A Geometrically Nonlinear Theory of Elastic Plates

Dewey H. Hodges
Professor.

Ali R. Atilgan[1]
Post Doctoral Fellow.

School of Aerospace Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332-0150

D. A. Danielson
Professor,
Mathematics Department,
Naval Postgraduate School,
Monterey, CA

A set of kinematical and intrinsic equilibrium equations are derived for plates undergoing large deflection and rotation but with small strain. The large rotation is handled by means of the general finite rotation *of a frame in which the material points that are originally along a normal line in the undeformed plate undergo only small displacements. The unit vector fixed in this frame, which coincides with the normal when the plate is undeformed, is not in general normal to the deformed plate average surface because of transverse shear. The arbitrarily large displacement and rotation of this frame, which vary over the surface of the plate, are termed global deformation; the small relative displacement is termed warping. It is shown that rotation of the frame about the normal is not zero and that it can be expressed in terms of other global deformation variables. Exact intrinsic virtual strain-displacement relations are derived; based on a reduced two-dimensional strain energy function from which the warping has been systematically eliminated, a set of intrinsic equilibrium equations follow. It is shown that* only five equilibrium equations can be derived *in this manner, because the component of virtual rotation about the normal is not independent. These equilibrium equations contain terms which cannot be obtained without the use of a finite rotation vector which contains three nonzero components. These extra terms correspond to the difference of in-plane shear stress resultants in other theories; this difference is a reactive quantity in the present theory.*

Introduction

This paper focuses on geometrically nonlinear analysis (i.e., small strain with possibly large deflections and rotations) of plates—solids with one small dimension and without initial curvature. We note that if all three dimensions of a flexible body are comparable, and if the body is undergoing small-strain deformation, then the displacement field in addition to rigid-body motion must be small. If, however, one or two dimensions of the body are small relative to the other(s), then even if the strains remain small the body can undergo large deflections.

Previous Work. Whenever a theory is formulated which is independent of any displacement or rotational variables, it is called an "intrinsic" theory. The earliest development of plates of an intrinsic nature is the work of Synge and Chien (1941). Therein, with the aid of series expansions in powers of a small thickness parameter, approximate theories are derived from three-dimensional elasticity. Sanders (1963) refers to this work as being "completely general" but also points out its being discussed and criticized in the literature. A quite recent intrinsic development is the work of Simo et al. (1989a, 1989b). They formulated a shell model in which finite deflections and rotations are taken into account. Their formulation regards the shell as a Cosserat surface, with an extensible director in the latter work, which is used in lieu of enforcing the usual plane stress assumption; thus, three-dimensional constitutive models can be employed, as done by Koiter and Simmonds (1972).

Some intrinsic theories are strictly two dimensional. That is, they provide no information about the three-dimensional behavior of the plate. Furthermore, they may rely on a two-dimensional constitutive law but not consider how to obtain such a law. For example, Reissner (1974) developed linear and nonlinear two-dimensional theories of shells in this manner. This was also the case when Ericksen and Truesdell (1958) formulated elegant theories for thin shells and curved rods without attempting to deduce them from the three-dimensional theory of elasticity. See Naghdi (1972) for an extensive review of this kind of approach, in which the constitutive relations are deliberately left out of consideration because they are unnecessary for the description of strain and the derivation of equilibrium equations. This approach appears to have originated with the Cosserats (1909) and is able to account for transverse shear and normal strains and rotations associated with couple stresses.

Writing plate/shell equations in a compact and concise form, even when finite deflections and rotations are considered, was

[1]Presently Assistant Professor, Mechanics Division, Department of Civil Engineering, Istanbul Technical University, Istanbul, Turkey.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

Manuscript received by the ASME Applied Mechanics Division, Dec. 3, 1990; final revision, Sept. 16, 1991. Associate Technical Editor: L. M. Keer.

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first suggested by Reissner (1950). Simmonds and Danielson (1970, 1972) also developed a general theory of shells which allowed large deflections in which the rotation was represented with a finite rotation vector. Other recent developments in shell theory are exemplified by Reissner (1982, 1986), Pietraszkiewicz (1979, 1989), Libai and Simmonds (1983), and Axelrad (1986). All of these more recent works develop the theory in terms of displacement of the reference surface and finite rotation of a frame. For the historical development as well as the state of the art in the development of two-dimensional plate/shell theories until 1972, the reader is referred to the review paper of Koiter and Simmonds (1972). For computational aspects of the problem, the reader should consult works by Noor et al. (1989a, 1989b).

Present Approach. The purpose of the present investigation is to find an appropriate set of equations for geometrically nonlinear two-dimensional plate analysis when the plate undergoes large displacements and rotations. Although the use of finite rotation vectors is rather common in plate/shell analyses, the present work attempts to refine the current understanding of the kinematics of rotation—including the "drilling rotation." A complete analysis of plates and shells includes the treatment of *surface kinematics*, *surface equilibrium*, and *modeling*. The term modeling refers to the determination of an appropriate two-dimensional constitutive law (from three-dimensional theory) and accurate *three-dimensional distributions* of displacement, strain, and stress *throughout the plate* in terms of surface (two-dimensional) deformation quantities. Modeling is an inherently approximate process.

Although the framework derived in this paper can be used to develop a modeling approach, that is not the main focus. The variational-asymptotical method of Berdichevsky (1979) is a viable possibility for modeling. This method allows one to calculate asymptotically correct two-dimensional strain energy and three-dimensional displacement, strain, and stress fields. Atılgan and Hodges (1992) applied this method to certain anisotropic plates. Only two assumptions were made regarding deformation: (1) strain is small compared to unity; (2) the wavelength of the deformation is long relative to the thickness of the plate. The first assumption allows them to treat only *geometrically* nonlinear behavior. The second leads to a hierarchy of approximations to general three-dimensional nonlinear theory, the first approximation being a Kirchhoff-like theory and the second analogous to Reissner-Mindlin theory. Although their analysis is applicable to laminated plates, their strain energy is asymptotically exact only if the plate is homogeneous with monoclinic material symmetry about its midplane. This way, there is no elastic coupling in the three-dimensional strain energy between transverse normal and transverse shear strains or between in-plane and transverse shear strains. Their resulting strain energy is the same as that obtained for nonlinear shell theory by Berdichevsky (1980), when appropriately specialized for homogeneous plates, and is the starting point for the plate deformation analysis herein.

We first develop a displacement field which is valid for any plate. Then we identify relevant two-dimensional intrinsic strain measures. Compatibility equations and global strain-displacement relations are then developed. A description of finite rotation for the plate is adopted which gives new insight into the relationship between drilling rotation and the other kinematical variables. After determining the virtual strain-displacement relations, the nonlinear intrinsic equlibrium equations are found. Finally, these equations are compared with published plate equations.

Three-Dimensional Displacement Field

A plate is a flexible body in which matter is distributed about a planar surface so that one dimension is significantly smaller than the other two. Throughout the analysis, Greek indices assume values 1 or 2; Latin indices assume values 1, 2, and 3; and repeated indices are summed over their ranges. In this section, we develop a general expression for the three-dimensional displacement field. This will later lead naturally to a general requirement of equality of the two intrinsic inplane shear strain measures.

