Workspaces of planar parallel manipulators

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Abstract: This paper presents geometrical algorithms for the determination of various workspaces of planar parallel manipulators. Workspaces are defined as regions which can be reached by a reference point C located on the mobile platform. First, the *maximal* workspace is determined as the region which can be reached by point C with at least one orientation. From the above regions, the *inclusive* workspace, i.e., the region which can be attained by point C with at least one orientation in a given range, can be obtained. Then, the *total orientation* workspace, i.e., the region which can be reached by point C with every orientation of the platform in a given range, is determined. Three types of planar parallel manipulators are described and one of them is used to illustrate the algorithms.

1 Introduction

Parallel manipulators have been proposed as mechanical architectures which can overcome the limitations of serial robots [5]. Parallel manipulators lead to complex kinematic equations and the determination of their workspace is a challenging problem. Some researchers have addressed the problem of the determination of the workspace of parallel manipulators ([2]; [8]; [11]; [12]), especially for computing the workspace of the robot when its orientation is fixed.

In this paper, the problem of the determination of the workspaces of planar 3 d.o.f parallel manipulators is addressed. Algorithms are proposed for the determination of the maximal workspace, a problem which has been elusive to previous analyses.

Planar 3 d.o.f. parallel manipulators are composed of three kinematic chains connecting a mobile platform to a fixed base. The manipulator of particular interest in this study is referred to as the $3-RPR$ manipulator. In this manipulator, the mobile platform is connected to the base via three identical chains consisting of a revolute joint attached to the ground followed by an actuated prismatic joint which is connected to the platform by a revolute joint (figure 1). Henceforth, the center of the joint connecting the ith chain to the ground will be denoted A_i and the center of the joint connecting the ith chain to the platform will be referred to as B_i. Others types of planar parallel manipulators are: the $3-RRR$ robot ([1];[9];[5]) in which the joints attached to the ground are the only actuated joints and the $3-PRR$ robot in which the prismatic joints are actuated.

A fixed reference frame is defined on the base and a moving reference frame is attached to the platform with its origin at a reference point C. The position of the moving platform is defined by the coordinates of point C in the fixed reference frame and its orientation is given by the angle θ between one axis of the fixed reference frame and the corresponding axis of the moving frame.

For the $3-RPR$ manipulator, the workspace limitations are due to the limitations of the prismatic actuators. The maximum and minimum lengths of the prismatic actuator of the jth

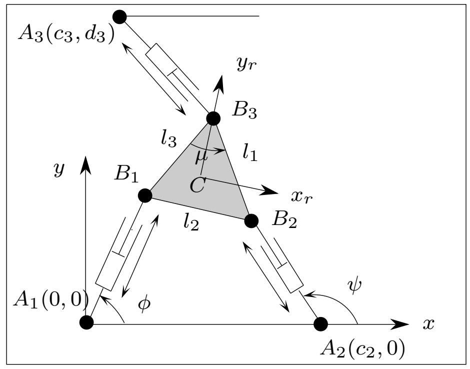

Figure 1: The $3-RPR$ parallel manipulator.

chain are denoted ρ^j_{max}, ρ^j_{min}. These values will be referred to as *extreme values* of the joint coordinates.

Furthermore, an *annular region* $\mathcal{E}$ is defined as the region which lies between two concentric circles with different radii. The circle $\mathcal{E}^e$ with the largest radius will be referred to as the *external circle* and the smaller circle $\mathcal{E}^i$ (if it exists) will be referred to as the *internal circle*. The dimensions of the manipulators which are used in the examples are given in the appendix. In what follows, the presentation of the various workspaces will focus on the $3-RPR$ manipulator.

2 Maximal workspace

The *maximal workspace* is defined as the region which the reference point C can reach with at least one orientation. It shall be noted that the maximal workspace will depend upon the choice of the reference point. One of the objectives of the present work is to determine geometrically the boundary of the maximal workspace.

The determination of the maximal workspace has been addressed by Kassner [7] who pointed out that the boundary of this workspace is composed of circular arcs and of portions of sextic curves, but was only able to compute them with a discretization method. The same observation was made by Kumar [8] but his method, based on screw analysis, cannot be used for a manipulator with prismatic actuators.

2.1 Determining if a point is in the maximal workspace

First, a simple algorithm is derived to determine if a location of the reference point is in the maximal workspace, this being equivalent to determining if there is at least one possible orientation of the platform for this location.

