Topological Hochschild homology and the Bass trace conjecture

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Abstract. We use the methods of topological Hochschild homology to shed new light on groups satisfying the Bass trace conjecture. Factorization of the Hattori-Stallings rank map through the Bökstedt-Hsiang-Madsen cyclotomic trace map leads to Linnell's restriction on such groups. As a new consequence of this restriction, we show that the conjecture holds for any group G wherein every subgroup isomorphic to the additive group of rational numbers has nontrivial and central image in some quotient of G.

Introduction

Let A be a ring, let $K_0(A)$ be the projective class group, and let $\mathrm{HH}_0(A)$ be the zeroth Hochschild homology group defined by the quotient $A/[A,A]$ of A by the additive subgroup $[A,A]$ generated by all elements of the form $ab-ba$ with $a,b \in A$. The Hattori-Stallings rank [16, 31], also known as the Hattori-Stallings trace, is the map

$$r\colon K_0(A) \longrightarrow \mathrm{HH}_0(A)$$

that takes the class of the finitely generated projective right A-module P to the image $r(P)$ of the identity endomorphism of P by the trace homomorphism

$$\mathrm{Hom}_A(P,P) \xleftarrow{\sim} P \otimes_A \mathrm{Hom}_A(P,A) \xrightarrow{\mathrm{ev}} A/[A,A].$$

Let G be a discrete group, and let $\mathbb{Z}[G]$ be the integral group ring of G. In this case, the group $\mathrm{HH}_0(\mathbb{Z}[G])$ is canonically isomorphic to the free abelian group generated by the set $C(G)$ of conjugacy classes of elements in G. Hence, we can write $r(P)$ uniquely as

$$r(P) = \sum_{[g]\in C(G)} r(P)(g)[g]$$

with $r(P)(g) \in \mathbb{Z}$. It was conjectured by Bass [2, Strong Conjecture] that $r(P)(g)$ is always zero, unless $g=1$, and it is this conjecture that we refer to as the Bass trace conjecture.

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In this paper, we consider the following factorization of the Hattori-Stallings rank through the cyclotomic trace map of Bökstedt-Hsiang-Madsen [6].

$$K_0(A) \xrightarrow{\mathrm{tr}} \mathrm{TR}_0(A) \longrightarrow \mathrm{HH}_0(A)$$

The middle group is equipped with operators F_s called the Frobenius operators, one for every positive integer s, and the image of the cyclotomic trace map is contained in the subgroup fixed by all Frobenius operators. In the case $A = \mathbb{Z}[G]$, we show that this leads to the following restriction on the elements $g \in G$ for which the Hattori-Stallings rank $r(P)(g)$ can be nonzero.

Theorem A. *Let G be a group, let $g \in G$ and suppose that $r(P)(g) \neq 0$, for some finitely generated projective $\mathbb{Z}[G]$-module P. Then there exists a positive integer $m = m(g)$ such that for all positive integers s, the elements g and g^{s^m} are conjugate.*

This restriction is not new, but was found earlier by P. A. Linnell [21, Lemma 4.1] by very different methods, using work of G. H. Cliff [7] on matrices over group rings. Strictly speaking, the statement we prove is stronger than Linnell's statement, in that in Theorem A, as in Emmanouil [11, Theorem 3.32], we have $m = m(g)$ (independently of s) whereas the original proofs presented in [21, Lemma 4.1] and [32] yield only that $m = m(g,s)$. A well-known consequence of the restriction is that if a nontrivial element g has $r(P)(g) \neq 0$, then g lies in a subgroup isomorphic to the additive group of rationals $\mathbb{Q}$. We present the following stronger, new consequence.

Theorem B. *Let G be a group such that every subgroup of G that is isomorphic to $\mathbb{Q}$ has nontrivial and central image in some quotient of G. Then the Bass trace conjecture holds for the group G.*

In particular, the Bass trace conjecture holds for any group such that the intersection of the transfinite lower central series contains no copy of $\mathbb{Q}$. This is proved in Section 3 below, where other consequences are explored, including the relation between the strong and weak forms of the Bass trace conjecture; the weak form asserts that $\sum_{[g] \in C(G) \smallsetminus \{[1]\}} r(P)(g) = 0$.

Below, we use simplicial and cyclic sets and their geometric realization. We use the notation $X[-]$ to indicate a simplicial or cyclic set with set of n-simplices $X[n]$ and refer the reader to [22, Chapter 7] and [8] for the properties of its geometric realization $|X[-]|$.

1. The cyclotomic trace map

In this section, we recall the cyclotomic trace map and give a thorough treatment of the factorization of the Hattori-Stallings rank through said map. We begin with a discussion of the groups $\mathrm{TR}^r_q(A)$ and of the operators that relate these groups as the integers $r \geqslant 1$ and q vary. The reader is referred to [18, Section 1] for further details. The definition of these groups given in loc. cit. uses very large constructions. The advantage of this is that the various maps that we need all have good point-set level models.

Let $\mathscr{C}$ be a (small) exact category in the sense of Quillen [27, §2], and let $K(\mathscr{C})$ be the algebraic K-theory spectrum as defined by Waldhausen [33, Section 1.3], the definition of which depends on a choice of null-object 0 in $\mathscr{C}$. As pointed out in [13, Section 6], the spectrum $K(\mathscr{C})$ is a symmetric spectrum in the sense of [20]. The algebraic K-groups of $\mathscr{C}$ are defined to be the homotopy groups

$$K_q(\mathscr{C}) = [S^q, K(\mathscr{C})]$$

given by the set of maps in the homotopy category of symmetric spectra from the suspension spectrum of the sphere S^q (which, by abuse of notation, we again denote by S^q) to the algebraic K-theory spectrum $K(\mathscr{C})$. For later use, we notice that the zeroth space of the symmetric spectrum $K(\mathscr{C})$ is defined to be the geometric realization

$$K(\mathscr{C})_0 = |N(i\mathscr{C})[-]|$$

of the nerve of the subcategory $i\mathscr{C}$ of isomorphisms in $\mathscr{C}$. There is a canonical map

$$\sigma\colon \pi_q(K(\mathscr{C})_0) \longrightarrow K_q(\mathscr{C})$$

from the homotopy groups of the zeroth space with the chosen null-object 0 as the basepoint to the homotopy groups of the K-theory spectrum. It is not an isomorphism, except in trivial cases, but if $\mathscr{C}$ is split-exact, then it is a group-completion for $q = 0$.