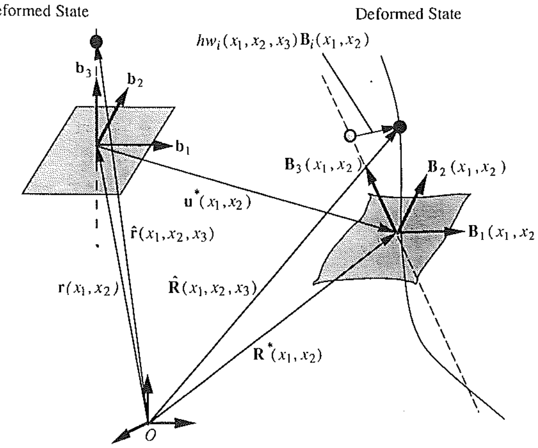

Fig. 1 Schematic of plate deformation

Let us introduce Cartesian coordinates x_i so that x_α denotes lengths along orthogonal straight lines in the midsurface of the undeformed plate, and $x_3 = h\zeta$ is the distance of an arbitrary point to the midsurface in the undeformed plate; See Fig. 1. Here h is the thickness of the plate, a constant, and $-1/2 \le \zeta \le 1/2$. Let $\mathbf{b}_i$ denote an orthogonal reference triad along the undeformed coordinate lines. The position vector from a fixed point O to an arbitrary point is

$$\hat{\mathbf{r}}(x_1,x_2,\zeta)=x_\alpha\mathbf{b}_\alpha+h\zeta\mathbf{b}_3=\mathbf{r}(x_1,x_2)+h\zeta\mathbf{b}_3. \tag{1}$$

Covariant and contravariant undeformed base vectors both reduce to $\mathbf{b}_i$ since the coordinate system is Cartesian. The position vector to the midsurface is also the average position of points along the normal line, at a particular value of x_1 and x_2, so that

$$\mathbf{r}=\int_{-1/2}^{1/2}\hat{\mathbf{r}}d\zeta=\langle\hat{\mathbf{r}}\rangle. \tag{2}$$

The angle brackets $\langle\ \rangle$ are used throughout the paper to denote the integral through the thickness.

Now consider the deformed state configuration. The particle which had position vector $\hat{\mathbf{r}}(x_1, x_2, \zeta)$ in the undeformed plate now has position vector $\hat{\mathbf{R}}(x_1, x_2, \zeta)$. We consider the set N of material points along a typical line normal to the undeformed plate, parallel to $\mathbf{b}_3$. During deformation of the plate, N undergoes a displacement which can be described as a small deformation superimposed on a possibly large rigid-body displacement and rotation. The rigid-body displacement and rotation of N vary over the surface of the plate. For the purpose of describing the global deformation in generic terms, we introduce a frame B (kinematically a rigid body) whose position and orientation vary over the plate. At any arbitrary point in the deformed plate, the points belonging to N are seen in B to be displaced only very slightly form a straight line fixed in B. This small displacement is termed warping.

To facilitate a mathematically simple description of the deformation, we introduce another orthonormal triad $\mathbf{B}_i(x_1, x_2)$ fixed in the frame B. We call $\mathbf{B}_i$ the deformed plate triad, whose orientation relative to $\mathbf{b}_i$ can be specified by an arbitrarily large rotation. The orientation of $\mathbf{B}_i$ is coincident with $\mathbf{b}_i$ when the plate is undeformed and must be defined in terms of the plate deformation. Rotation from $\mathbf{b}_i$ to $\mathbf{B}_i$ is described in terms of a matrix $C(x_1, x_2)$ the elements of which, C_{ij}, are the direction cosines

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$$\mathbf{B}_i = C_{ij}\mathbf{b}_j \quad C_{ij} = \mathbf{B}_i \cdot \mathbf{b}_j. \tag{3}$$

Now, we can write the position vector from O to any point in the deformed plate (Fig. 1) as

$$\hat{\mathbf{R}}(x_1,x_2,\zeta) = \mathbf{R}^*(x_1,x_2) + h\zeta\mathbf{B}_3(x_1,x_2) + hw_i(x_1,x_2,\zeta)\mathbf{B}_i(x_1,x_2) \tag{4}$$

where hw_i is the warping displacement field. The displacement field implied by Eq. (4) accounts for all possible deformations, but it is not yet well defined because the warping is measured in a translated and rotated frame B. The position ($\mathbf{R}^*$) and orientation ($\mathbf{B}_i$) of this frame must be defined in order to make the field unique. Thus, it is necessary to identify six dependency relations between $\mathbf{R}^*$ and the triad $\mathbf{B}_i$; in all but one case these relations will lead to equations which the warping must satisfy.

First, in order for Eq. (4) to be analogous to Eq. (1), $\mathbf{R}^*$ is defined as the average position of points parallel to the vector $\mathbf{B}_3$ corresponding to particular values of x_1 and x_2 in the plate.

$$\mathbf{R}^*(x_1,x_2) \triangleq \langle \hat{\mathbf{R}}(x_1,x_2,\zeta) \rangle = \mathbf{r}(x_1,x_2) + \mathbf{u}^*(x_1,x_2) \tag{5}$$

In order for this definition to hold, the warping must satisfy these three equations

$$\langle w_i(x_1,x_2,\zeta) \rangle = 0. \tag{6}$$

Equation (4) is still indeterminate until three additional dependency relations are placed either on $\mathbf{B}_i$ or on $\mathbf{R}^*$. Consider now the inner product of this vector and $\mathbf{B}_\alpha$ one obtains

$$\mathbf{B}_\alpha \cdot \langle \zeta \hat{\mathbf{R}} \rangle = h \langle \zeta w_\alpha \rangle. \tag{7}$$

To set the direction of $\mathbf{B}_3$ so that it is parallel to $\langle \zeta \hat{\mathbf{R}} \rangle$, we simply require the warping to satisfy the two equations

$$\langle \zeta w_\alpha \rangle = 0. \tag{8}$$

Now only one more dependency relation is needed to make Eq. (4) determinant. Recall that the amount of rigid-body translation of the normal line element is determined by the first three relations, Eqs. (6); the direction of $\mathbf{B}_3$ is determined by the fourth and fifth, Eqs. (8). However, the direction of $\mathbf{B}_\alpha$ (i.e., the amount which B must be rotated about $\mathbf{B}_3$) is not yet settled. It is clear that we are free to set this rotation so that

$$\mathbf{B}_1 \cdot \mathbf{R}^*_{,2} = \mathbf{B}_2 \cdot \mathbf{R}^*_{,1} \tag{9}$$

where $(\ )_{,\alpha}$ denotes differentiation with respect to x_α. The corresponding relations which the warping must satisfy, so that these relations hold, are identically satisfied because of Eqs. (6). Now the displacement field is well defined and unique, assuring continuity of deformation. This final relation will be written quite simply in terms of generalized strain measures, defined below.

Note that the triad $\mathbf{B}_i$ is not exactly the same as the triad ($\mathbf{A}_\alpha$, $\mathbf{N}$) used in Reissner (1974), Simmonds and Danielson (1972), and Libai and Simmonds (1983), unless the average surface of the deformed plate is the image of the midsurface of the undeformed plate. Also note that $\mathbf{R}^*$ is the position vector from O to the points of the *average surface of the deformed plate* and is not in general equal to $\hat{\mathbf{R}}(x_1, x_2, 0)$.

