DENSE C-EMBEDDED SUBSPACES OF PRODUCTS

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Abstract. Throughout this paper the spaces X_i are assumed Tychonoff, $(X_I)_\kappa$ denotes $X_I := \Pi_{i\in I} X_i$ with the κ-box topology (so $(X_I)_\omega = X_I$), and $\Sigma_\lambda(p) := \{x \in X_I : |\{i \in I : x_i \neq p_i\}| < \lambda\}$ whenever $p \in X_I$.

Milton Ulmer proved in 1970/1973 that in a product of first-countable spaces, each Σ-product is C-embedded. The authors generalize this theorem in several ways.

Theorem. Let κ be a regular cardinal. If $q \in (X_I)_\kappa \backslash \Sigma_{\kappa^+}(p)$ and each q_i is a $P(\kappa)$-point in X_i with $\chi(q_i, X_i) \leq \kappa$, then $\Sigma_{\kappa^+}(p)$ is C-embedded in $\Sigma_{\kappa^+}(p) \cup \{q\}$.

Corollary. Let κ be a regular cardinal. If each X_i is a $P(\kappa)$-space such that $\chi(X_i) \leq \kappa$, then in $(X_I)_\kappa$ each set of the form $\Sigma_{\kappa^+}(p)$ is C-embedded.

In the same works, Ulmer constructed an example showing that a Σ-product in a product of spaces of countable pseudocharacter need not be C-embedded. Again the authors modify and extend his work, this time as follows.

Theorem. For every $\kappa \geq \omega$ there are a set $\{X_i : i \in I\}$ of Tychonoff spaces with $|I| = \kappa$, $q \in X_I$ and $f \in C(X_I \backslash \{q\}, \{0, 1\})$ such that no continuous function from X_I to $[0, 1]$ extends f. One may arrange further that $|X_i| = \kappa$ for every $i \in I$, and either

(i) there is $i_0 \in I$ such that $\psi(X_{i_0}) = \omega$ and for $i_0 \neq i \in I$ the space X_i is the one-point compactification of a discrete space with cardinality κ; or

(ii) the spaces X_i are pairwise homeomorphic, with $\psi(X_i) = \omega$, and:
either (a) all but one point in each X_i is isolated, or (b) each X_i is dense-in-itself.

1. Introduction

Throughout this paper, ω is the least infinite cardinal, $\mathfrak{c} = 2^\omega = |[0, 1]|$, and I is an infinite index set. κ and λ are infinite cardinals, and $[I]^{<\kappa} := \{J \subseteq I : |J| < \kappa\}$. All topological spaces are assumed Tychonoff, $(X_I)_\kappa$ denotes $X_I := \Pi_{i\in I} X_i$ with the κ-box topology (so $(X_I)_\omega = X_I$), and $\Sigma_\lambda(p) := \{x \in X_I : |\{i \in I : x_i \neq p_i\}| < \lambda\}$ whenever $p \in X_I$. By a (canonical) basic open set in $(X_I)_\kappa$ we mean a set of the form $U = U_I = \Pi_{i\in I} U_i$ with U_i open in X_i and with $R(U) := \{i \in I : U_i \neq X_i\} \in [I]^{<\kappa}$. (In the terminology of [2], $R(U)$ is the *restriction set* of the open set U.) The symbol $\chi(x, X)$ denotes the character (i.e., the local weight) of the point x in the space X, $\chi(X) := \sup\{\chi(x, X) : x \in X\}$, and $\psi(X)$ denotes the pseudocharacter of the space X.

A point $x \in X$ is said to be a $P(\kappa)$-*point* provided the intersection of any family of fewer than κ-many neighborhoods of x is also a neighborhood of x. A space X is said to be a $P(\kappa)$-*space* provided each point $x \in X$ is a $P(\kappa)$-point. Clearly, every topological space is a $P(\omega)$-space. The $P(\omega^+)$-spaces are also called P-*spaces*.