We let $\mathbb{T}$ be the circle group given by the multiplicative group of complex numbers of modulus 1. We use the model of the topological Hochschild $\mathbb{T}$-spectrum $T(\mathscr{C})$ which is given in [18, Section 1.2] and which produces a symmetric orthogonal $\mathbb{T}$-spectrum in the sense of [19, 23]; it is based on the Dundas-McCarthy model [9, Section 2.1.7] generalizing Bökstedt's original construction [5] from rings to exact categories. The symmetric spectrum structure is used to construct the cyclotomic trace map, and the orthogonal spectrum structure is used to define the maps R_s, F_s, and V_s which we now explain. First, for every positive integer r, the group $\mathrm{TR}^r_q(\mathscr{C})$ is defined to be the equivariant homotopy group

$$\mathrm{TR}^r_q(\mathscr{C}) = [S^q \wedge (\mathbb{T}/C_r)_+, T(\mathscr{C})]_{\mathbb{T}}$$

given by the set of maps in the homotopy category of symmetric orthogonal $\mathbb{T}$-spectra from the suspension $\mathbb{T}$-spectrum of the pointed $\mathbb{T}$-space $S^q \wedge (\mathbb{T}/C_r)_+$ to $T(\mathscr{C})$. Here $C_r \subseteq \mathbb{T}$ denotes the subgroup of order r and the subscript "+" indicates the addition of a disjoint basepoint. If s divides r, then there are natural additive maps

$$\begin{aligned} R_s\colon\ & \mathrm{TR}^r_q(\mathscr{C}) \longrightarrow \mathrm{TR}^{r/s}_q(\mathscr{C}) && \text{(restriction)} \\ F_s\colon\ & \mathrm{TR}^r_q(\mathscr{C}) \longrightarrow \mathrm{TR}^{r/s}_q(\mathscr{C}) && \text{(Frobenius)} \\ V_s\colon\ & \mathrm{TR}^{r/s}_q(\mathscr{C}) \longrightarrow \mathrm{TR}^r_q(\mathscr{C}) && \text{(Verschiebung).} \end{aligned}$$

The map F_s is induced by the canonical projection $f_s\colon \mathbb{T}/C_{r/s} \to \mathbb{T}/C_r$, and the map V_s is induced by the associated transfer map $v_s\colon S^q \wedge (\mathbb{T}/C_r)_+ \to S^q \wedge (\mathbb{T}/C_{r/s})_+$. The definition of the map R_s is more subtle and uses the so-called cyclotomic structure of $T(\mathscr{C})$. We briefly recall the definition and refer to [17, Section 2] for details. In general, if T is a symmetric orthogonal $\mathbb{T}$-spectrum, then the geometric C_s-fixed point spectrum $\Phi^{C_s}T$ is a symmetric orthogonal $\mathbb{T}/C_s$-spectrum. In the special case of a suspension $\mathbb{T}$-spectrum $S^q \wedge X$, there is a canonical weak equivalence of symmetric orthogonal $\mathbb{T}/C_s$-spectra

$$i_s\colon S^q \wedge (X^{C_s}) \xrightarrow{\sim} \Phi^{C_s}(S^q \wedge X).$$

The group isomorphism $\rho_s\colon \mathbb{T} \to \mathbb{T}/C_s$ defined by the sth root gives rise to a change-of-groups functor ρ_s^* from the category of symmetric orthogonal $\mathbb{T}/C_s$-spectra to the category of symmetric orthogonal $\mathbb{T}$-spectra. In particular, we obtain the symmetric orthogonal $\mathbb{T}$-spectrum $\rho_s^*\Phi^{C_s}T$. Now, the cyclotomic structure of the topological Hochschild $\mathbb{T}$-spectrum $T(\mathscr{C})$ is a collection of compatible weak equivalences

$$r_s\colon \rho_s^*\Phi^{C_s}T(\mathscr{C}) \longrightarrow T(\mathscr{C})$$

of symmetric orthogonal $\mathbb{T}$-spectra, one for every positive integer s, and the restriction map

$$R_s\colon \operatorname{TR}_q^r(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r/s}(\mathscr{C})$$

is defined to be the composition

$$\begin{aligned}&[S^q\wedge(\mathbb{T}/C_r)_+, T(\mathscr{C})]_{\mathbb{T}} \xrightarrow{\Phi^{C_s}} [\Phi^{C_s}(S^q\wedge(\mathbb{T}/C_r)_+), \Phi^{C_s}T(\mathscr{C})]_{\mathbb{T}/C_s}\\ &\xrightarrow{\rho_s^*} [\rho_s^*\Phi^{C_s}(S^q\wedge(\mathbb{T}/C_r)_+), \rho_s^*\Phi^{C_s}T(\mathscr{C})]_{\mathbb{T}} \xrightarrow{r_{s*}} [S^q\wedge(\mathbb{T}/C_{r/s})_+, T(\mathscr{C})]_{\mathbb{T}}\end{aligned}$$

of the geometric C_s-fixed point functor, the change-of-groups isomorphism ρ_s^*, and the isomorphism induced by the weak equivalences $\rho_s^* i_s$ and r_s. We record the following basic properties of the maps R_s, F_s, and V_s.

Lemma 1.1. *Let r, s, and t be positive integers, and let $d = (s,t)$ be the greatest common divisor. The following relations (i)–(iv) hold.*

(i) *The maps $R_1, F_1, V_1\colon \operatorname{TR}_q^r(\mathscr{C}) \to \operatorname{TR}_q^r(\mathscr{C})$ are the identity maps.*

(ii) *If st divides r, then*

$$\begin{aligned}R_s\circ R_t = R_{st}&\colon \operatorname{TR}_q^r(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r/st}(\mathscr{C})\\ F_s\circ F_t = F_{st}&\colon \operatorname{TR}_q^r(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r/st}(\mathscr{C})\\ V_s\circ V_t = V_{st}&\colon \operatorname{TR}_q^{r/st}(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r}(\mathscr{C})\\ R_s\circ F_t = F_t\circ R_s&\colon \operatorname{TR}_q^r(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r/st}(\mathscr{C})\\ R_s\circ V_t = V_t\circ R_s&\colon \operatorname{TR}_q^{r/t}(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r/s}(\mathscr{C})\end{aligned}$$

(iii) *If both s and t divide r, then*

$$F_s\circ V_t = dV_{t/d}\circ F_{s/d}\colon \operatorname{TR}_q^{r/t}(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r/s}(\mathscr{C}).$$

(iv) *If both s and t but not st divide r, then*

$$R_s\circ V_t = 0\colon \operatorname{TR}_q^{r/t}(\mathscr{C}) \longrightarrow \operatorname{TR}_q^{r/s}(\mathscr{C}).$$