Global Deformation Analysis

After identifying the set of intrinsic strain measures which appear in the two-dimensional energy, compatibility equations and plate strain-displacement measures are finally developed in terms of the displacement of the average surface and rotation of B. We then develop a set of relationships that give variations of intrinsic strain measures in terms of virtual displacements, which are independent of the displacement or rotational variables chosen.

Two-Dimensional Strain Measures and Strain Energy. The strain field derived from the above displacement field can be put into very simple form if, similar to the one-dimensional development in Danielson and Hodges (1987), we introduce the generalized (two-dimensional) strain measures analogous to those of Reissner (1974), Simmonds and Danielson (1972), and Libai and Simmonds (1983) such that

$$\mathbf{R}^*_{,\alpha} = \mathbf{B}_\alpha + \epsilon_{\alpha\beta}\mathbf{B}_\beta + 2\gamma^*_{\alpha 3}\mathbf{B}_3 \tag{10}$$

and components of the curvature vectors such that

$$\mathbf{B}_{i,\alpha} = (-K_{\alpha\beta}\mathbf{B}_\beta \times \mathbf{B}_3 + K_{\alpha 3}\mathbf{B}_3) \times \mathbf{B}_i. \tag{11}$$

It is clear from Eq. (10) that the sixth dependency relation, Eq. (9), reduces to

$$\epsilon_{12} = \epsilon_{21}. \tag{12}$$

It is important to note that *no assumptions were made which in any way limit the validity of this condition. It is completely general and valid for all plates.*

Note that $\mathbf{R}^*_{,1} \times \mathbf{R}^*_{,2}$ is normal to the average surface of the deformed plate and is not in general parallel to $\mathbf{B}_3$ unless the Kirchhoff hypothesis is adopted which constrains $2\gamma^*_{\alpha 3} = 0$. We will henceforth use the term normal to refer to $\mathbf{B}_3$ (although it may only be approximately normal to the deformed plate average surface).

The presence of the warping variable w_i in $\hat{\mathbf{R}}$, while accounting for all possible deformations, creates practical difficulties. Indeed, for one to use a strain field containing w_i is tantamount to using three-dimensional elasticity. To be a true plate theory, the warping must be eliminated. There are several options for elimination of the warping, the most common approaches being based on asymptotical methods. A variety of such methods are referred to in Reissner (1974) and Libai and Simmonds (1983).

For example, the variational asymptotical method of Berdichevsky (1980) was applied to plates by Atılgan and Hodges (1991). Therein, the warping is eliminated in a systematic way. The three-dimensional strain energy per unit volume is expressed in terms of the three-dimensional strain field. The strain field is in turn expressed in terms of ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, hK_{11}, $h(K_{12} + K_{21})$, hK_{22}, $2\gamma^*_{13}$, and $2\gamma^*_{23}$, each of which is taken to be $O(\epsilon)$. Here ϵ and h/l are the small parameters and l is the wavelength of the deformation pattern. An asymptotical analysis leads to a hierarchical approximation for the two-dimensional strain energy, where the first approximation is classical theory; the energy is $O(\mu\epsilon^2)$ and is independent of the transverse shear strains $2\gamma^*_{\alpha 3}$. Here, μ represents the order of the material constants. Each succeeding approximation is of a higher power of h/l than the last, but no higher powers of ϵ are retained, which is appropriate for geometrically nonlinear theory. They carried out the analysis through the second approximation, where their resulting strain energy is asymptotically correct to $O(\mu\epsilon^2 h^2/l^2)$ for homogeneous monoclinic plates. The transverse shear strain measures turn out to be $O(\epsilon h/l)$.

Rather than repeat that entire analysis here, we will simply state the results in symbolic form and build the present analysis on them. Atılgan and Hodges (1991) showed that the approximate two-dimensional strain energy derived from the above displacement field can be put into very simple form if the reference surface of the deformed plate is shifted. The position of this new surface is denoted by $\mathbf{R}(x_1, x_2)$ where

$$\mathbf{R} = \mathbf{R}^* - 12h\mathbf{B}_3 z(\epsilon_{11}, 2\epsilon_{12}, \epsilon_{22}, K_{11}, K_{12}+K_{21}, K_{22}) \tag{13}$$

and where z is a dimensionless function of $O(\epsilon)$, the form of which is not important in the present context. This also implies a shifted displacement vector such that $\mathbf{R} = \mathbf{r} + \mathbf{u}$. Then, use of a transformed transverse shear strain $2\gamma_{\alpha 3}$ defined as

$$\mathbf{R}_{,\alpha} \cdot \mathbf{R}_3 = 2_{\gamma\alpha 3} \tag{14}$$

allows the asymptotically correct two-dimensional strain energy to be expressed to $O(\mu\epsilon^2 h^2/l^2)$ for homogeneous monoclinic plates independent of the derivatives of the two-

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dimensional strain measures. This new position vector will change ϵ_{11}, $2\epsilon_{12}$, and ϵ_{22} only by terms of order ϵ^2, but the curvature components are unaffected. In Atılgan and Hodges (1991) expressions for the warping and strain fields are given in terms of these measures.

The two-dimensional strain energy function is a quadratic form

$$U = U(\epsilon_{11}, 2\epsilon_{12}, \epsilon_{22}, 2\gamma_{13}, 2\gamma_{23}, K_{11}, K_{12}+K_{21}, K_{22}). \quad (15)$$

This way the plate elastic law can be put into a form so that the conjugate stress resultants denoted by N_{11}, N_{12}, N_{22}, Q_α, M_{11}, M_{12}, and M_{22}, are linear functions of ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{\alpha 3}$, K_{11}, $K_{12} + K_{21}$, and K_{22}. The quantities N_{11} and N_{22} are the in-plane stretching stress resultants, N_{12} is the in-plane shear stress resultant, Q_α are the transverse shear stress resultants; M_{11} and M_{22} are the bending moment resultants and M_{12} is the twisting moment resultant, all expressed in the $\mathbf{B}_i$ basis. This is physically equivalent to the energy functional suggested in the first two lines of Eq. (34) of Reissner (1986) (these two lines do not consider the effects of laminations which are not parallel to the plane of the undeflected plate).

We note that a materially nonlinear theory would require the retention of higher-order terms in ϵ. This would, in general, introduce terms which involve the difference $K_{12} - K_{21}$. However, these terms do not enter a *geometrically* nonlinear plate theory because they contribute terms to the strain energy that are of higher order than quadratic in ϵ (a more precise statement is given below). Apparently, the difference $K_{12} - K_{21}$ shows up in a *physically linear* theory only by consideration of couple-stress elasticity, such as in Reissner (1972).

Compatibility Equations. It is well known that the 11 quantities ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{\alpha 3}$, $K_{\alpha\beta}$, and $K_{\alpha 3}$ are not independent. First, it is clear that the kinematics of the deformed plate reference surface can be expressed in terms of *at most* six independent quantities: three measures of displacement, say $\mathbf{u}\cdot\mathbf{b}_i$, and three measures of the rotation of B (since the elements of C can be expressed in terms of three independent quantities). In Simmonds and Danielson (1972) and Reissner (1974), appropriate compatibility equations are derived by first enforcing the equalities

$$\mathbf{R}_{,12} = \mathbf{R}_{,21} \quad \mathbf{B}_{i,12} = \mathbf{B}_{i,21}. \quad (16)$$

These two vector equations lead to six independent compatibility equations equivalent to a form of those found in Reissner (1974) specialized for plates. These equations are rewritten here for convenience in the present notation.