Recall that a subspace Y of X is C-*embedded* in X provided each real-valued continuous function on Y extends continuously over X. The set of all real-valued continuous functions on a space Y is denoted by $C(Y)$.

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We use below the simple fact (which we will not mention again explicitly) that when Y is dense in X a function $f \in C(Y)$ extends continuously over X if and only if f extends continuously to each point of $X \backslash Y$; in this connection see [1], [9].

For additional topological definitions not given here see [5], [6], or [2].

The problem of determining conditions on a space X and a proper, dense subspace Y under which Y is C-embedded in X has generated considerable attention in the literature. H. Corson [3], I. Glicksberg [7], R. Engelking [4], N. Noble [10], N. Noble and M. Ulmer [11], M. Ulmer [12], [13], M. Hušek [8], and many others have achieved nontrivial results for the case where X is a product space X_I and Y is a Σ-product (that is, a subspace of the form $Y = \Sigma_{\omega^+}(p) \subseteq X_I$). As is indicated in [2], most of their results generalize to product spaces with the κ-box topology.

In this paper we continue the study of the question: When is a Σ-product C-embedded in the full product space? We give some positive and some negative answers. In Section 2 we give sufficient conditions under which a Σ_{κ^+}-product is C-embedded in a product space X_I equipped with the κ-box topology (Theorem 2.3). This generalizes a theorem of Ulmer [12], [13]: in a product of first-countable spaces every Σ-product is C-embedded. In the same works, Ulmer also showed that not every Σ-product in every (Tychonoff) product space is C-embedded. In Section 3, modifying Ulmer's example, we show (see Theorem 3.2) that in a (suitably constructed) product space X_I, even a set of the form $X_I \backslash \{q\}$ need not be C-embedded. Our argument indicates further that the apparent dependence in Ulmer's works on the existence of many isolated points is in part illusory.

2. Positive results

The following theorem is due to Ulmer [12], [13].

Theorem 2.1. *Let $\{X_i : i \in I\}$ be a set of first-countable spaces and $p \in X_I$. Then $\Sigma(p)$ is C-embedded in X_I.*

A careful reading of the proof of Theorem 2.1 as exposed in [2, Theorem 10.17] shows that those authors actually proved the following theorem.

Theorem 2.2. *Let $\{X_i : i \in I\}$ be a set of topological spaces, $p \in X_I$, and let $\Sigma(p) \subseteq Y \subseteq X_I$. If $q \in X_I \setminus Y$ is such that $\chi(q_i, X_i) \leq \omega$ for every $i \in I$, then Y is C-embedded in $Y \cup \{q\}$.*

Now by modifying the indicated proofs, we generalize Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Let $\{X_i : i \in I\}$ be a set of topological spaces and let $p, q \in (X_I)_\kappa$, where κ is regular. If for every $i \in I$ the point q_i is a $P(\kappa)$-point in X_i with $\chi(q_i, X_i) \leq \kappa$, then $\Sigma_{\kappa^+}(p)$ is C-embedded in $\Sigma_{\kappa^+}(p) \cup \{q\}$.*

Proof. If the statement fails then some $f \in C(\Sigma_{\kappa^+}(p))$ does not extend continuously to q; since $\mathbb{R}$ is complete there is $\epsilon > 0$ such that the oscillation of f exceeds ϵ on every neighborhood in $(X_I)_\kappa$ of q. (See in this connection [5, Lemma 4.3.16].) For U a neighborhood of q in $(X_I)_\kappa$ we choose $x_U, y_U \in U \cap \Sigma_{\kappa^+}(p)$ such that

$$|f(x_U) - f(y_U)| > \epsilon;$$

we assume without loss of generality, since f is continuous and $\Sigma_\kappa(p)$ is a dense subspace of $\Sigma_{\kappa^+}(p)$ in $(X_I)_\kappa$, that $x_U, y_U \in \Sigma_\kappa(p)$.