Proof. The statements (i)–(iii) are immediate consequences of the definitions and of [17, Proposition 3.2]; compare loc. cit., Lemma 3.3. To prove (iv), we consider the diagram

$$\begin{array}{ccc}[S^q\wedge(\mathbb{T}/C_{r/t})_+, T(\mathscr{C})]_{\mathbb{T}} & \xrightarrow{\Phi^{C_s}} & [\Phi^{C_s}(S^q\wedge(\mathbb{T}/C_{r/t})_+), \Phi^{C_s}T(\mathscr{C})]_{\mathbb{T}/C_s}\\ \big\downarrow{\scriptstyle V_t = v_t^*} & & \big\downarrow{\scriptstyle (\Phi^{C_s}v_t)^*}\\ [S^q\wedge(\mathbb{T}/C_r)_+, T(\mathscr{C})]_{\mathbb{T}} & \xrightarrow{\Phi^{C_s}} & [\Phi^{C_s}(S^q\wedge(\mathbb{T}/C_r)_+), \Phi^{C_s}T(\mathscr{C})]_{\mathbb{T}/C_s}\end{array}$$

which commutes since Φ^{C_s} is a functor. Moreover, since s was assumed not to divide r/t, the fixed set $(\mathbb{T}/C_{r/t})^{C_s}$ is empty. It follows that the upper right-hand term in the diagram is zero, which proves (iv). □

For completeness, we note that $\mathrm{TR}^r_q(\mathscr{C})$ is a module over the ring $\mathbb{W}_{\langle r\rangle}(\mathbb{Z})$ of big Witt vectors in $\mathbb{Z}$ indexed on the set $\langle r\rangle$ of divisors in r, and that if s divides r, then

$$V_s \circ F_s\colon\ \mathrm{TR}^r_q(\mathscr{C}) \longrightarrow \mathrm{TR}^r_q(\mathscr{C})$$

is equal to multiplication by the element $V_s(1) \in \mathbb{W}_{\langle r\rangle}(\mathbb{Z})$. This fact, however, is not used in this paper. On the other hand, we need the following result.

Proposition 1.2. *Let r be a positive integer, let p be a prime number dividing r, and write $r = p^u d$ with d not divisible by p. In this situation, the sequence*

$$\mathrm{TR}^d_0(\mathscr{C}) \xrightarrow{V_{p^u}} \mathrm{TR}^r_0(\mathscr{C}) \xrightarrow{R_p} \mathrm{TR}^{r/p}_0(\mathscr{C}) \longrightarrow 0$$

is exact.

Proof. As a particular case of [1, Proposition 1.1] with $q = 0$, $v = 1$, and $\lambda = 0$, we have the exact sequence

$$\mathbb{H}_0(C_{p^u}, \mathrm{TR}^d(\mathscr{C})) \xrightarrow{N_{p^u}} \mathrm{TR}^r_0(\mathscr{C}) \xrightarrow{R_p} \mathrm{TR}^{r/p}_0(\mathscr{C}) \longrightarrow 0.$$

The left-hand term is the 0th Borel homology group of C_{p^u} with coefficients in the symmetric orthogonal $\mathbb{T}$-spectrum $\mathrm{TR}^d(\mathscr{C}) = \rho_d^*(T(\mathscr{C})^{C_d})$ and is defined by

$$\mathbb{H}_q(C_{p^u}, \mathrm{TR}^d(\mathscr{C})) = [S^q \wedge (\mathbb{T}/C_{p^u})_+, E_+ \wedge \rho_d^*(T(\mathscr{C})^{C_d})]_{\mathbb{T}},$$

where E is a free $\mathbb{T}$-CW-complex whose underlying space is contractible. If E' is another such $\mathbb{T}$-CW-complex, then there is a $\mathbb{T}$-homotopy equivalence $f\colon E \to E'$, the $\mathbb{T}$-homotopy class of which is unique, and hence, the Borel homology group is well-defined up to canonical isomorphism. The skeleton filtration of E gives rise to a first quadrant spectral sequence

$$E^2_{s,t} = H_s(C_{p^u}, \mathrm{TR}^d_t(\mathscr{C})) \Rightarrow \mathbb{H}_{s+t}(C_{p^u}, \mathrm{TR}^d(\mathscr{C}))$$

from the group homology of C_{p^u} with coefficients in the trivial C_{p^u}-module $\mathrm{TR}^d_t(\mathscr{C})$. In particular, the edge homomorphism defines an isomorphism

$$\mathrm{TR}^d_0(\mathscr{C}) = H_0(C_{p^u}, \mathrm{TR}^d_0(\mathscr{C})) \xrightarrow{\sim} \mathbb{H}_0(C_{p^u}, \mathrm{TR}^d(\mathscr{C})).$$

Finally, as was noted in [17, Lemma 3.2], the composition of the edge homomorphism and the left-hand map N_{p^u} in the exact sequence at the top of the proof is equal to the Verschiebung map V_{p^u}. This completes the proof. □

We next recall the cyclotomic trace map of Bökstedt-Hsiang-Madsen [6], but use the technically better construction given by Dundas-McCarthy [9]. The $\mathbb{T}$-fixed point spectrum $T(\mathscr{C})^{\mathbb{T}}$ is a symmetric orthogonal spectrum. We further replace the symmetric spectrum $K(\mathscr{C})$

by its suspension symmetric orthogonal spectrum which we again write $K(\mathscr{C})$; compare [19]. There is a natural map of symmetric orthogonal spectra

$$\tau\colon K(\mathscr{C}) \longrightarrow T(\mathscr{C})^{\mathbb{T}}$$

given by the canonical map defined in [18, p. 14, middle]. Now the cyclotomic trace map is defined to be the natural additive map

$$\mathrm{tr}_r\colon K_q(\mathscr{C}) \longrightarrow \mathrm{TR}^r_q(\mathscr{C})$$

given by the composition

$$[S^q, K(\mathscr{C})] \xrightarrow{\tau_*} [S^q, T(\mathscr{C})^{\mathbb{T}}] \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/\mathbb{T})_+,\, T(\mathscr{C})]_{\mathbb{T}} \xrightarrow{p_r^*} [S^q \wedge (\mathbb{T}/C_r)_+,\, T(\mathscr{C})]_{\mathbb{T}}$$

of the map induced by the map τ, the canonical isomorphism, and the map induced by the canonical projection $p_r\colon \mathbb{T}/C_r \to \mathbb{T}/\mathbb{T}$. The next result is essential for our purposes here.