First, from the $\mathbf{B}_3$ component of Eq. (16*a*) we obtain

$$(1+\epsilon_{22})K_{12} - (1+\epsilon_{11})K_{21} = 2\gamma_{23,1} - 2\gamma_{13,2} + \epsilon_{12}(K_{22} - K_{11}). \quad (17)$$

Next, from the $\mathbf{B}_\alpha$ components of Eq. (16*a*) we obtain two equations, for $\alpha = 1$ and 2, respectively,

$$(1+\epsilon_{22})K_{13} - \epsilon_{12}K_{23} = \epsilon_{12,1} - \epsilon_{11,2} - 2\gamma_{13}K_{21} + 2\gamma_{23}K_{11}$$
$$(1+\epsilon_{11})K_{23} + \epsilon_{12}K_{13} = \epsilon_{22,1} - \epsilon_{12,2} - 2\gamma_{13}K_{22} + 2\gamma_{23}K_{12}. \quad (18)$$

Finally, from the three components of Eq. (16*b*) we obtain

$$K_{11,2} - K_{21,1} + K_{13}K_{22} - K_{12}K_{23} = 0$$
$$K_{22,1} - K_{12,2} + K_{23}K_{11} - K_{21}K_{13} = 0$$
$$K_{23,1} - K_{13,2} + K_{11}K_{22} - K_{12}K_{21} = 0. \quad (19)$$

Equations (17)–(19) are similar to those of Reissner (1974) when specialized for plates. Equations (18) can also be derived by enforcing $\epsilon_{12,\alpha} = \epsilon_{21,\alpha}$.

There are now 11 quantities which are related by six compatibility equations. This means that these strain measures can be determined in terms of *only five* independent quantities—*not six*.

We have established above that the strain energy is a function only of the sum $K_{12} + K_{21}$. To take advantage of this, we introduce a twist measure

$$2\kappa_{12} = K_{12} + K_{21}. \quad (20)$$

Now the individual twisting curvatures can be expressed in terms of $2\kappa_{12}$ as

$$K_{12} = \frac{2\gamma_{23,1} - 2\gamma_{13,2} + \epsilon_{12}(K_{22} - K_{11}) + (1+\epsilon_{11})2\kappa_{12}}{2 + \epsilon_{11} + \epsilon_{22}}$$
$$K_{21} = \frac{2\gamma_{13,2} - 2\gamma_{23,1} + \epsilon_{12}(K_{22} - K_{11}) + (1+\epsilon_{22})2\kappa_{12}}{2 + \epsilon_{11} + \epsilon_{22}}. \quad (21)$$

The difference of these can now be found such that

$$\frac{K_{12} - K_{21}}{2} = \frac{2\gamma_{23,1} - 2\gamma_{13,2} + \epsilon_{12}(K_{22} - K_{11}) + \kappa_{12}(\epsilon_{11} - \epsilon_{22})}{2 + \epsilon_{11} + \epsilon_{22}}. \quad (22)$$

This difference is clearly $O(\epsilon h/l^2)$, and one can show that it contributes terms that are $O(\mu\epsilon^3 h^2/l^2)$ to the strain energy. Retention of such terms is inappropriate in a physically linear theory.

Clearly, Eqs. (18) can be solved for the in-plane curvatures K_{13} and K_{23}. Now, using these expressions for K_{13} and K_{23}, along with Eqs. (21), one can rewrite the *three* Eqs. (19) entirely in terms of the *eight* strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, $2\gamma_{23}$, K_{11}, $2\kappa_{12}$, and K_{22}. This confirms that *only five independent measures of displacement and rotation are necessary to define these strain measures as we will demonstrate conclusively below by deriving such measures.*

Global Displacement and Rotation Variables. There is no unique choice for the global deformation variables. For this reason, the importance (not to mention the beauty) of an intrinsic formulation is widely appreciated. On the other hand, for the purpose of understanding the displacement field more fully, for actual computational algorithms, and for easy derivation of virtual strain-displacement relations, Hodges (1990), which are needed below for the intrinsic analysis, it is expedient to introduce a suitable set of displacement measures.

The displacement measures we choose are derived by expressing $\mathbf{R}$ in terms of $\mathbf{r}$ plus a displacement vector so that

$$\mathbf{R}(x_1,x_2) = \mathbf{r}(x_1,x_2) + \mathbf{u}(x_1,x_2)$$
$$= x_\alpha \mathbf{b}_\alpha + u_i \mathbf{b}_i. \quad (23)$$

This allows the determination of the strain measures $\epsilon_{\alpha\beta}$ and $2\gamma_{\alpha 3}$ in terms of C and the derivatives of u_i. Introducing column matrices $u = \lfloor u_1\ u_2\ u_3 \rfloor^T$, $e_1 = \lfloor 1\ 0\ 0 \rfloor^T$, $e_2 = \lfloor 0\ 1\ 0 \rfloor^T$, $\gamma_1 = \lfloor \epsilon_{11}\ \epsilon_{12}\ 2\gamma_{13} \rfloor^T$, and $\gamma_2 = \lfloor \epsilon_{21}\ \epsilon_{22}\ 2\gamma_{23} \rfloor^T$, we use Eqs. (10), (13), and (14) to determine

$$e_\alpha + \gamma_\alpha = C(e_\alpha + u_{,\alpha}) \quad (24)$$

where C is the matrix of direction cosines from Eq. (3).

As rotation measures Rodrigues parameters, Kane et al. (1983), allow the compact expression of C. Euler's theorem shows that any rotation can be represented as a rotation of magnitude Θ about a line parallel to a unit vector $\mathbf{e}$. Defining $\rho_i = 2\mathbf{e}\cdot\mathbf{b}_i \tan(\Theta/2)$ and arranging these in a column matrix $\rho = \lfloor \rho_1\ \rho_2\ \rho_3 \rfloor^T$, the matrix C is simply

$$C = \frac{\left(1 - \frac{\rho^T\rho}{4}\right)I - \tilde{\rho} + \frac{\rho\rho^T}{2}}{1 + \frac{\rho^T\rho}{4}} \quad (25)$$

where here and throughout the paper $\tilde{(\ )}$ is the antisymmetric matrix whose elements are $\tilde{(\ )}_{12} = -(\ )_3$, $\tilde{(\ )}_{13} = (\ )_2$, $\tilde{(\ )}_{23} = -(\ )_1$.