It follows from the hypotheses that for $i \in I$ there is a local base $\{U_\alpha(q_i) : \alpha < \kappa\}$ at q_i such that $U_{\alpha'}(q_i) \subseteq U_\alpha(q_i)$ whenever $\alpha < \alpha' < \kappa$. (If $\chi(q_i, X_i) < \kappa$ then q_i is isolated in X_i and we take $U_\alpha(q_i) = \{q_i\}$ for all $\alpha < \kappa$.)

For $\alpha < \kappa$ we define $J(\alpha) \in [I]^{<\kappa}$, a neighborhood $U(\alpha)$ of q, and $x(\alpha), y(\alpha) \in \Sigma_\kappa(p)$ as follows:

(i) $J(0) = \emptyset$;
(ii) $U(0) = X_I$;
(iii) $x(0) = x_{U(0)}, y(0) = y_{U(0)}$;
(iv) $J(\alpha + 1) = J(\alpha) \cup \{i \in I : x(\alpha)_i \neq p_i \text{ or } y(\alpha)_i \neq p_i\}$ for $\alpha < \kappa$;
(v) $J(\alpha) = \cup_{\beta<\alpha} J(\beta)$ for $\alpha < \kappa$ and α limit ordinal;
(vi) $U(\alpha) = \{z \in X_I : z_i \in U_\alpha(q_i) \text{ for } i \in J(\alpha)\}$ for $\alpha < \kappa$; and
(viii) $x(\alpha) = x_{U(\alpha)}, y(\alpha) = y_{U(\alpha)}$ for $\alpha < \kappa$.

For every α we have $x(\alpha), y(\alpha) \in \Sigma_\kappa(p)$, and $|J(\alpha)| < \kappa$ since κ is regular; hence $U(\alpha)$ is a basic open set in $(X_I)_\kappa$. We set $J(\kappa) = \cup_{\alpha<\kappa} J(\alpha)$, so $|J(\kappa)| \leq \kappa$. We define $\overline{z} \in X_I$ by

$$\overline{z}_i = q_i \text{ for } i \in J(\kappa) \text{ and}$$
$$\overline{z}_i = p_i \text{ for } i \in I \setminus J(\kappa).$$

Since $|J(\kappa)| \leq \kappa$ we have $\overline{z} \in \Sigma_{\kappa^+}(p)$.

Now let $V = (f(\overline{z}) - \frac{\epsilon}{2}, f(\overline{z}) + \frac{\epsilon}{2}) \subseteq \mathbb{R}$. Since f is continuous at $\overline{z}$ there exists a basic neighborhood U of $\overline{z}$ such that $f(U \cap \Sigma_{\kappa^+}(p)) \subset V$. For $i \in R(U) \cap J(\kappa)$ choose $\alpha_i < \kappa$ such that $U_{\alpha_i}(q_i) \subset U_i$, and set $\alpha := \sup_i \alpha_i$; then $\alpha < \kappa$, since $|R(U)| < \kappa$ and κ is regular. Then $x(\alpha), y(\alpha) \in U$, so $f(x(\alpha)), f(y(\alpha)) \in V$ and we have the contradiction
$\epsilon < |f(x(\alpha)) - f(y(\alpha))| \leq |f(x(\alpha)) - f(\overline{z})| + |f(y(\alpha)) - f(\overline{z})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Corollary 2.4. *Let κ be regular, $\{X_i : i \in I\}$ a set of topological spaces, and $p \in (X_I)_\kappa$. If for every $q \in (X_I)_\kappa \setminus \Sigma_{\kappa^+}(p)$ and $i \in I$ the point q_i is a $P(\kappa)$-point in X_i with $\chi(q_i, X_i) \leq \kappa$, then $\Sigma_{\kappa^+}(p)$ is C-embedded in $(X_I)_\kappa$.*