Lemma 1.3. *If r and s are positive integers with s dividing r, then*

$$F_s \circ \mathrm{tr}_r = \mathrm{tr}_{r/s} = R_s \circ \mathrm{tr}_r\colon K_q(\mathscr{C}) \longrightarrow \mathrm{TR}^{r/s}_q(\mathscr{C}).$$

Proof. The left-hand equality follows immediately from the definitions since

$$p_r \circ f_s = p_{r/s}\colon \mathbb{T}/C_{r/s} \longrightarrow \mathbb{T}/\mathbb{T}.$$

Similarly, the right-hand equality follows from the definition of the restriction map, which we recalled above, since the diagram

$$\begin{array}{ccc} S^q \wedge (\mathbb{T}/C_{r/s})_+ & \xrightarrow{\quad p_{r/s} \quad} & S^q \wedge (\mathbb{T}/\mathbb{T})_+ \\ \big\downarrow{\scriptstyle i_s} & & \big\downarrow{\scriptstyle i_s} \\ \rho_s^*\Phi^{C_s}(S^q \wedge (\mathbb{T}/C_r)_+) & \xrightarrow{\rho_s^*\Phi^{C_s} p_r} & \rho_s^*\Phi^{C_s}(S^q \wedge (\mathbb{T}/\mathbb{T})_+) \end{array}$$

commutes. □

We next recall the definition of the $(0,0)$ th space of $T(\mathscr{C})$, which we use below. The cyclic nerve of the exact category $\mathscr{C}$ or, more generally, of any (small) category $\mathscr{C}$ enriched in pointed sets is defined to be the cyclic pointed set $N^{\mathrm{cy}}(\mathscr{C})[-]$ with n-simplices

$$N^{\mathrm{cy}}(\mathscr{C})[n] = \bigvee_{(P_0,\dots,P_n)} \mathrm{Hom}_{\mathscr{C}}(P_1,P_0) \wedge \mathrm{Hom}_{\mathscr{C}}(P_2,P_1) \wedge \dots \wedge \mathrm{Hom}_{\mathscr{C}}(P_0,P_n),$$

the wedge sum ranging over $(n+1)$-tuples of objects of $\mathscr{C}$, and with cyclic structure maps

$$\begin{aligned} d_i(f_0 \wedge \dots \wedge f_n) &= \begin{cases} f_0 \wedge \dots \wedge f_i \circ f_{i+1} \wedge \dots \wedge f_n & \text{if } 0 \leqslant i < n \\ f_n \circ f_0 \wedge \dots \wedge f_{n-1} & \text{if } i = n \end{cases} \\ s_i(f_0 \wedge \dots \wedge f_n) &= \begin{cases} f_0 \wedge \dots \wedge f_i \wedge \mathrm{id}_{P_{i+1}} \wedge f_{i+1} \wedge \dots \wedge f_n & \text{if } 0 \leqslant i < n \\ f_0 \wedge \dots \wedge f_n \wedge \mathrm{id}_{P_0} & \text{if } i = n \end{cases} \\ t_n(f_0 \wedge \dots \wedge f_n) &= f_n \wedge f_0 \wedge f_1 \wedge \dots \wedge f_{n-1}. \end{aligned}$$

We also define $\mathscr{N}(\mathscr{C})[k]$ be the category of functors from the category $[k]$ generated by the graph $0 \leftarrow 1 \leftarrow \cdots \leftarrow k$ and define $\mathscr{N}^i(\mathscr{C})[k]$ to be the *full* subcategory whose objects are all functors $P\colon [k] \to \mathscr{C}$ that take morphisms in $[k]$ to isomorphisms in $\mathscr{C}$. In particular,

$$\mathrm{ob}(\mathscr{N}^i(\mathscr{C})[k]) = N(i\mathscr{C})[k],$$

but morphisms in $\mathscr{N}^i(\mathscr{C})[k]$ need not be isomorphisms. The cyclic nerve of $\mathscr{N}^i(\mathscr{C})[k]$ is a cyclic pointed set, covariantly functorial in $[k]$. Hence, the collection of these cyclic pointed sets form a simplicial cyclic pointed set $N^{\mathrm{cy}}(\mathscr{N}^i(\mathscr{C})[-])[-]$, and

$$T(\mathscr{C})_{(0,0)} = |N^{\mathrm{cy}}(\mathscr{N}^i(\mathscr{C})[-])[-]|$$

is defined to be the geometric realization. Again, there is a canonical map

$$\sigma_r\colon \pi_q((T(\mathscr{C})_{(0,0)})^{C_r}) \longrightarrow \mathrm{TR}^r_q(\mathscr{C})$$

from the equivariant homotopy groups of the $(0,0)$ th space to the equivariant homotopy groups of the symmetric orthogonal $\mathbb{T}$-spectrum $T(\mathscr{C})$, and this map, too, is typically not a bijection. We note that, while the definition of the domain of this map uses only the underlying category enriched in pointed sets of the exact category $\mathscr{C}$, the definition of the target uses the exact category structure. We also note that, on $(0,0)$ th spaces, the map tr_1 is induced by the map

$$\mathrm{tr}_1\colon \mathrm{ob}(\mathscr{N}^i(\mathscr{C})[-]) \longrightarrow N^{\mathrm{cy}}(\mathscr{N}^i(\mathscr{C})[-])[0]$$

that to an object P associates its identity morphism id_P.

We recall that McCarthy [24] has defined the Hochschild homology of a (small) exact category $\mathscr{C}$. Proceeding as in [18, Section 1.2], this construction gives rise to a symmetric orthogonal $\mathbb{T}$-spectrum $\mathrm{HH}(\mathscr{C})$, whose equivariant homotopy groups we write

$$\mathrm{HH}^r_q(\mathscr{C}) = [S^q \wedge (\mathbb{T}/C_r)_+, \mathrm{HH}(\mathscr{C})]_{\mathbb{T}}.$$

There is no cyclotomic structure on $\mathrm{HH}(\mathscr{C})$, however, and the groups $\mathrm{HH}^r_q(\mathscr{C})$ with $r > 1$ appear to be of little use. Hence, we consider only the groups $\mathrm{HH}_q(\mathscr{C}) = \mathrm{HH}^1_q(\mathscr{C})$ which are McCarthy's Hochschild homology groups of $\mathscr{C}$. To define the $(0,0)$ th space, we recall that the additive cyclic nerve $N^{\mathrm{cy}}_{\oplus}(\mathscr{C})[-]$ is the cyclic abelian groups with n-simplices

$$N^{\mathrm{cy}}_{\oplus}(\mathscr{C})[n] = \bigoplus_{(P_0,\dots,P_n)} \mathrm{Hom}_{\mathscr{C}}(P_1,P_0) \otimes \mathrm{Hom}_{\mathscr{C}}(P_2,P_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{C}}(P_0,P_n)$$

and with the cyclic structure maps defined as for the cyclic nerve, replacing smash products by tensor products. Now the $(0,0)$ th space of $\mathrm{HH}(\mathscr{C})$ is defined to be the geometric realization

$$\mathrm{HH}(\mathscr{C})_{(0,0)} = |N^{\mathrm{cy}}_{\oplus}(\mathscr{N}^i(\mathscr{C})[-])[-]|$$

of the indicated simplicial cyclic abelian group. There is a canonical map

$$\sigma_1\colon \pi_q(\mathrm{HH}(\mathscr{C})_{(0,0)}) \longrightarrow \mathrm{HH}_q(\mathscr{C})$$

from the (equivariant) homotopy groups of the $(0,0)$ th space to the (equivariant) homotopy groups of the symmetric orthogonal $\mathbb{T}$-spectrum $\mathrm{HH}(\mathscr{C})$, but in contrast to $K(\mathscr{C})$ and $T(\mathscr{C})$, it follows from [24, Corollary 3.3.4] that this map is an isomorphism, if $\mathscr{C}$ is split-exact.