For a given position of C, point B_1 can move on a circle C^1_B with center C and radius $||\mathbf{CB_1}||$. We first verify if C^1_B is completely inside or outside the annular region $\mathcal{E}_1$, corresponding to the constraint for leg 1, by checking the distance between the centers of C^1_B and $\mathcal{E}_1$ with respect to their radii. If C^1_B is inside $\mathcal{E}_1$, then any orientation is allowed for the platform, with respect to the constraints on leg 1. If C^1_B is outside $\mathcal{E}_1$ then no orientation is allowed for the platform and C is outside the maximal workspace.

If the preceding test fails it may be assumed that there are intersection points between C_B^1 and $\mathcal{E}_1^e$, $\mathcal{E}_1^i$. For each of these intersection points there is a unique orientation angle possible for the platform. These angles are ordered in the interval $[0, 2\pi]$ in order to obtain a set of consecutive intervals. Then, in order to determine which intervals define valid orientations for the platform, the middle value of each interval is used as the orientation of the platform and the constraints on leg 1 are tested for the corresponding configuration. A similar procedure is performed for the legs 2 and 3. For leg i, n various intervals are obtained and the set I_n^i of possible orientations of the platform with respect to the constraint on the leg can be determined.

The intersection $I_\cap$ of these lists is then determined as the intersection of all the sets of three intervals $\{I_1 \in I_{n_1}^1, I_2 \in I_{n_2}^2, I_3 \in I_{n3}^3\}$. If $I_\cap$ is not empty then C belongs to the maximal workspace and $I_\cap$ defines the possible orientation for the moving platform at this point.

2.2 Determination of the boundary of the maximal workspace

For purposes of simplification, it is first assumed that the reference point on the platform is chosen as one of the B_i's, for example point B_3. The general case will be presented later on as a generalization. If a location of B_3 belongs to the boundary of the maximal workspace then at least one of the legs is at an extreme value (otherwise B_3 will be capable of moving in any direction which is in contradiction with B_3 being on the boundary of the workspace). Note that the configuration with three legs in an extreme extension defines only isolated points of the boundary since they are solutions of the direct kinematics of the manipulator, a problem which admits at most 6 different solutions [3].

2.2.1 Boundary points with one extreme leg length

In order to geometrically determine the points of the boundary for which one leg length of the manipulator is at an extreme value, the kinematic chain $A_iB_iB_3$ is considered as a planar serial 2 d.o.f. manipulator whose joint at A_i is fixed to the ground. It is well known that the positions of B_3 belonging to the boundary of the workspace are such that A_i, B_i, B_3 lie on the same line. For instance, consider leg 1: two types of alignment are possible. Either $A_1B_1B_3$ or $A_1B_3B_1$ (or $B_3A_1B_1$) are aligned in this order. Consequently, point B_3 lies on a circle C_{B_3} centered at A_i. As B_3 moves on C_{B_3}, points B_1, B_2 will move on circles denoted C_{B_1}, C_{B_2}. Valid positions of B_3 on its circle are such that the corresponding positions of B_1, B_2, B_3 respectively belong to the annular regions $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$.

Let α denote the rotation angle of leg 1 around A_1. The intersection points of circle C_{B_i} with the annular region $\mathcal{E}_i$ are then computed. All the intersection points define specific values for the angle α and the orientation of the platform. These values are ordered in a list leading to a set of intervals I^i. It is then possible to determine which intervals are components of the boundary of the workspace by taking the middle point of the arc and verifying if the corresponding pose of the platform belongs to the workspace.

As mentioned previously various types of alignment with various extreme values of the leg length are possible and some of them will lead to a component of the boundary. The arcs which are obtained after studying these different cases are placed in an appropriate structure and will be denoted *phase 1 arcs*.

2.2.2 Boundary points with two extreme leg lengths

The case for which the reference point lies on the boundary of the workspace while two leg lengths of the manipulator are in an extreme extension is now investigated. Since the reference point is point B_3, only the cases where the legs with extreme lengths are legs 1 and 2 need to be considered.

When legs 1 and 2 have a fixed length, the trajectory of point B_3 is the coupler curve of a four-bar mechanism. This mechanism has been well studied [4] and it is well known that the coupler curve is a sextic. Consequently it can be deduced that the boundary of the maximal workspace will be constituted of circular arcs and of portions of sextics.

Four sextics will play an important role in this study. They are the coupler curves of the four-bar mechanisms with leg lengths corresponding to the various combinations of extreme lengths of legs 1 and 2, i.e., $(\rho^1_{max}, \rho^2_{max})$, $(\rho^1_{max}, \rho^2_{min})$, $(\rho^1_{min}, \rho^2_{min})$, $(\rho^1_{min}, \rho^2_{max})$.