However, in the present context it is expedient to work in terms of the direction cosines of $\mathbf{B}_3$ and a rotation about $\mathbf{B}_3$. Clearly this is also a general way to represent finite rotation. Let us denote the Kronecker symbol by δ_{ij}, the direction cosines of $\mathbf{B}_3$ by

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$$\theta_i = C_{3i} - \delta_{3i} \tag{26}$$

and the angle of rotation about $\mathbf{B}_3$ by ϕ_3. Hodges (1987) has shown that the Rodrigues parameters can be uniquely expressed in terms of θ_i and ϕ_3 as

$$\rho_1 = \frac{\rho_3\theta_1 - 2\theta_2}{2+\theta_3}$$
$$\rho_2 = \frac{\rho_3\theta_2 + 2\theta_1}{2+\theta_3}$$
$$\rho_3 = 2\tan\left(\frac{\phi_3}{2}\right). \tag{27}$$

The three rotational parameters θ_i are not independent. They satisfy the constraint

$$\theta_1^2 + \theta_2^2 + (1+\theta_3)^2 = 1. \tag{28}$$

Recall that Eq. (9) affects only the choice of $\mathbf{B}_\alpha$ and reduces

$$\epsilon_{12} = \frac{[-\theta_1\theta_2(1+u_{1,1}) + (2+\theta_3-\theta_2^2)u_{2,1} - \theta_2 u_{3,1}(2+\theta_3)]\cos\phi_3}{2+\theta_3} - \frac{[(2+\theta_3-\theta_1^2)(1+u_{1,1}) - \theta_1\theta_2 u_{2,1} - \theta_1 u_{3,1}(2+\theta_3)]\sin\phi_3}{2+\theta_3}$$
$$\epsilon_{21} = \frac{[-\theta_1\theta_2(1+u_{2,2}) + (2+\theta_3-\theta_1^2)u_{1,2} - \theta_1 u_{3,2}(2+\theta_3)]\cos\phi_3}{2+\theta_3} + \frac{[(2+\theta_3-\theta_2^2)(1+u_{2,2}) - \theta_1\theta_2 u_{1,2} - \theta_2 u_{3,2}(2+\theta_3)]\sin\phi_3}{2+\theta_3} \tag{30}$$

$$2\gamma_{13} = \theta_1(1+u_{1,1}) + \theta_2 u_{2,1} + (1+\theta_3)u_{3,1}$$
$$2\gamma_{23} = \theta_1 u_{1,2} + \theta_2(1+u_{2,2}) + (1+\theta_3)u_{3,2}.$$

These expressions explicitly depend on $\sin\phi_3$ and $\cos\phi_3$. It is evident that one can choose ϕ_3 so that $\epsilon_{12} = \epsilon_{21}$, yielding

$$\tan\phi_3 = \frac{(2+\theta_3)(u_{2,1} - u_{1,2} + \theta_1 u_{3,2} - \theta_2 u_{3,1}) - \theta_2^2 u_{2,1} + \theta_1^2 u_{1,2} + \theta_1\theta_2(u_{2,2} - u_{1,1})}{(2+\theta_3)(2+\theta_3 + u_{1,1} + u_{2,2} - \theta_1 u_{3,1} - \theta_2 u_{3,2}) - \theta_1^2 u_{1,1} - \theta_2^2 u_{2,2} - \theta_1\theta_2(u_{1,2} + u_{2,1})}. \tag{31}$$

to $\epsilon_{12} = \epsilon_{21}$. The above characterization of finite rotation will allow this choice to be easily expressed in terms of ϕ_3.

When Eq. (27) is substituted into Eq. (25), the resulting elements of C can be expressed as functions of θ_i and ϕ_3

$$C_{11} = \frac{(2+\theta_3-\theta_1^2)\cos\phi_3 - \theta_1\theta_2\sin\phi_3}{2+\theta_3}$$
$$C_{12} = \frac{(2+\theta_3-\theta_2^2)\sin\phi_3 - \theta_1\theta_2\cos\phi_3}{2+\theta_3}$$
$$C_{13} = -\theta_1\cos\phi_3 - \theta_2\sin\phi_3$$
$$C_{21} = \frac{-(2+\theta_3-\theta_1^2)\sin\phi_3 - \theta_1\theta_2\cos\phi_3}{2+\theta_3}$$
$$C_{22} = \frac{(2+\theta_3-\theta_2^2)\cos\phi_3 + \theta_1\theta_2\sin\phi_3}{2+\theta_3}$$
$$C_{23} = \theta_1\sin\phi_3 - \theta_2\cos\phi_3$$
$$C_{31} = \theta_1$$
$$C_{32} = \theta_2$$
$$C_{33} = 1+\theta_3. \tag{29}$$

This representation reduces to those of Reissner (1982, 1986) when considering small, finite rotations. There is an apparent singularity in the present scheme when $\theta_3 = -2$ (i.e., when the plate deforms in such a way that $\mathbf{B}_3$ is pointed in the opposite direction of $\mathbf{b}_3$). This should pose no practical problem, however, since θ_1 and $\theta_2 = 0$ for that condition, and none of the kinematical relations become infinite in the limit as $\theta_3 \to -2$.

When these expressions for the direction cosines are substituted into Eq. (24), explicit expressions for the strain measures can be found as

$$1+\epsilon_{11} = \frac{[(2+\theta_3-\theta_1^2)(1+u_{1,1}) - \theta_1\theta_2 u_{2,1} - \theta_1 u_{3,1}(2+\theta_3)]\cos\phi_3}{1+\theta_3} + \frac{[-\theta_1\theta_2(1+u_{1,1}) + (2+\theta_3-\theta_2^2)u_{2,1} - \theta_2 u_{3,1}(2+\theta_3)]\sin\phi_3}{2+\theta_3}$$
$$1+\epsilon_{22} = \frac{[-\theta_1\theta_2 u_{1,2} + (2+\theta_3-\theta_1^2)(1+u_{2,2}) - \theta_2 u_{3,2}(2+\theta_3)]\cos\phi_3}{2+\theta_3} - \frac{[(2+\theta_3-\theta_1^2)u_{1,2} - \theta_1\theta_2(1+u_{2,2}) - \theta_1 u_{3,2}(2+\theta_3)]\sin\phi_3}{2+\theta_3}$$

It is now clear that once the functions u_1, u_2, u_3, θ_1, and θ_2 are known, the entire displacement field is determined. Because of this, one should expect that a variational formulation would yield only five equilibrium equations—not six. Indeed, five intrinsic equilibrium equations are derived below from Eq. (15).

For small displacement, we note that the rotation about the normal reduces to

$$\phi_3 = \frac{1}{2}(u_{2,1} - u_{1,2}) \text{ (small deflection, small strain)} \tag{32}$$

as expected.

Although one can now find exact expressions for ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, and $2\gamma_{23}$ which are independent of ϕ_3, such expressions are rather lengthy and are not given here. Alternatiively, one could leave ϕ_3 in the equations and regard, say, Eq. (31) as a constraint. This would allow the construction of a plate finite element which would be compatible with beam elements which have three rotational degrees-of-freedom at the nodes.

Expressions for the curvatures can also be found in terms of C from Eqs. (11).

$$\tilde{K}_\alpha = -C_{,\alpha}C^T \tag{33}$$

where the column matrices $K_\alpha = \lfloor -K_{\alpha 2}\ K_{\alpha 1}\ K_{\alpha 3} \rfloor^T$. Using the form of C from Eqs. (29), the following expressions for the curvatures result:

$$K_{\alpha 1} = \theta_{1,\alpha}\cos\phi_3 + \theta_{2,\alpha}\sin\phi_3 - \frac{\theta_{3,\alpha}(\theta_1\cos\phi_3 + \theta_2\sin\phi_3)}{2+\theta_3}$$
$$K_{\alpha 2} = -\theta_{1,\alpha}\sin\phi_3 + \theta_{2,\alpha}\cos\phi_3 - \frac{\theta_{3,\alpha}(-\theta_1\sin\phi_3 + \theta_2\cos\phi_3)}{2+\theta_3}$$
$$K_{\alpha 3} = \phi_{3,\alpha} + \frac{\theta_{1,\alpha}\theta_2 - \theta_1\theta_{2,\alpha}}{2+\theta_3} \tag{34}$$

As before, ϕ_3 can be eliminated from these expressions so that all six curvatures can be expressed in terms of five independent quantities. Note that $K_{\alpha 3}$ does not appear in the strain field. It will, however, appear in the equilibrium equations because of its appearance in the virtual strain-displacement relations derived below.