Finally, there is a canonical map of symmetric orthogonal $\mathbb{T}$-spectra

$$\ell \colon T(\mathscr{C}) \longrightarrow \mathrm{HH}(\mathscr{C})$$

called the linearization map that on $(0,0)$ th spaces is induced by the map

$$\ell \colon N^{\mathrm{cy}}(\mathscr{N}^i(\mathscr{C})[-])[-] \longrightarrow N^{\mathrm{cy}}_{\oplus}(\mathscr{N}^i(\mathscr{C})[-])[-]$$

of simplicial cyclic pointed sets that to $f_0 \wedge \cdots \wedge f_n$ associates $f_0 \otimes \cdots \otimes f_n$ and replaces wedge sums by direct sums; compare [10, Definition IV.1.3.5]. To summarize the situation, we have the following commutative diagram, which we need only in the case $q = 0$.

$$\begin{array}{ccccc} \pi_q(K(\mathscr{C})_0) & \xrightarrow{\mathrm{tr}_1} & \pi_q(T(\mathscr{C})_{(0,0)}) & \xrightarrow{\ell} & \pi_q(\mathrm{HH}(\mathscr{C})_{(0,0)}) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_1} \\ K_q(\mathscr{C}) & \xrightarrow{\mathrm{tr}_1} & \mathrm{TR}^1_q(\mathscr{C}) & \xrightarrow{\ell} & \mathrm{HH}_q(\mathscr{C}) \end{array} \tag{1.4}$$

The lower right-hand horizontal map is an isomorphism, for $q \leqslant 1$; see loc. cit. Moreover, if the exact category $\mathscr{C}$ is split-exact, then the right-hand vertical map is an isomorphism, for all $q \geqslant 0$, and the left-hand vertical map is a group-completion, for $q = 0$.

We now specialize to the case where $\mathscr{C} = \mathscr{P}(A)$ is the (split-)exact category of finitely generated projective right modules over a unital associative ring A. We define

$$K_q(A) = K_q(\mathscr{P}(A)), \qquad \mathrm{TR}^r_q(A) = \mathrm{TR}^r_q(\mathscr{P}(A)), \qquad \mathrm{HH}_q(A) = \mathrm{HH}_q(\mathscr{P}(A)).$$

The map left-hand vertical map σ in (1.4) induces an isomorphism of the projective class group of A onto the group $K_0(A)$. Moreover, we have canonical isomorphisms

$$A/[A,A] \xrightarrow{j} \pi_0(\mathrm{HH}(\mathscr{P}(A))_{(0,0)}) \xrightarrow{\sigma_1} \mathrm{HH}_0(\mathscr{P}(A))$$

which justifies our writing $\mathrm{HH}_0(A)$ for the right-hand group. Here, the map j takes the class of $a \in A$ to the class of the map of right A-modules $l_a \colon A \to A$ defined by $l_a(b) = ab$, and its inverse takes the class of $f \in \mathrm{Hom}_A(P_0, P_0)$ to the trace $\mathrm{tr}(f) \in A/[A,A]$. We refer the reader to [24, Proposition 2.4.3] for further discussion.

We record the following result. Henceforth, we refer to the composite map in the statement as the Hattori-Stallings rank.

Lemma 1.5. *The following composite map agrees, under the above identification of its target group with $A/[A,A]$, with the Hattori-Stallings rank.*

$$K_0(A) \xrightarrow{\mathrm{tr}_1} \mathrm{TR}^1_0(A) \xrightarrow{\ell} \mathrm{HH}_0(A)$$

Proof. Comparing definitions, we see that it suffices to show that the composition of the top horizontal maps in (1.4) takes the class of P to the class of id_P. It does. □

It follows from Lemmas 1.1 and 1.3 that the maps tr_r define a map

$$\mathrm{tr} \colon K_0(A) \longrightarrow \mathrm{TR}_0(A) = \lim_r \mathrm{TR}^r_0(A)$$

to the inverse limit with respect to the restriction maps, the limit indexed by set of the positive integers ordered under division. Moreover, it follows from Lemma 1.1 that the collection of Frobenius maps $F_s\colon\ \mathrm{TR}_0^r(A) \to \mathrm{TR}_0^{r/s}(A)$ induces a Frobenius map

$$F_s\colon\ \mathrm{TR}_0(A) \longrightarrow \mathrm{TR}_0(A)$$

and that these maps, in turn, define an action of the multiplicative monoid of positive integers $\mathbb{N}$ on $\mathrm{TR}_0(A)$. We write $\mathrm{TR}_0(A)^{\mathbb{N}}$ for the subgroup of elements fixed by the $\mathbb{N}$-action. Finally, Lemma 1.3 shows that the image of the map tr is contained in $\mathrm{TR}_0(A)^{\mathbb{N}} \subseteq \mathrm{TR}_0(A)$, and Lemma 1.5 shows that the Hattori-Stallings rank is equal to the composite map

$$K_0(A) \xrightarrow{\ \mathrm{tr}\ } \mathrm{TR}_0(A)^{\mathbb{N}} \xrightarrow{\ i\ } \mathrm{TR}_0(A) \xrightarrow{\ \mathrm{pr}_1\ } \mathrm{TR}_0^1(A) \xrightarrow{\ \ell\ } \mathrm{HH}_0(A),$$

where i is the canonical inclusion. This factorization of the Hattori-Stallings rank map leads to restrictions on its image, as we next see.

The cyclotomic structure map $r_s\colon\ \rho_s^*\Phi^{C_s}T(\mathscr{C}) \to T(\mathscr{C})$, on the level of $(0,0)$ th spaces, is a $\mathbb{T}$-equivariant homeomorphism, the inverse of which is given by the composition of the homeomorphisms Δ_r and D_r defined in [6, Sections 1–2]. In particular, the middle left-hand horizontal map in the following diagram is a bijection. We use this fact in the case $A = \mathbb{Z}[G]$ and $\mathscr{C} = \mathscr{P}(A)$, where we consider the commutative diagram