Some particular points, referred to as the *critical points* will determine the circular arcs and the portions of sextic which define the boundary of the maximal workspace. The critical points can be of five different types, thereby defining five sets of such points.

The first set consists of the intersection points of the sextics and the annular region $\mathcal{E}_3$: in this case the three leg lengths are at an extreme value. Therefore, these points are solutions of the *direct kinematics* and can be found numerically. The second set of critical points consists of the intersection points of the sextics with the phase 1 arcs. In this case the length of legs 1 and 2 are defined by the sextics and the length of leg 3 is the radius of the arc. These points will also be critical points for the arcs. A third set of critical points are the multiple points of the sextics. Indeed we may have only one critical point on a circuit on the sextic: therefore introducing the multiple points as critical points enables to define two arcs of sextic on the circuit, one of them being a member of the boundary of the workspace. Finding these multiple points is a well known problem [4]. The fourth set of critical points for the sextics will be the limit points of the coupler curve. Indeed, for some value of the leg lengths the four-bar mechanism may not be a crank, i.e., the angle ϕ is restricted to belong to some intervals. Consequently the sextic is not continuous and each position of B_3 corresponding to one of the bounds of the intervals is a critical point. The last set of critical points for the sextics consists of the set of intersection points between the sextics. Recently Innocenti has proposed an algorithm to solve this problem [6].

2.2.3 Determination of the portions of sextic belonging to the boundary

Any portion of sextic belonging to the boundary must lie between two critical points. For each critical point T_i the unique pair of angles ϕ_i, ψ_i corresponding to T_i is determined (note that for the critical points which are multiple points of the coupler curve although the coupler point is identical, the angles ϕ_i, ψ_i are different). For a given value of ϕ, there are in general two possible solutions for ψ which are obtained by solving a second order equation in the tangent of the half-angle of ψ. Consequently ψ is determined using one of the two expressions of the tangent of the half-angle.

First, the T_i's are sorted according to the expression which is used for determining the corresponding angle ψ_i, thereby giving rise to two sets of T_i's. Each of these sets is then sorted according to an increasing value of the angle ϕ. Consequently, the sextics are split into *arcs* of sextics, some of which are components of the boundary of the maximal workspace.

A component of the boundary will be such that for any point on the arc a motion along one of the normals to the sextic will lead to a violation of the constraints while a motion along the other normal will lead to feasible values for the link lengths. Any other combination implies that the arc is not a component of the boundary of the maximal workspace. In order to perform this test, the inverse jacobian matrix for a point on the arc (for example the middle point i.e. the coupler point obtained for ψ as the middle value between the angles ψ of the extreme points of the arc) is computed as well as the unit normal vectors $\mathbf{n}_1, \mathbf{n}_2$ of the sextic at this point. Then the joint velocities are calculated for a cartesian velocity directed along $\mathbf{n}_1, \mathbf{n}_2$. The sign of the joint velocities obtained indicates whether or not the arc is a component of the boundary. A similar procedure is used to identify the circular arcs which are components of the boundary. To this end, the phase 1 arcs, for which the critical points — the intersection points with the sextics and the extreme points of the arcs — have been determined, are considered. Each of the arcs between two critical points is examined to determine if the arc is a component of the boundary by using the same test as for the arcs of sextic. The boundary of the maximal workspace is finally obtained as a list of circular arcs and portions of sextics. The maximal workspace of the manipulators described in the appendix are shown in figure 2.

The computation time of the boundary of the maximal workspace is heavily dependent on the result. On a SUN 4-60 workstation this time may vary from 1500 to 15000 ms. The most expensive part of the procedure is the calculation of the intersection of the sextics.