Corollary 2.5. *Let κ be a regular cardinal and let $\{X_i : i \in I\}$ be a set of $P(\kappa)$-spaces such that $\chi(X_i) \leq \kappa$ for every $i \in I$. Then each space of the form $\Sigma_{\kappa^+}(p)$ (with $p \in (X_I)_\kappa$) is C-embedded in $(X_I)_\kappa$.*

A stronger version of the case $\kappa = \omega^+$ of Corollary 2.5 appears in the cited works of Ulmer:

Theorem 2.6. *Let $\{X_i : i \in I\}$ be a set of P-spaces and $p \in X_I$. Then $\Sigma_{\omega_1}(p)$ is C-embedded in X_I.*

The attention which the Σ-products $\Sigma_{\omega^+}(p) \subseteq X_I$ have attracted with respect to questions of C-embedded subspaces of product spaces might lead one to believe that every G_δ-dense C-embedded subspace must contain such a space. We show now that this is by no means the case. Indeed, in appropriate circumstances a G_δ-dense, C-embedded subspace of a product space X_I may meet each Σ-product $\Sigma_{\omega^+}(p) \subseteq X_I$ in at most one point.

Example 2.7. *There exists a product space X and a G_δ-dense subspace D such that D intersects every Σ-product $\Sigma(p) \subseteq X$ in at most one point.*

Proof. Let $X = X_I = \mathbb{T}^\mathfrak{c}$ (with $\mathbb{T}$ the unit circle). Every nonempty G_δ-set in X contains, for some countable $C \subseteq \mathfrak{c}$ and some $p \in \mathbb{T}^C$, a set of the form $\{p\} \times \mathbb{T}^{\mathfrak{c}\setminus C}$. There are $\mathfrak{c}^\omega$ possible choices for C, and $|\mathbb{T}^C|$-many choices (i.e., $\mathfrak{c}$-many) for $p \in \mathbb{T}^C$. Thus, there are $\mathfrak{c}$-many "basic G_δ" subsets of $\mathbb{T}^\mathfrak{c}$. Let $\{A_\eta : \eta < \mathfrak{c}\}$ index these. From the same observation, each nonempty G_δ-subset of $\mathbb{T}^\mathfrak{c}$ has cardinality (at least) $|\{p\} \times \mathbb{T}^{\mathfrak{c}\setminus C}| = 2^\mathfrak{c}$.

Now fix $q \in \mathbb{T}^\mathfrak{c}$. Each point $r \in \Sigma(q)$ differs from q in at most ω-many coordinates (which may be chosen in $\mathfrak{c}^\omega = \mathfrak{c}$-many ways), so $|\Sigma(q)| \leq \mathfrak{c}$ (equality is easy to see). Thus there are $2^\mathfrak{c}$-many (pairwise disjoint) Σ-product subsets of $\mathbb{T}^\mathfrak{c}$, say $\{S_\eta : \eta < 2^\mathfrak{c}\}$. Each S_η is G_δ-dense, so for $\eta < \mathfrak{c}$ we can choose a point $r_\eta \in A_\eta \cap S_\eta$. Let $D := \{r_\eta : \eta < \mathfrak{c}\}$. Then D is G_δ-dense in X, and D meets each Σ-product in X in at most one point. (In fact we did not "use" the sets S_η with $\mathfrak{c} \leq \eta < 2^\mathfrak{c}$, i.e., D is disjoint from "most" of the Σ-products in X.) That D is C-embedded in X is immediate from the theorem of Noble [10]. □

3. Negative results

The following result, exposed in [2, Theorem 10.18], also takes its root in Ulmer [12], [13]. So far as we have been able to determine, this was the first construction proving that not every Σ-product in every (Tychonoff) product space is C-embedded.