(1.6)

$$\begin{array}{ccccc}
 & & C(G) & \xrightarrow{\quad i \quad} & A/[A,A] \\
 & & \big\downarrow{\scriptstyle j} & & \sim\big\downarrow{\scriptstyle j} \\
\pi_0((T(\mathscr{C})_{(0,0)})^{C_s}) & \underset{\sim}{\xrightarrow{\ r_s\ }} & \pi_0(T(\mathscr{C})_{(0,0)}) & \xrightarrow{\ \ell\ } & \pi_0(\mathrm{HH}(\mathscr{C})_{(0,0)}) \\
\big\downarrow{\scriptstyle \sigma_s} & & \big\downarrow{\scriptstyle \sigma_1} & & \sim\big\downarrow{\scriptstyle \sigma_1} \\
\mathrm{TR}_0^s(\mathscr{C}) & \xrightarrow{\quad R_s \quad} & \mathrm{TR}_0^1(\mathscr{C}) & \underset{\sim}{\xrightarrow{\quad \ell \quad}} & \mathrm{HH}_0(\mathscr{C})
\end{array}$$

in which the top middle vertical map j takes the class of $g \in G$ to the class of the map of right A-modules $l_g\colon A \to A$ defined by $l_g(b) = gb$, and the top vertical map i is induced by the canonical inclusion of G in A. We define the natural set map

$$[-]_s\colon C(G) \longrightarrow \mathrm{TR}_0^s(A)$$

to be the composition of the map j, the inverse of the map r_s, and the map σ_s.

Lemma 1.7. *Let G be a group. For every $g \in G$, and every divisor t of r,*

$$R_t([g]_r) = [g]_{r/t}, \qquad F_t([g]_r) = [g^t]_{r/t}.$$

Proof. The first statement is immediate from the definitions, and the second statement follows from the definitions and [17, Lemma 3.3, p. 54]. □

It follows from (1.6) that the map $[-]_1$ induces an isomorphism from the free abelian group generated by $C(G)$ onto the group $\mathrm{TR}_0^1(\mathbb{Z}[G])$. We have the following generalization.

Proposition 1.8. *Let G be a group, and let r be a positive integer. The abelian group $\mathrm{TR}_0^r(\mathbb{Z}[G])$ is free with a basis consisting of the elements $V_t([g]_{r/t})$, where t and $[g]$ range over the divisors of r and the elements of $C(G)$, respectively.*

Proof. The proof is by induction on r; the case $r = 1$ was established above. So we fix r and assume that the statement holds for all proper divisors of r. We choose a prime p that divides r and consider the exact sequence

$$\mathrm{TR}_0^d(\mathbb{Z}[G]) \xrightarrow{V_{p^u}} \mathrm{TR}_0^r(\mathbb{Z}[G]) \xrightarrow{R_p} \mathrm{TR}_0^{r/p}(\mathbb{Z}[G]) \longrightarrow 0$$

from Proposition 1.2 with $r = p^u d$ and d not divisible by p. It will suffice to show that the left-hand map V_{p^u} is injective. But it follows from Lemma 1.1 that the composite map

$$\mathrm{TR}_0^d(\mathbb{Z}[G]) \xrightarrow{V_{p^u}} \mathrm{TR}_0^r(\mathbb{Z}[G]) \xrightarrow{F_{p^u}} \mathrm{TR}_0^d(\mathbb{Z}[G])$$

is equal to multiplication by p^u, and since the group $\mathrm{TR}_0^d(\mathbb{Z}[G])$, inductively, is a free abelian group, we conclude that V_{p^u} is injective as desired. □

We use Proposition 1.8 to evalute the group $\mathrm{TR}_0(\mathbb{Z}[G])$ which, we recall, is defined to be the inverse limit of the groups $\mathrm{TR}_0^r(\mathbb{Z}[G])$ with respect to the restriction maps. To this end, we abuse notation and write $[g]$ for the element $([g]_r)$ of $\mathrm{TR}_0(\mathbb{Z}[G])$. We also remark that the maps $V_t\colon \mathrm{TR}_0^{r/t}(\mathbb{Z}[G]) \to \mathrm{TR}_0^r(\mathbb{Z}[G])$ give rise to a map $V_t\colon \mathrm{TR}_0(\mathbb{Z}[G]) \to \mathrm{TR}_0(\mathbb{Z}[G])$.

Corollary 1.9. *Every element $a \in \mathrm{TR}_0(\mathbb{Z}[G])$ admits a unique expression as a series*

$$a = \sum a_{t,[g]} V_t([g])$$

where t and $[g]$ range over the positive integers and the conjugacy classes of elements in G, respectively, and where $a_{t,[g]}$ are integers with the property that for every positive integer t, the coefficient $a_{t,[g]}$ is nonzero for only finitely many $[g] \in C(G)$.

Proof. By Proposition 1.8, each group $\mathrm{TR}_0^r(\mathbb{Z}[G])$ in the inverse limit is a free abelian group with a basis consisting of the elements $V_t([g]_{r/t})$, where t and $[g]$ range over the divisors of r and the conjugacy classes of elements in G, respectively. Moreover, if s divides r, then Lemmas 1.1 and 1.7 show that the restriction map

$$R_s\colon \mathrm{TR}_0^r(\mathbb{Z}[G]) \longrightarrow \mathrm{TR}_0^{r/s}(\mathbb{Z}[G])$$

is equal to the $\mathbb{Z}$-linear map given by

$$R_s(V_t([g]_{r/t})) = \begin{cases} V_t([g]_{r/st}) & \text{if } st \text{ divides } r, \\ 0 & \text{otherwise.} \end{cases}$$

We see that the statement follows, since $V_t([g]) \in \mathrm{TR}_0(\mathbb{Z}[G])$ is the unique element that projects to $V_t([g]_{r/t}) \in \mathrm{TR}_0^r(\mathbb{Z}[G])$, if t divides r, and to $0 \in \mathrm{TR}_0^r(\mathbb{Z}[G])$, otherwise. □

2. Proofs of Theorems A and B

We first use the factorization of the Hattori-Stallings rank map through the cyclotomic trace map to prove the slightly strengthened version of Linnell's theorem [21, Lemma 4.1] that we stated as Theorem A in the introduction.

Proof of Theorem A. Let P be a finitely generated projective $\mathbb{Z}[G]$-module. We may write the Hattori-Stallings rank of P uniquely as

$$r(P) = a_0[1] + a_1[g_1] + \cdots + a_n[g_n],$$

where $[1], [g_1], \ldots, [g_n]$ are distinct elements of $C(G)$, where a_0 is an arbitrary integer, and where $a_1, \ldots, a_n$ are nonzero integers. Let m be the minimal exponent of the symmetric group on n letters (the least common multiple of $\{1, 2, 3, \ldots, n\}$). We claim that

$$[g_i] = [g_i^{s^m}]$$

for all positive integers s and for $i = 1, 2, \ldots, n$. This will prove the theorem. To prove the claim, we consider the image $a = \mathrm{tr}(P) \in \mathrm{TR}_0(\mathbb{Z}[G])$ by the cyclotomic trace map. We write the element a uniquely as a series

