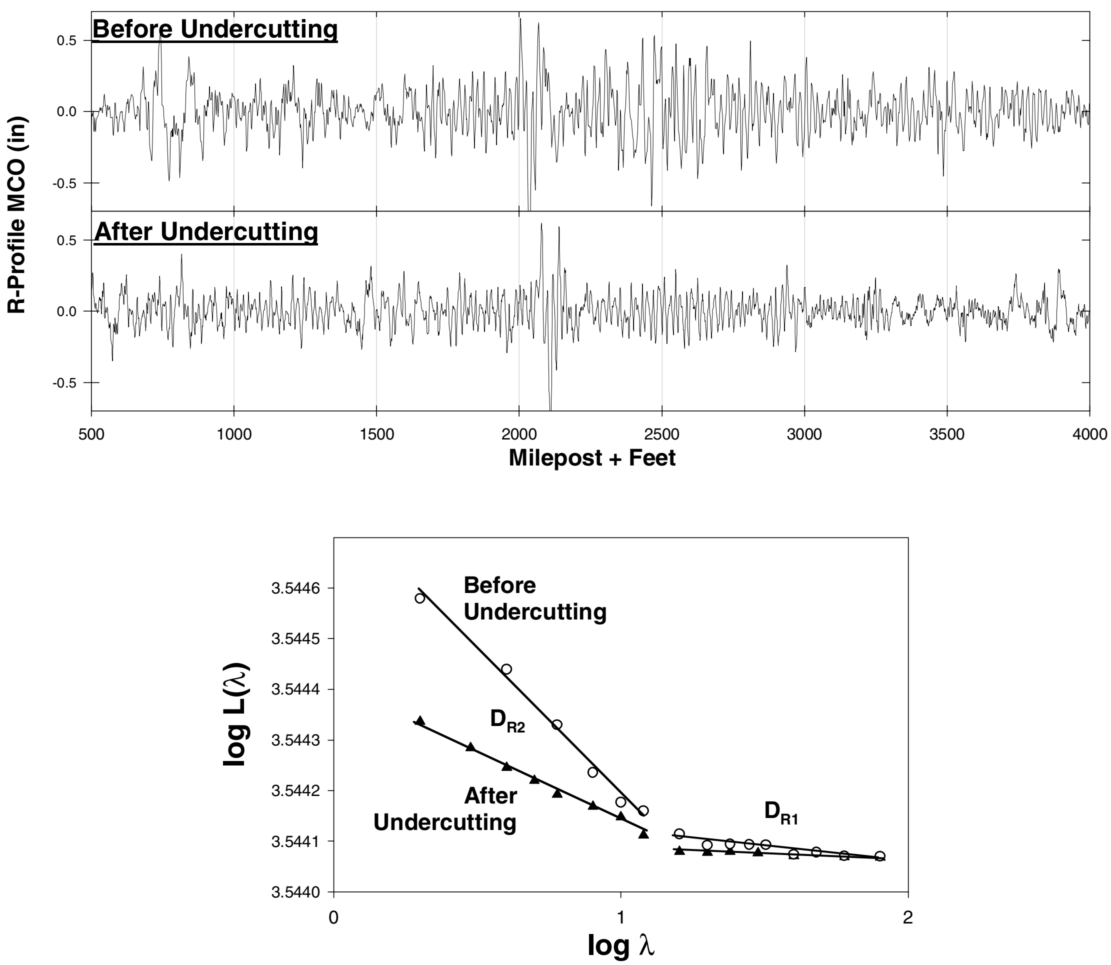

FIGURE 11 Example of fractal analysis results for before and after undercutting.

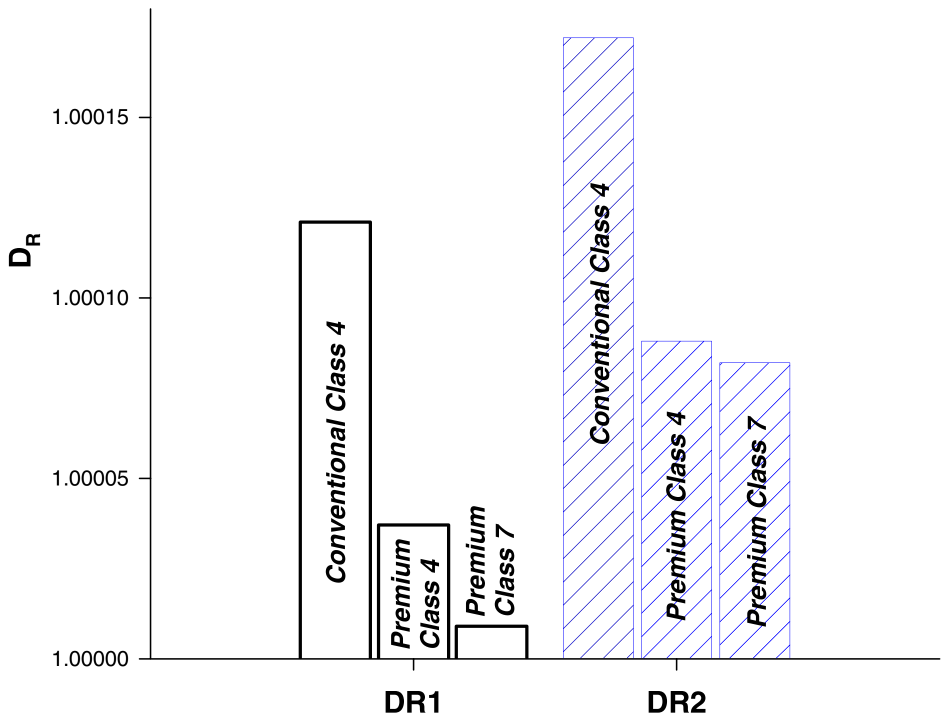

FIGURE 12 Comparison of Premium and Conventional track.