Theorem 3.1. *For every $\kappa \geq \omega$ there are a set $\{X_i : i \in I\}$ of Tychonoff spaces with $|I| = \kappa$, $p, q \in X_I$ and $f \in C(\Sigma_\kappa(p), \{0,1\})$ such that no continuous function from $\Sigma_\kappa(p) \cup \{q\}$ to $[0,1]$ extends f. One may arrange further that $|X_i| = \kappa$ for every $i \in I$, and either*

(i) *all but one point of each X_i is isolated, in fact there is $i_0 \in I$ such that for $i_0 \neq i \in I$ the space X_i is the one-point compactification of a discrete space; or*
(ii) *$\psi(X_{i_0}) = \omega$ and for $i_0 \neq i \in I$ the space X_i is first countable; or*
(iii) *$\psi(X_{i_0}) = \omega$ and for $i_0 \neq i \in I$ the space X_i is a P-space.*

A routine modification of the construction which establishes Theorem 3.2 shows that in the spaces hypothesized there even the space $X_I \backslash \{q\}$ is not $[0,1]$-embedded in X_I. Our attempts to determine the full scope of applicability of the argument used have not succeeded, but we have established that the apparent dependence on the existence of many isolated points is in part illusory. In this connection we say as usual that a space is *dense-in-itself* if it contains no isolated points.

Theorem 3.2. *For every $\kappa \geq \omega$ there are a set $\{X_i : i \in I\}$ of Tychonoff spaces, with $|I| = \kappa$, $q \in X_I$ and $f \in C((X_I \setminus \{q\}), \{0,1\})$, such that no continuous function from X_I to $[0,1]$ extends f. One may arrange further that $|X_i| = \kappa$ for every $iinI$, and either*

(i) *there is $i_0 \in I$ such that $\psi(X_{i_0}) = \omega$, and for $i_0 \neq i \in I$ the space X_i is the one-point compactification of a discrete space with cardinality κ; or*
(ii) *the spaces X_i are pairwise homeomorphic, with $\psi(X_i) = \omega$ and either*
  (a) *all but one point in each space X_i is isolated; or*
  (b) *each space X_i is dense-in-itself.*

Proof. We note first that there is a family $\mathcal{F}$ of subsets of κ such that

(1) $|\mathcal{F}| = \kappa$;
(2) $|A| = \kappa$ for $A \in \mathcal{F}$;
(3) if $\mathcal{G} \subset \mathcal{F}$ and $|\mathcal{G}| < \omega$, then $|\cap \mathcal{G}| = \kappa$; and
(4) if $\mathcal{G} \subset \mathcal{F}$ and $|\mathcal{G}| \geq \omega$, then $\cap \mathcal{G} = \emptyset$.

(To define such a family $\mathcal{F}$, let $\{B_\xi : \xi < \kappa\}$ be a faithful indexing of $[\kappa]^{<\omega} \backslash \{\emptyset\}$ and set $A_\zeta = \{\xi < \kappa : \zeta \in B_\xi\}$ for $\zeta < \kappa$ and $\mathcal{F} = \{A_\zeta : \zeta < \kappa\}$.)

To initiate each of the three constructions which follow, we choose and fix an ultrafilter $q_0 \in \beta(\kappa)$ such that $\mathcal{F} \subseteq q_0$.

(i) Let $X_0 = X_{i_0} = \kappa \cup \{q_0\}$ with the topology inherited from $\beta(\kappa)$ (κ with the discrete topology), and for $A \in \mathcal{F}$ let X_A denote the one-point compactification $A \cup \{q_A\}$ of the (discrete) space A. Let $I := \{i_0\} \cup \mathcal{F}$ and

$$X = X_0 \times \prod_{A \in \mathcal{F}} X_A,$$

and set

$$K = \{x \in X : x_0 \neq q_0, \text{ and } x_A = x_0 \text{ whenever } x_0 \in A \in \mathcal{F}\}.$$

Because no element of κ lies in infinitely many elements of $\mathcal{F}$, the set K is open in X. (In detail: if $x \in K$ and $x_0 = \xi \in \kappa$, then

$$\pi_0^{-1}(\{\xi\}) \cap (\bigcap \{\pi_A^{-1}(\{\xi\}) : \xi \in A \in \mathcal{F}\})$$

is a neighborhood in X of x which is a subset of K.)