Virtual Strain-Displacement Relations. In order to derive intrinsic equilibrium equations from the strain energy, it is necessary to express the variations of generalized strain meas-

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ures in terms of virtual displacements and virtual rotations.

The variation of the strain energy can be written from Eq. (15) as

$$\delta U=\frac{\partial U}{\partial \epsilon_{11}}\,\delta\epsilon_{11}+\frac{\partial U}{\partial \epsilon_{12}}\,\delta\epsilon_{12}+\frac{\partial U}{\partial \epsilon_{22}}\,\delta\epsilon_{22}+\frac{\partial U}{\partial \gamma_{13}}\,\delta\gamma_{13}+\frac{\partial U}{\partial \gamma_{23}}\,\delta\gamma_{23}+\frac{\partial U}{\partial K_{11}}\,\delta K_{11}+\frac{\partial U}{\partial \kappa_{12}}\,\delta\kappa_{12}+\frac{\partial U}{\partial K_{22}}\,\delta K_{22}. \quad (35)$$

It is obvious now that one must express $\delta\epsilon_{11}, \ldots, \delta K_{22}$, in terms of virtual displacements and rotations in order to obtain the final Euler-Lagrange equations of the energy functional in their intrinsic form. Following Hodges (1990), we introduce measures of virtual displacement and rotation which are "compatible," Kane (1968), with the intrinsic strain measures. For the virtual displacement, we note the form of Eq. (24) and choose

$$\overline{\delta q}=C\delta u. \quad (36)$$

Similarly, for the virtual rotation, we note the form of Eq. (33) and write

$$\widetilde{\overline{\delta\psi}}=-\delta C C^T. \quad (37)$$

The bars indicate that these quantities are not variations of functions. Using these relations it is clear that

$$\delta u=C^T\overline{\delta q} \quad (38)$$

and

$$\delta C=-\widetilde{\overline{\delta\psi}}C. \quad (39)$$

Let us begin with the generalized strain-displacement relationship, Eq. (24). A particular in-plane strain element can be written as

$$\epsilon_{\alpha\beta}=e_\beta^T[C(e_\alpha+u_{,\alpha})-e_\alpha]. \quad (40)$$

Taking a straightforward variation, one obtains

$$\delta\epsilon_{\alpha\beta}=e_\beta^T[\delta C(e_\alpha+u_{,\alpha})+C\delta u_{,\alpha}]. \quad (41)$$

The right-hand side contains $u_{,\alpha}$ and $\delta u_{,\alpha}$ which must be eliminated in order to obtain variations of the strain which are independent of displacement. These are needed to derive intrinsic equilibrium equations.

Premultiplying both sides of Eq. (24) by C^T, making use of Eq. (39), and finally using a property of the tilde operator that, for arbitrary column matrices Y land Z, $\tilde{Y}Z=-\tilde{Z}Y$, one can make the first term in brackets on the right-hand side independent of $u_{,\alpha}$. After all this, one can obtain

$$\begin{aligned}\delta C(e_\alpha+u_{,\alpha})&=\delta CC^T(e_\alpha+\gamma_\alpha)\\&=-\widetilde{\overline{\delta\psi}}(e_\alpha+\gamma_\alpha)\\&=(\tilde{e}_\alpha+\tilde{\gamma}_\alpha)\overline{\delta\psi}.\end{aligned} \quad (42)$$

An expression for the second term in brackets on the right-hand side of Eq. (41) can now be obtained by differentiating Eq. (38) with respect to x_α and premultiplying by C. This yields

$$C\delta u_{,\alpha}=C(C^T\overline{\delta q})_{,\alpha}=\overline{\delta q}_{,\alpha}-C_{,\alpha}C^T\overline{\delta q}. \quad (43)$$

Now, recalling Eq. (33), one can simplify the second term on the right-hand side so that

$$C\delta u_{,\alpha}=\overline{\delta q}_{,\alpha}+\tilde{K}_\alpha\overline{\delta q} \quad (44)$$

Substituting Eqs. (42) and (44) into Eq. (41), one can obtain an intrinsic expression for the variation of the in-plane strain components as

$$\delta\epsilon_{\alpha\beta}=e_\beta^T[\overline{\delta q}_{,\alpha}+\tilde{K}_\alpha\overline{\delta q}+(\tilde{e}_\alpha+\tilde{\gamma}_\alpha)\overline{\delta\psi}] \quad (45)$$

where $e_\beta^T\,\tilde{e}_\alpha$ vanishes when $\alpha=\beta$. This equation can be written as four scalar equations

$$\begin{aligned}\delta\epsilon_{11}&=\overline{\delta q}_{1,1}-K_{13}\overline{\delta q}_2+K_{11}\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_1+\epsilon_{12}\overline{\delta\psi}_3\\ \delta\epsilon_{12}&=\overline{\delta q}_{2,1}+K_{13}\overline{\delta q}_1+K_{12}\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_2-(1+\epsilon_{11})\overline{\delta\psi}_3\\ \delta\epsilon_{21}&=\overline{\delta q}_{1,2}-K_{23}\overline{\delta q}_2+K_{21}\overline{\delta q}_3-2\gamma_{23}\overline{\delta\psi}_1+(1+\epsilon_{22})\overline{\delta\psi}_3\\ \delta\epsilon_{22}&=\overline{\delta q}_{2,2}+K_{23}\overline{\delta q}_1+K_{22}\overline{\delta q}_3-2\gamma_{23}\overline{\delta\psi}_2-\epsilon_{12}\overline{\delta\psi}_3.\end{aligned} \quad (46)$$

Now, by virtue of Eq. (12),

$$\delta\epsilon_{12}=\delta\epsilon_{21}. \quad (47)$$

From this, one can solve for the virtual rotation component about $\mathbf{B}_3$ as

$$\overline{\delta\psi}_3=\frac{\overline{\delta q}_{2,1}-\overline{\delta q}_{1,2}+K_{13}\overline{\delta q}_1+K_{23}\overline{\delta q}_2+(K_{12}-K_{21})\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_2+2\gamma_{23}\overline{\delta\psi}_1}{2+\epsilon_{11}+\epsilon_{22}}. \quad (48)$$

It is now possible to write the variations of all strain measures in terms of three virtual displacement and two virtual rotation components as

$$\begin{aligned}\delta\epsilon_{11}&=\overline{\delta q}_{1,1}-K_{13}\overline{\delta q}_2+K_{11}\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_1+\epsilon_{12}\overline{\delta\psi}_3\\ \delta\epsilon_{22}&=\overline{\delta q}_{2,2}+K_{23}\overline{\delta q}_1+K_{22}\overline{\delta q}_3-2\gamma_{23}\overline{\delta\psi}_2-\epsilon_{12}\overline{\delta\psi}_3\\ 2\delta\epsilon_{12}&=\overline{\delta q}_{2,1}+\overline{\delta q}_{1,2}+K_{13}\overline{\delta q}_1-K_{23}\overline{\delta q}_2+(K_{12}+K_{21})\overline{\delta\psi}_3\\&\quad-2\gamma_{13}\overline{\delta\psi}_2-2\gamma_{23}\overline{\delta\psi}_1+(\epsilon_{22}-\epsilon_{11})\overline{\delta\psi}_3\end{aligned} \quad (49)$$

with $\overline{\delta\psi}_3$ taken from Eq. (48).