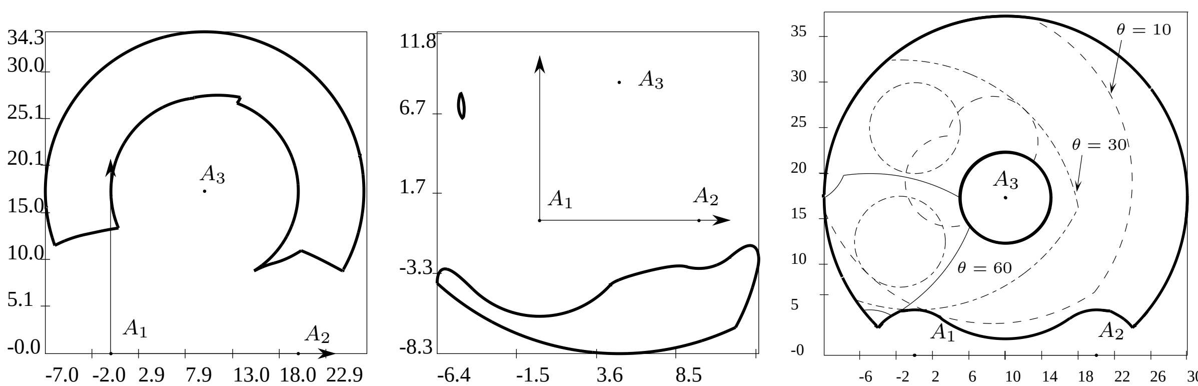

Figure 2: Left: the maximal workspace for manipulator 3 with $\rho_1 \in [8, 12]$, $\rho_2 \in [5, 15]$, $\rho_3 \in [10, 17]$. Middle: the maximal workspace for manipulator 4, $\rho_1 \in [8, 12]$, $\rho_2 \in [5, 15]$, $\rho_3 \in [10, 17]$. Right: the area within the thick lines is the maximal workspace of manipulator 3 with $\rho_1 \in [5, 20]$, $\rho_2 \in [5, 20]$, $\rho_3 \in [5, 20]$. The dashed and thin lines represent the constant orientation workspace for various orientations of the platform.

2.2.4 Maximal workspace for any reference point

To compute the maximal workspace for a reference point different from B_3, a similar algorithm can be used. Basically, only the complexity of the algorithm will be increased. Indeed, not only the eight circles of type C_1, C_2 have to be considered but also the four circles centered at A_3 which correspond to the case where the length of link 3 has an extreme value. Similarly, the twelve sextics which can be obtained from all the possible values for the extreme lengths of links 1, 2, 3 must now be considered.

3 Inclusive maximal workspace

The *inclusive maximal workspace* (denoted IMW) is defined as the set of all the positions which can be reached by the reference point with at least one orientation of the platform in a given interval referred to as the *orientation interval*. Hence, the maximal workspace is simply a particular case of IMW for which the prescribed orientation interval is $[0, 2\pi]$. In what follows, it is assumed that the orientation of the moving platform is defined by the angle between the x axis and the line B_3B_1. Moreover, it is also assumed that the reference point of the moving platform is B_3.

The computation of the boundary of the IMW is similar to the computation of the boundary of the maximal workspace. First, it is recalled that it is simple to determine if a point belongs to the IMW since one can compute the possible orientations of the moving platform at this point. It is also clear that a point lies on the boundary if and only if at least one of the link lengths is at an extreme value.

Consider first the circles described by B_3 when points A_i, B_i, B_3 lie on the same line. For each position of B_3 on the circles, the orientation of the moving platform is uniquely defined. The valid circular arcs must satisfy the following constraints: the points B_1, B_2, B_3 lie inside the annular regions $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and the orientation of the moving platform belongs to the orientation interval.

The determination of these arcs is thus similar to obtaining the arcs when computing the maximal workspace boundary. The main difference is that building the I^i intervals involves the consideration of the rotation angle α such that the orientation of the moving platform corresponds to one of the limits of the orientation interval.

Similarly, when the sextics are considered, the positions of B_3 for which the orientation of the moving platform is at one of the limits of the orientation interval will be added in the set of critical points.

To verify if a particular arc is a component of the boundary, the orientation for a point taken at random on the arc is examined to determine if it belongs to the orientation interval. Then the test using the inverse jacobian matrix allows to determine if the arc is a component of the boundary.

Typically the computation time for an IMW is about 1000 to 20000 ms on a SUN 4-60 workstation. Figure 3 presents some IMW for various orientation intervals.

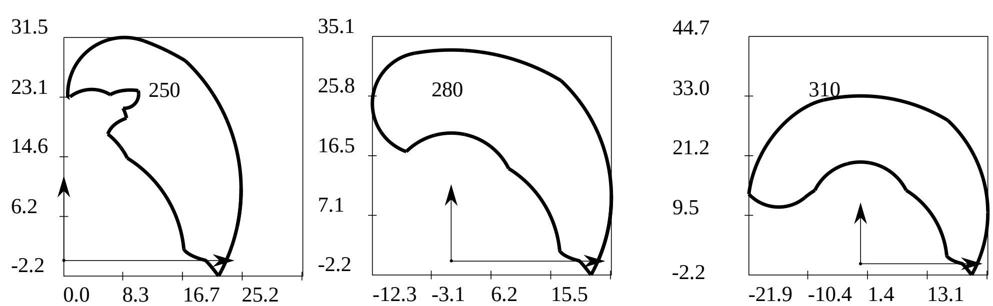

Figure 3: IMW of manipulator 1 for various orientation intervals (the orientation intervals always begin at 0). The limits are $\rho_1 \in [2, 8]$, $\rho_2 \in [5, 25]$, $\rho_3 \in [10, 25]$.