$$a = \sum a_{t,[g]} V_t([g])$$

as in the statement of Corollary 1.9. By Lemma 1.5, we have $\ell(\mathrm{pr}_1(a)) = r(P)$, which shows that the coefficient $a_{1,[g]}$ is equal to a_i, if $[g] = [g_i]$, and zero, otherwise. Since a is in the image of the cyclotomic trace map, we have from Lemma 1.3 that $a = F_s(a)$, for all positive integers s. Let $s = p$ be a prime number. In this case, we have by Lemma 1.1 that

$$F_p(a) = \sum a_{t,[g]} V_t([g^p]) + \sum p a_{t,[g]} V_{t/p}([g]),$$

where, in the left-hand sum, t ranges over all positive integers prime to p, where, in the right-hand sum, t ranges over all positive integers divisible by p, and where, in both sums, $[g]$ ranges over the conjugacy classes of elements in G. We note that the generators $V_{t/p}([g])$ that appear in the right-hand sum are pairwise distinct, while the generators $V_t([g^p])$ that appear in the left-hand sum need not be distinct. In particular, every generator of the form $V_{t/p}([g])$ with t divisible by p^2 appears only in the right-hand sum and appears with the coefficient $pa_{t,[g]}$. Therefore, the equation $a = F_p(a)$ implies that, for t divisible by p^2,

$$a_{t/p,[g]} = pa_{t,[g]}.$$

Equivalently, on writing $t = pu$, we have that, for u divisible by p,

$$a_{u,[g]} = pa_{pu,[g]}.$$

Iterating this equation, we find that, for u divisible by p, the coefficient $a_{u,[g]}$ can be divided by p arbitrarily often, and therefore, is equal to zero. Since this is true for all prime numbers p, we conclude that the coefficient $a_{t,[g]}$ is equal to zero, unless $t = 1$. Hence,

$$a = a_0[1] + a_1[g_1] + \cdots + a_n[g_n],$$

and by Lemma 1.7, the equation $a = F_s(a)$ becomes

$$a_0[1] + a_1[g_1] + \cdots + a_n[g_n] = a_0[1] + a_1[g_1^s] + \cdots + a_n[g_n^s].$$

It follows, by the uniqueness of this expression, that for every positive integer s, the map

$$\varphi_s \colon C(G) \longrightarrow C(G)$$

that takes $[g]$ to $[g^s]$ restricts to a bijection $\varphi_s|_S$ of the subset $S = \{[g_1], \ldots, [g_n]\}$ onto itself. But $(\varphi_s|_S)^m = \mathrm{id}_S$, by the definition of m, so the claim follows. □

Corollary 2.1. *Let g be a nontrivial element of the group G for which there exists a finitely generated projective $\mathbb{Z}[G]$-module P with $r(P)(g) \neq 0$. In this situation, the element g lies in a subgroup C of G that:*

(i) *is isomorphic to $\mathbb{Q}$,*

(ii) *is generated by conjugates of g,*

(iii) *is contained in a finitely generated subgroup H of G,*

(iv) *has its elements lying in finitely many H-conjugacy classes, and*

(v) *has normal closure the subgroup $[G, C] = [G, g]$ of G generated by all commutators of the form $[h, g]$ with $h \in G$.*

Proof. Parts (i)–(iv) may be found in [21, Lemma 4.1] and [12, Theorem 3.32]; only part (v) is new. By Theorem A, there exists a positive integer m such that

$$[g] = [g^{s^m}],$$

for every positive integer s. In particular, we set $s = 2^m - 1$ so that $(2^m - 1, s^m - 1) = 1$, and choose integers k and l such that

$$k(2^m - 1) + l(s^m - 1) = 1.$$

Now, there exist elements $x, y \in G$ such that $g^{2^m} = xgx^{-1}$ and $g^{s^m} = ygy^{-1}$, or equivalently, such that $g^{2^m - 1} = [x, g]$ and $g^{s^m - 1} = [y, g]$. Therefore,

$$g = [x, g]^k [y, g]^l.$$

This shows that $g \in [G, g]$. Therefore, the smallest normal subgroup of G that contains g, namely $\langle g \rangle [G, \langle g \rangle]$, reduces to $[G, g]$. By (ii) above, $C \subseteq [G, g] = [G, C]$, which is thereby also the normal closure of C. □

Proof of Theorem B. Let $g \in G$ be a nontrivial element for which there exists a finitely generated projective $\mathbb{Z}[G]$-module P with $r(P)(g)$ nontrivial, and let $C \subseteq G$ be the subgroup specified in Corollary 2.1. We must show that if $N \subseteq G$ is a normal subgroup and if the image $\bar{C}$ of C in G/N is contained in the center, then that image is necessarily trivial. Now, by the assumption on C, we have $[G/N, \bar{C}\,] = 1$. Hence,

$$C \subseteq [G, g] = [G, C] \subseteq N$$

which shows that the image $\bar{C}$ in G/N is trivial as desired. □

3. Further consequences

In this section, we explore some consequences of the main theorems.

We recall that the lower central series of the group G is defined to be the descending sequence of subgroups

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$$

with $G_n = [G, G_{n-1}]$. We write $G_\omega = \bigcap_{n \geqslant 1} G_n$, and recall that the group G is defined to be residually nilpotent if G_ω is trivial. Continuing, the transfinite lower central series is defined as follows. For a successor ordinal $\alpha + 1$, we have $G_{\alpha+1} = [G, G_\alpha]$, while, for a limit ordinal β, we have $G_\beta = \bigcap_{\alpha < \beta} G_\alpha$. The series stabilizes once α is larger than the cardinality of G. Its intersection is sometimes called the relatively perfect radical, and is the maximal subgroup $P \subseteq G$ with the property that $P = [G, P]$. From the five-term homology exact sequence, it is the maximal subgroup P such that the canonical projection $G \to G/P$ induces an epimorphism on $H_2(-, \mathbb{Z})$ and an isomorphism on $H_1(-, \mathbb{Z})$. For further discussion of this group, see [29].

Corollary 3.1. *Suppose that the intersection P of the transfinite lower central series of the group G does not contain a subgroup isomorphic to $\mathbb{Q}$. Then the group G satisfies the Bass trace conjecture.*

Proof. We show that G satisfies the hypothesis of Theorem B. So let $D \subseteq G$ be a subgroup isomorphic to $\mathbb{Q}$ and let $N \subseteq G$ be the normal closure of D. We have $[G, N] \subseteq N$, and if also $N \subseteq [G, N]$, then the subgroup N, and hence D, would be contained in the intersection of the lower transfinite central series, contradicting our hypothesis on G. So we conclude that N, and hence D, have nontrivial image in $G/[G, N]$. Since this image is central, the hypothesis of Theorem B is fulfilled, as was to be shown. □

The class of groups with the property of Corollary 3.1 of course includes all hypocentral groups (those where the intersection of the transfinite lower central series is trivial), and so in particular all residually nilpotent groups. Previously, it was shown by Emmanouil [11] that residually nilpotent groups satisfy the Bass trace conjecture, if they are of finite rational homological dimension. However, as Guido Mislin has kindly pointed out (private communication), because of Proposition 3.8 (i) below, to affirm the conjecture for all residually nilpotent groups, it suffices to check it for all finitely generated residually nilpotent groups. Such groups are residually finitely generated nilpotent, and hence, contain no divisible elements, because all subgroups of finitely generated nilpotent groups are finitely generated. Thus, the Bass trace conjecture for residually nilpotent groups follows from Theorem A. For further example of groups whose satisfaction of the Bass trace conjecture follows from Theorem A, we have the following.