Let us now consider the transverse shear strains

$$2\gamma_{\alpha 3}=e_3^T[C(e_\alpha+u_{,\alpha})-e_\alpha]. \quad (50)$$

Taking a straightforward variation and making use of Eq. (39) with the above tilde identity, one can find the virtual strain-displacement equation for transverse shear strains as follows:

$$2\delta\gamma_{\alpha 3}=e_3^T[\overline{\delta q}_{,\alpha}+\tilde{K}_\alpha\overline{\delta q}+(\tilde{e}_\alpha+\tilde{\gamma}_\alpha)\overline{\delta\psi}]. \quad (51)$$

Explicit expressions for the variations of the shear strain components are now easily written as

$$2\delta\gamma_{\alpha 3}=\overline{\delta q}_{3,\alpha}+\overline{\delta\psi}_\alpha+\epsilon_{\alpha\beta}\overline{\delta\psi}_\beta-K_{\alpha\beta}\overline{\delta q}_\beta. \quad (52)$$

Finally, the variations of the curvatures are found. First, taking the straightforward variation of Eq. (33), one obtains

$$\delta\tilde{K}_\alpha=-\delta C_{,\alpha}C^T-C_{,\alpha}\delta C^T. \quad (53)$$

In order to eliminate $\delta C_{,\alpha}$, we differentiate Eq. (37) with respect to x_α:

$$\widetilde{\overline{\delta\psi}}_{,\alpha}=-\delta C_{,\alpha}C^T-\delta CC^T_{,\alpha}. \quad (54)$$

In order to eliminate δC, we can use Eq. (39). Then Eq. (53) becomes

$$\delta\tilde{K}_\alpha=\widetilde{\overline{\delta\psi}}_{,\alpha}+\tilde{K}_\alpha\widetilde{\overline{\delta\psi}}-\widetilde{\overline{\delta\psi}}\tilde{K}_\alpha. \quad (55)$$

Using another tilde identity ($\widetilde{\tilde{Y}Z}=\tilde{Y}\tilde{Z}-\tilde{Z}\tilde{Y}$) one can find the following virtual strain-displacement relation:

$$\delta K_\alpha=\overline{\delta\psi}_{,\alpha}+\tilde{K}_\alpha\overline{\delta\psi}. \quad (56)$$

In explicit form

$$\begin{aligned}\delta K_{11}&=\overline{\delta\psi}_{1,1}-K_{13}\overline{\delta\psi}_2+K_{12}\overline{\delta\psi}_3\\ \delta K_{22}&=\overline{\delta\psi}_{2,2}+K_{23}\overline{\delta\psi}_1-K_{21}\overline{\delta\psi}_3\\ 2\delta\kappa_{12}&=\overline{\delta\psi}_{1,2}+\overline{\delta\psi}_{2,1}+K_{13}\overline{\delta\psi}_1-K_{23}\overline{\delta\psi}_2+(K_{22}-K_{11})\overline{\delta\psi}_3,\end{aligned} \quad (57)$$

where $\overline{\delta\psi}_3$ should again be eliminated by using Eq. (48).

Intrinsic Equilibrium Equations

In this section, we will make use of the virtual strain-displacement relations in the variation of the internal strain energy

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in order to derive the intrinsic equilibrium equations. Here we define the generalized forces as

$$\frac{\partial U}{\partial \epsilon_{11}} = N_{11} \qquad \frac{\partial U}{\partial \epsilon_{22}} = N_{22} \qquad \frac{1}{2}\frac{\partial U}{\partial \epsilon_{12}} = N_{12}$$

$$\frac{\partial U}{\partial K_{11}} = M_{11} \qquad \frac{\partial U}{\partial K_{22}} = M_{22} \qquad \frac{1}{2}\frac{\partial U}{\partial \kappa_{12}} = M_{12}$$

$$\frac{1}{2}\frac{\partial U}{\partial \gamma_{13}} = Q_1 \qquad \frac{1}{2}\frac{\partial U}{\partial \gamma_{23}} = Q_2. \tag{58}$$

If the influence of warping on the virtual work of body forces and surface tractions is ignored, body forces and applied tractions over the upper and lower surfaces of the plate can be shown to have virtual work equivalent to that of applied forces and moments distributed over the plate reference surface. The virtual work per unit area can then be written as

$$\overline{\delta W} = \overline{\delta q_i} f_i + \overline{\delta \psi_i} m_i \tag{59}$$

where $\overline{\delta\psi_3}$ is given by Eq. (48). It is now possible to obtain intrinsic equilibrium equations and consistent edge conditions, as in Basar (1987), by use of the principle of virtual work and the virtual strain-displacement relations derived in the previous section. The equilibrium equations are

$$N_{11,1} + (N_{12} + \mathfrak{N})_{,2} - K_{13}(N_{12} - \mathfrak{N}) - K_{23}N_{22} + Q_1K_{11} + Q_2K_{21} + f_1 = 0$$

$$N_{22,2} + (N_{12} - \mathfrak{N})_{,1} + K_{23}(N_{12} + \mathfrak{N}) + K_{13}N_{11} + Q_1K_{12} + Q_2K_{22} + f_2 = 0$$

$$Q_{1,1} + Q_{2,2} - K_{11}N_{11} - K_{22}N_{22} - 2\kappa_{12}N_{12} + (K_{12} - K_{21})\mathfrak{N} + f_3 = 0$$

$$M_{11,1} + M_{12,2} - Q_1(1+\epsilon_{11}) - Q_2\epsilon_{12} + 2\gamma_{13}N_{11} + 2\gamma_{23}(N_{12} + \mathfrak{N}) - M_{12}K_{13} - M_{22}K_{23} + m_1 = 0$$

$$M_{12,1} + M_{22,2} - Q_1\epsilon_{12} - Q_2(1+\epsilon_{22}) + 2\gamma_{13}(N_{12} - \mathfrak{N}) + 2\gamma_{23}N_{22} + M_{11}K_{13} + M_{12}K_{23} + m_2 = 0 \tag{60}$$

where

$$(2 + \epsilon_{11} + \epsilon_{22})\mathfrak{N} = (N_{22} - N_{11})\epsilon_{12} + N_{12}(\epsilon_{11} - \epsilon_{22}) + M_{22}K_{21} - M_{11}K_{12} + M_{12}(K_{11} - K_{22}) + m_3. \tag{61}$$

When displacements and rotations are free along an edge, the boundary conditions for a rectangular domain allow for specification of N_{11}, $N_{12} - \mathfrak{N}$, Q_1, M_{11}, and M_{12} on an edge where x_1 = constant; similarly $N_{12} + \mathfrak{N}$, N_{22}, Q_2, M_{12}, and M_{22} can be specified on an edge where x_2 = constant. We note that an applied "drilling moment" m_3, stemming only from tractions on the upper and lower surface of the plate, is hard to physically realize. It is retained in the equations for completeness, however.