We claim also that K is closed in $X \setminus \{q\}$. Let $x \in X \setminus (K \cup \{q\})$. If $x_0 \neq q_0$, there is $A \in \mathcal{F}$ such that $x_0 \in A$ and $x_A \neq x_0$. Then

$$\pi_0^{-1}(\{x_0\}) \cap \pi_A^{-1}(\{x_A\})$$

is a neighborhood in X of x disjoint from K. If $x_0 = q_0$, then since $x \neq q$ there is $A \in \mathcal{F}$ such that $x_A \neq q_A$ and there is $B \in \mathcal{F}$ such that $x_A \in \kappa \setminus B$; we set

$$U = \{q_0\} \cup (A \cap B).$$

Then U is a neighborhood in X_0 of q_0 and hence

$$\pi_0^{-1}(U) \cap \pi_A^{-1}(\{x_A\})$$

is a neighborhood in X of x; this misses K, for if $y_0 \in U$ then either $y_0 = q_0$ (hence $y \notin K$) or $y_0 \in A \cap B$ (hence $y_0 \neq y_A = x_A$). The proof that K is closed in $X \setminus \{q\}$ is complete.

To complete the proof it is enough to show that every neighborhood in X of q meets both K and $X \setminus (K \cup \{q\})$, for then the function which is 1 on K and 0 on $X \setminus (K \cup \{q\})$ will admit no continuous extension to q.

Let $V = V_0 \times \prod_{A \in \mathcal{F}} V_A$ be a neighborhood in X of q with (finite) restriction set $R(V)$, let $A' \in \mathcal{F}$ be such that $A' \notin R(V)$, and fix $x_{A'} \in A'$. Define $y \in X$ by the rules

$$\begin{gathered} y_0 = q_0 \\ y_{A'} = x_{A'} \\ y_A = q_A \text{ for } A \in \mathcal{F} \setminus \{A'\}; \end{gathered}$$

then $y \in V \cap (X \setminus (K \cup \{q\})$.

We define $z \in V \cap K$. Since $|A \setminus V_A| < \omega$ for $A \in \mathcal{F}$, we have $V_A \in q_0$ for $A \in \mathcal{F} \subset q_0$; hence there is

$$z_0 \in V_0 \cap (\bigcap \{V_A : A \in R(V) \cap \mathcal{F}\}).$$

We set $\mathcal{F}' = \{A \in \mathcal{F} : z_0 \in A\}$, we note that $\mathcal{F}'$ is a finite subset of $\mathcal{F}$ such that $R(V) \cap \mathcal{F} \subset \mathcal{F}'$, and we set

$$\begin{gathered} z_A = z_0 \text{ for } A \in \mathcal{F}' \\ z_A = q_A \text{ for } A \in \mathcal{F} \setminus \mathcal{F}'. \end{gathered}$$

It is clear that $z \in V \cap K$, as required. The proof of (i) is complete.

(ii)(a) Define $X = X_0 = X_{i_0}$ as in part (i), and for $A \in \mathcal{F}$ let $q_A := \{B \cap A : B \in q_0\}$ and give $X_A := A \cup \{q_A\}$ the topology inherited from $\beta(A)$ (A with the discrete topology). Then X_0 and the spaces X_A ($A \in \mathcal{F}$) are pairwise homeomorphic spaces with countable pseudocharacter. Now we set

$$X = X_0 \times \prod_{A \in \mathcal{F}} X_A,$$

and

$$K = \{x \in X : x_0 \neq q_0, \text{ and } x_A = x_0 \text{ whenever } x_0 \in A \in \mathcal{F}\}.$$

The proof now proceeds without change and *verbatim* as in the proof of (i), beginning with the paragraph "Because no element . . .."