Example 3.2. The mapping class group Γ_g of a smooth orientable closed surface of genus $g \geqslant 3$ is a finitely generated perfect group. Therefore, every element of Γ_g lies in the intersection of the transfinite lower central series. However, Γ_g contains no copy of $\mathbb{Q}$ because it is residually finite [14]. Hence, the Bass trace conjecture holds for this group; see also [25, Corollary 7.17].

Example 3.3. We let $G = SL(2, \mathbb{Q})$ and consider the elements

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The calculation

$$g^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad hgh^{-1} = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}$$

shows that g^k is conjugate to g if and only if k is a square. In particular, for every positive integer r, we have

$$[g] = [g^{r^2}].$$

So the element g satisfies the conclusion of Theorem A with $m(g) = 2$. Now, the element g clearly lies in the subgroup

$$D = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q} \right\},$$

which is isomorphic to $\mathbb{Q}$. Nevertheless, we may conclude from Corollary 2.1 that the Bass trace conjecture holds for G. Indeed, since G is a linear group, it follows from [2, Theorem 9.6] that for every finitely generated subgroup $H \subseteq G$, the divisible elements in H have finite order. Therefore, the group G does not contain any subgroup C that satisfies both of the properties (i) and (iii) of Corollary 2.1. (For an alternative route, using (v) of Corollary 2.1, we can use the well-known fact that finitely generated linear groups are residually nilpotent, a situation discussed above. The affirmation of the Bass trace conjecture for the general case of linear groups then follows from Proposition 3.8 below. Of course, it was also established in Bass' original paper [2].)

That the groups in the next example satisfy the Bass trace conjecture does not evidently follow from Theorem A, because they contain numerous subgroups isomorphic to $\mathbb{Q}$.

Example 3.4. Let F be a field of characteristic zero, and let Λ be a well-ordered set. In this situation, the McLain group $M(\Lambda; F)$ is known to be hypocentral [28, Theorem 6.22], and so, by Corollary 3.1 satisfies the Bass trace conjecture.

Next, we have two examples with $m(g) = 1$.

Example 3.5. As pointed out in [30], the conclusion of Theorem A holds for the following group of P. Hall [15] with $m = 1$. First, index all prime numbers p_n by the set of integers. Let C be the direct sum of countably many copies of $\mathbb{Q}$, again indexed by the set of integers. Now let W be the group generated by elements ξ, η subject only to the requirement that all conjugates of η commute; write A for the abelian group that they generate. Hall's finitely generated group H is the semidirect product formed by letting ξ act on C by shifting the nth copy of $\mathbb{Q}$ to the $(n+1)$th copy and by letting η act on all elements of the nth copy of $\mathbb{Q}$ by raising to the p_nth power. The commutator subgroup H' is the semidirect product of C and A, while $H'' = C$ is a minimal normal subgroup of H. Thus, the subgroup C satisfies all of Corollary 2.1. Nevertheless, the Bass trace conjecture holds for H since it is soluble and hence amenable [4].

Example 3.6. By [26, Corollary 1.2], every countable torsion-free group can be embedded into a (torsion-free) 2-generated group G with a unique nontrivial conjugacy class. In particular, every element of $g \in G$ satisfies the conclusion of Theorem A with $m(g) = 1$. If $C \subseteq G$ is a subgroup isomorphic to $\mathbb{Q}$, then all the conditions of Corollary 2.1 hold with respect to $g \in C$ embedded in G (since obviously the conjugates of any nontrivial element generate the whole group, making G simple). Since G has no quotient with nontrivial center, the hypothesis of Theorem B fails to hold. It would be interesting to decide whether or not this group G satisfies the Bass trace conjecture.

Example 3.7. It is observed in [3] that the Bass trace conjecture holds for all groups if and only if it holds for all binate groups. Now, binate groups are perfect (indeed, acyclic, with every element a commutator); and so again every element lies in the intersection of the transfinite lower central series, whence Theorem B is not in general applicable.

In this context, it is worth recording the following known results on the class $\mathscr{B}$ of groups for which the Bass trace conjecture holds. The reader may find the proofs of (i) and (ii) in [11, Remark 1.5] and [12, Proposition 3.39], respectively.

Proposition 3.8. *The Bass trace conjecture holds for the group G, if either*

(i) *it holds for every finitely generated subgroup $H \subseteq G$, or*

(ii) *it holds for every proper subgroup $H \subseteq G$ of finite index.*

Thus, the class $\mathscr{B}$ is locally and virtually closed. However, it is not known whether the class $\mathscr{B}$ is residually closed, that is, whether a group G with the property that for every $g \in G$, there exists a group homomorphism $f \colon G \to G'$ with $G' \in \mathscr{B}$ and $f(g)$ nontrivial is necessarily in $\mathscr{B}$. On the other hand, it is easy to verify that the class of all groups G in which every divisible element maps nontrivially to the center of some quotient of G is a residually closed class.

Finally, we make the following amusing observation concerning the relation between the strong and weak forms of the Bass trace conjecture. (It is known that to affirm the weak Bass trace conjecture for all groups, it suffices to do so for a single group, described in [3].)

Proposition 3.9. *The strong form of the Bass trace conjecture holds for all groups if*

(i) *the class $\mathscr{B}$ of groups for which the strong form of the Bass trace conjecture holds is closed under taking finitely generated subgroups; and*

(ii) *the weak form of the Bass trace conjecture holds for all finitely generated groups.*

Proof. By Proposition 3.8 (i), it suffices to verify the strong Bass trace conjecture for every finitely generated group G. Now, by [26, Theorem 1.1], the group G embeds in a finitely generated group T in which any two elements of the same order are conjugate. But from Corollary 2.1 applied to T, we know that for every finitely generated projective $\mathbb{Z}[T]$-module P, all nontrivial elements $x \in T$ with $r(P)(x)$ nonzero have the same, infinite, order. Thus, they lie in a single conjugacy class in T. Therefore, the assumption (ii) that the weak Bass trace conjecture holds for T, implies that $T \in \mathscr{B}$. Finally, from the assumption (i) applied to the group T, we conclude that $G \in \mathscr{B}$. □

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