The terms containing $\mathfrak{N}$ stem from consistent inclusion of the finite rotation of B although the *nonzero* rotation about $\mathbf{B}_3$ is expressed in terms of other kinematical quantities. Similar terms are found in the shell equations derived by Berdichevsky (1979, 1989) where only five equilibrium equations are derived.

In a mixed formulation, $\mathfrak{N}$ can be shown to be the Lagrange multiplier that enforces Eq. (48). To further understand the nature of $\mathfrak{N}$, one can undertake the following exercize: Specialize the six equilibrium equations of Reissner (1974) for plates, also setting $P_i = 0$ and $\epsilon_{12} = \epsilon_{21}$. The sixth equation then becomes zeroth-order and can be solved for $(N_{21} - N_{12})/2$. This shows that Reissner's $(N_{21} - N_{12})/2$ is the same as our $\mathfrak{N}$, and Reissner's $(N_{21} + N_{12})/2$ is the same as our N_{12}. Finally, substitution of this sixth equation into the other five yields our five equilibrium equations, Eq. (60).

A few investigators have noted an apparent conflict between the symmetry of the stress resultants and the satisfaction of moment equilibrium about the normal; for example, see Kraus (1967). In reality there is no conflict, but one must be careful. We show herein that the triad $\mathbf{B}_i$ can always be chosen so that $\epsilon_{12} = \epsilon_{21}$. If this relation is enforced strongly, there is only one in-plane shear stress resultant, N_{12}, that can be derived from the energy. In that case the physical quantity associated with the antisymmetric part of Reissner's in-plane stress resultants, while it is not available from the constitutive law, is nevertheless available as a reactive quantity from the moment equilibrium equation about the normal. However, it must be stressed that the moment equilibrium equation about the normal is not available from a conventional energy approach, in which virtual displacements and rotations must be independent.

A somewhat analogous situation arises in reducing Timoshenko beam theory to Euler-Bernoulli beam theory. Shear deformation is suppressed, and thus the shear force resultant is no longer available from the constitutive law. It must be determined from the equilibrium equations as the derivative of the bending moment. If the constraint of vanishing shear deformation is further used to eliminate the virtual rotation, the moment equilibrium equation cannot be derived from the energy.

In a somewhat similar vein, not being able to obtain the antisymmetric part of the moment stress resultants from derivatives of the two-dimensional strain energy is a result of the approximate dimensional reduction process in which it was determined that, based on asymptotical arguments and *geometrically* nonlinear three-dimensional elasticity, the antisymmetric term $K_{12} - K_{21}$ does not appear in the two-dimensional strain energy. When the foundation of the theory is broadened so as to include couple-stress considerations, then $K_{12} - K_{21}$ could appear; see Reissner (1972, 1982).

For practical computational schemes, these equilibrium equations need to be supplemented with additional sets of equations. Any approach will require a set of constitutive relations between the stress resultants N_{11}, N_{12}, N_{22}, Q_1, Q_2, M_{11}, M_{12}, and M_{22}, and the strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, $2\gamma_{23}$, K_{11}, $2\kappa_{12}$, and K_{22}; an example of such a set is given in Atılgan and Hodges (1991). Finally a set of kinematical equations is needed. The kinematical part of the analysis can be done in two fundamentally different wasy: a purely intrinsic form and a mixed form.

In the intrinsic form we have five equilibrium equations, Eqs. (60); six compatibility equations, Eqs. (17)–(19); and the eight constitutive equations—a total of 19 equations. The 19 unknowns are the eight stress resultants, N_{11}, N_{12}, N_{22}, Q_1, Q_2, M_{11}, M_{12}, and M_{22}; and 11 strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, $2\gamma_{23}$, K_{11}, $2\kappa_{12}$, and K_{22}, along with K_{13}, K_{23}, and $K_{12} - K_{21}$. The last three strain measures appear in the equilibrium equations but not in the constitutive law.

In the mixed form we use the same five equilibrium equations and eight constitutive equations. We also need a set of strain-displacement relations between the 11 strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, $2\gamma_{23}$, K_{11}, $2\kappa_{12}$, and K_{22}, along with K_{13}, K_{23}, and $K_{12} - K_{21}$, and the five global displacement and rotational variables u_1, u_2, u_3, θ_1, and θ_2. One possible set of such equations is as follows: use five of Eqs. (30), using either ϵ_{12} or ϵ_{21}; use the six Eqs. (34), with Eq. (20). There are also the two other rotational variables θ_3 and ϕ_3 which are governed by Eqs. (28) and (31), respectively. This way there are 26 equations and 26 unknowns. This mixed formualtion is capable of handling boundary conditions on two-dimensional stress resultants and displacement/rotation variables. At least in principle, one could derive a displacement formulation by eliminating all the unknowns except the displacement and rotation variables.

Equations (60) and (61) contain terms which could be disregarded because of the original assumption of small strain. We will not undertake this simplification here, because it is not necessary for using the equations in computational algorithms. Therefore, our equilibrium equations and kinematical

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equations are geometrically exact; all approximations stem from the reduction process used to obtain the two-dimensional constitutive law.

Lest it be thought that the phenomena associated with the drilling rotation not being independent affects only nonlinear problems, Kanok-Nukulchai (1979) developed a linear shell finite element including three rotations at each node. It was shown that unless the constraint expressed in Eq. (32) is enforced on the rotation about the normal, the shell element derived therein fails to converge to the exact solution. This result would appear to confirm the present conclusion concerning the importance of dealing properly with the rotation about the normal.

Conclusions

Governing equations for the small-strain deformation of an elastic plate are derived. Expansion of the displacement field in terms of the normal coordinate is not needed; rather, the warping of the line of material points normal to the plate is determined by Atılgan and Hodges (1991), based on the variational asymptotical method. The two-dimensional strain and curvature measures which remain in the strain energy are the same as those in Berdichevsky (1980). Global strain-displacement relations, compatibility equations, and exact intrinsic virtual strain-displacement (transpositional) relations are derived from these measures. Intrinsic equilibrium equations are then derived from the approximate two-dimensional strain energy function, making use of the transpositional relations. The resulting intrinsic equilibrium and compatibility equations are compared with others in the literature.

The following conclusions are drawn in this paper:

1 A full finite rotation must be permitted so that the displacement field is fully specified. However, the rotation about the normal is not independent, but rather is known in terms of other quantities. Indeed, it can always be chosen, for any plate, so that the two-dimensional in-plane shear strain measures are equal. This way, all the strain measures can be expressed in terms of five independent quantities: three displacement and two rotation measures.

2 Equality of the surface in-plane shear strain measures leads to only five independent virtual displacement and rotation measures in an energy-based development. This implies that only five equilibrium equations are obtainable in a displacement-based variational formulation. Moment equilibrium about the normal is satisfied implicitly. If one does not include the full finite rotation, but rather sets the rotation about the normal equal to zero, the correct equilibrium equations cannot be obtained.

Acknowledgments

The first and second authors were supported by the U.S. Army Aerostructures Directorate, Langley Research Center, under contract NASA Grant NAG-1-1094. The technical monitor is Howard E. Hinnant. The third author was supported by the Naval Postgraduate School Research Program. Technical discussions between the first two authors and V. L. Berdichevsky are gratefully acknowledged. The authors thank J. G. Simmonds and an anonymous reviewer for insightful criticisms.

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