(ii)(b) Let $C = [0, 1]$, and define $X_0 = \{q_0\} \cup (\kappa \times C)$ where each set $\{\xi\} \times C$ is open-and-closed in X_0 and (each copy of) C has the usual topology of $[0, 1]$; X_0 is topologized so that sets of the form $\{q_0\} \cup (B \times C)$ with $B \in q_0$ are a local basis at q_0.

For $A \in \mathcal{F}$ again define $q_A := \{B \cap A : B \in q_0\}$ where each set $\{\xi\} \times C$ is open-and-closed in X_0 and (each copy of) C has the usual topology of $[0, 1]$; $X_A := \{q_A\} \cup (\kappa \times C)$

is topologized so that sets of the form $\{q_A\} \cup (B \times C)$ with $B \in q_A$ are a local basis at q_A. Then X_0 and the spaces X_A $(A \in \mathcal{F})$ have countable pseudocharacter. Now we set

$$X = X_0 \times \prod_{A \in \mathcal{F}} X_A$$

and

$$K = \{x \in X : x_0 \neq q_0, \text{ and } (x_A)_0 = (x_0)_0 \text{ whenever } (x_0)_0 \in A \in \mathcal{F}\}$$

(in the case when $x_A = q_A$ by $(x_A)_0$ and $(x_A)_1$ we mean q_A and in the case when $x_A = (\xi, r)$ then $(x_A)_0 = \xi$ and $(x_A)_1 = r$).

The required verification now closely parallels the arguments already given. Again, the set K is open in X and is closed in $X \setminus \{q\}$. (When $x \in X \backslash (K \cup \{q\})$, if $x_0 \neq q_0$ the set

$$\pi_0^{-1}(\{\xi\} \times C) \cap \pi_A^{-1}(\{(x_A)_0\} \times C)$$

is a neighborhood in X of x disjoint from K, and if $x_0 = q_0$ there is $B \in \mathcal{F}$ such that $(x_A)_0 \in \kappa \setminus B$ and then

$$U = \{q_0\} \cup ((A \cap B) \times C)$$

is a neighborhood in X_0 of q_0 for which

$$\pi_0^{-1}(U) \cap \pi_A^{-1}(\{(x_A)_0\} \times C)$$

is a neighborhood in X of x which misses K.)

The fact that every neighborhood in X of q meets both K and $X \setminus (K \cup \{q\})$ also proceeds as before, *mutatis mutandis*.

Let $V = V_0 \times \prod_{A \in \mathcal{F}} V_A$ be a basic open neighborhood in X of q with (finite) restriction set $R(V)$, let $A' \in \mathcal{F}$ be such that $A' \notin R(V)$, and $x_{A'}$ be a point such that $(x_{A'})_0 \in A'$. Set

$$\begin{aligned} y_0 &= q_0 \\ y_{A'} &= x_{A'} \\ y_A &= q_A \text{ for } A \in \mathcal{F} \setminus \{A'\}; \end{aligned}$$

then $y \in V \cap (X \setminus (K \cup \{q\})$.

We define $z \in V \cap K$. For every $A \in \mathcal{F} \subset q_0$ we may assume that $V_A = \{q_A\} \cup (B_A \times C)$ where $B_A \in q_A$ and $V_0 = \{q_0\} \cup (B_0 \times C)$ where $B_0 \in q_0$. Thus, there is

$$\xi_0 \in B_0 \cap (\bigcap \{B_A : A \in R(V) \cap \mathcal{F}\}).$$

We set $\mathcal{F}' = \{A \in \mathcal{F} : \xi_0 \in A\}$, we note that $\mathcal{F}'$ is a finite subset of $\mathcal{F}$ such that $R(V) \cap \mathcal{F} \subset \mathcal{F}'$. If $r \in C$ we set

$$\begin{aligned} z_0 &= (\xi_0, r) \\ z_A &= (\xi_0, r) \text{ for } A \in \mathcal{F}' \\ z_A &= q_A \text{ for } A \in \mathcal{F} \setminus \mathcal{F}'. \end{aligned}$$

It is clear that $z \in V \cap K$, as required. $\square$

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