4 Total orientation workspace

This section addresses the problem of determining the region reachable by the reference point with all the orientations in a given set $[\theta_i, \theta_j]$ which will be referred to as the orientation interval. This workspace will be denoted as $\mathtt{TOW}$.

It is relatively easy to determine if a point belongs to this workspace since it is possible to compute the possible orientations for any position of the reference point. For a point belonging to the boundary of the $\mathtt{TOW}$, one leg will be at an extreme value. Indeed, two legs cannot be at an extreme value since in this case the orientation of the moving platform is unique and consequently the point cannot belong to the $\mathtt{TOW}$.

Assume that for a point on the boundary the orientation of the moving platform is one of the bounds of the interval, i.e., θ_i or θ_j while the length of leg i is at an extreme value. As B_i moves on the circle of the annular region $\mathcal{E}_i$ corresponding to the value of the leg length, point B_3 moves on a circle C_w^i with the same radius whose center is obtained by translating the center of $\mathcal{E}_i$ by the vector $\mathbf{B_i B_3}$, which is fixed since the orientation of the moving platform is known. Any point in the $\mathtt{TOW}$ must lie within the circle C_w^i. Therefore if the bounds θ_i, θ_j and all the possible B_i's are considered, any point of the $\mathtt{TOW}$ must be inside the 12 circles with center and radii (A_3,ρ^3_{max}),(A_3, ρ^3_{min}), ($A_1 + \mathbf{B_1 B_3}, \rho^1_{max}$),($A_1 + \mathbf{B_1 B_3}, \rho^1_{min}$), ($A_2 + \mathbf{B_2 B_3}, \rho^2_{max}$), ($A_2 + \mathbf{B_2 B_3}, \rho^2_{min}$).

Assume now that a point on the boundary is reached with an orientation different from θ_i, θ_j and that the length of link 1 has an extreme value, say ρ^1_{max}. When the orientation of the moving platform lies in the orientation interval, B_1 belongs to a circular arc defined by its center B_3, its radius $||\mathbf{B_3 B_1}||$ and the angles θ_i, θ_j. As the point belongs to the $\mathtt{TOW}$ the arc must lie inside the annular region $\mathcal{E}_1$. Furthermore this arc is tangent at some point to the external circle of $\mathcal{E}_1$ since B_3 lies on the boundary of the $\mathtt{TOW}$. This tangency implies that point B_3 lies on a circle of center A_1 and radius $\rho^1_{max} - ||\mathbf{B_1 B_3}||$. Any point within the $\mathtt{TOW}$ must be inside this circle. Four such circles may exist, whose center and radii are ($A_1, \rho^1_{max} - ||\mathbf{B_1 B_3}||$), ($A_1, \rho^1_{min} - ||\mathbf{B_1 B_3}||$), ($A_2, \rho^2_{max} - ||\mathbf{B_2 B_3}||$), ($A_2, \rho^2_{min} - ||\mathbf{B_2 B_3}||$).

If a point B_3 belongs to the $\mathtt{TOW}$ it is necessary that the point is included in the 16 circles which have been determined. Consequently the boundary of the $\mathtt{TOW}$ is the intersection of these sixteen circles. Note that a particular case of $\mathtt{TOW}$ is the *dextrous workspace*, which is the region which can be reached by the reference point with any orientation [8],[12],[11].

5 Conclusion

Geometrical algorithms for the determination of the boundary of various workspaces for planar parallel manipulators have been described. Basically the presented algorithms can be extended without any difficulties to other types of planar parallel robots [10]. The authors want to acknowledge that this work has been supported in part by the France-Canada collaboration contract n° 070191.

Appendix

The dimensions of the manipulators used in the examples of this paper are defined in figure 1 and their numerical values are presented in the following table.

Manipulator	Type	l_1	l_2	l_3	c_2	c_3	d_3	μ
1	RPR	25	25	25	20	0	10	60
2	RPR	20.839	17.045	16.54	15.91	0	10	52.74
3	RPR	25	25	25	20	10	17.32	60
4	RPR	2	2	2	10	5	8.66	60

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