High-frequency soliton-like waves in a relaxing medium

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A nonlinear evolution equation is suggested to describe the propagation of waves in a relaxing medium. It is shown that for low-frequency approach this equation is reduced to the KdVB equation. The high-frequency perturbations are described by a new nonlinear equation. This equation has ambiguous looplike solutions. It is established that a dissipative term, with a dissipation parameter less than some limit value, does not destroy these looplike solutions. © *1999 American Institute of Physics.* [S0022-2488(99)00503-4]

I. INTRODUCTION

As a rule the behavior of media under the action of high-frequency wave perturbations is not described in the framework of equilibrium models of continuum mechanics. So, to develop physical models for wave propagation through media with complicated inner kinetics, the notions based on the relaxational nature of a phenomenon are regarded to be promising and fruitful.

The description of nonlinear processes arising in different areas of research can often be reduced to the well-known Korteweg–de Vries (KdV) equation.[1,2] It turns out that low-frequency perturbations in a relaxing medium satisfy the KdV equation, too. The high-frequency perturbations are described by a new nonlinear evolution equation which has been investigated in Refs. 3 and 4. This equation has an ambiguous solution in the form of a solitary wave. This work deals with the looplike solutions of the model evolution equation. It is proved that the dissipative term, with a dissipation parameter less than some limit value, does not destroy the looplike solutions.

II. LOW-FREQUENCY AND HIGH-FREQUENCY PERTURBATIONS IN RELAXING MEDIUM

Thermodynamic equilibrium is disturbed owing to the propagation of fast perturbations in a medium. There are processes of the interaction that tend to return the equilibrium. The parameters characterizing this interaction are referred to as the inner variables unlike the macroparameters such as the pressure p, mass velocity u, and density ρ. In essence, the change of macroparameters caused by the changes of inner parameters is a relaxation process. From the nonequilibrium thermodynamics standpoint, the models of a relaxing medium are more general than the equilibrium models for describing the evolution of the wave perturbations.

We restrict our attention to barothropic media. An equilibrium state equation of a barothropic medium is a one-parameter equation. As a result of relaxation, an additional variable ξ (inner parameter) appears in the state equation. It defines the completeness of the relaxation process

$$p=p(\rho,\xi). \tag{2.1}$$

There are two limiting cases:

(*i*) the lack of the relaxation (inner interaction processes are frozen) $\xi=1$.

$$p=p(\rho,1)\equiv p_f(\rho), \tag{2.2}$$

(*ii*) the relaxation complete (there is the local thermodynamic equilibrium) $\xi=0$:

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$$p = p(\rho, 0) \equiv p_e(\rho). \tag{2.3}$$

These relationships enable us to introduce the sound velocities for fast processes

$$c_f^2 = dp_f/d\rho \tag{2.4}$$

and for slow processes

$$c_e^2 = dp_e/d\rho. \tag{2.5}$$

Slow and fast processes are compared by means of the relaxation time τ_p. The dynamic state equation is written down in the form of the differential first-order equation

$$\tau_p\left(\frac{dp}{dt} - c_f^2 \frac{d\rho}{dt}\right) + (p - p_e) = 0. \tag{2.6}$$

Clearly, for the fast processes ($\omega\tau_p \gg 1$) we have the relation (2.2), and for the slow ones ($\omega\tau_p \ll 1$) we obtain (2.3).

The substantiation of this equation within the framework of the thermodynamics of irreversible processes has been given in Refs. 5–8. The mechanism of the exchange (inner) processes is not defined concretely when the equation (2.6) is derived, and the thermodynamic and kinetic parameters appear in this equation only. These characteristics can be found by experiment. The dynamic state equation (2.6) enables us to take into account the exchange processes completely. We note that the phenomenological approach for describing the relaxation processes in hydrodynamics is developed in many works.[7–10] The dynamic state equation was used to describe the propagation of sound in a relaxing medium,[7] to take into account the exchange processes within media (gas–solid particles),[8] and to study wave fields in gas-liquid media[9] and in soils.[10] In most works the state equation has been derived from the concept of some concrete mechanism for the inner process.

To analyze the wave motion, we shall use the hydrodynamic equations: the law of the conservation of mass

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\rho_0 \partial x} = 0 \tag{2.7}$$

and the law of the conservation of momentum

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\rho_0 \partial x} = 0. \tag{2.8}$$

Here $V \equiv \rho^{-1}$ is specific volume and x is Lagrangian space coordinate.

The closed system of equations consists of two motion equations (2.7) and (2.8) and the dynamic state equation (2.6). The motion equations (2.7) and (2.8) are written in Lagrangian coordinates, since the state equation (2.6) is related to the element of the mass of medium.

Let us consider a small perturbation $p' < p_0$. The state equations for fast [(2.2)] and slow [(2.3)] processes are considered to be known. They can be expanded as the power series with accuracy $O(p'^2)$

$$V_f(p_0 + p') = V_0 - c_f^{-2} V_0^2 p' + \frac{1}{2} \left.\frac{d^2 V_f}{dp^2}\right|_{p=p_0} p'^2 + \ \dots,$$

$$V_e(p_0+p')=V_0-c_e^{-2}V_0^2p'+\frac{1}{2}\frac{d^2V_e}{dp^2}\bigg|_{p=p_0}p'^2+\ \ldots .$$

Hereafter, the velocities c_e, c_f are related to initial pressure p_0. Combining these two relationships with the motion equations (2.7) and (2.8), we obtain the equation in one unknown (the dash in p' is omitted):

$$\tau_p\frac{\partial}{\partial t}\left(\frac{\partial^2 p}{\partial x^2}-c_f^{-2}\frac{\partial^2 p}{\partial t^2}+\frac{1}{2V_0^2}\frac{d^2V_f}{dp^2}\bigg|_{p=p_0}\frac{\partial^2 p^2}{\partial t^2}\right)+\left(\frac{\partial^2 p}{\partial x^2}-c_e^{-2}\frac{\partial^2 p}{\partial t^2}+\frac{1}{2V_0^2}\frac{d^2V_e}{dp^2}\bigg|_{p=p_0}\frac{\partial^2 p^2}{\partial t^2}\right)=0. \tag{2.9}$$

A similar equation has been obtained in Ref. 5, though without nonlinear terms.

Now we shall show that for low-frequency perturbations the equation (2.9) is reduced to the Korteweg–de Vries–Burgers (KdVB) equation, while for high-frequency waves we shall obtain the equation with hydrodynamic nonlinearity and term that appeared in the Klein–Gordon equation.

To analyze the equation (2.9), let us apply the multiscale method.[11,12] The value $\varepsilon\equiv\tau_p\omega$ is chosen to be small (large) parameter where the quantity ω is the characteristic frequency of wave perturbation. For the sake of convenience we rewrite the equation (2.9) as follows:

$$\tau_p\omega\frac{\partial}{\partial t\omega}\left(\frac{\partial^2 p}{\partial(x\omega)^2}-c_f^{-2}\frac{\partial^2 p}{\partial(t\omega)^2}+\alpha_f\frac{\partial^2 p^2}{\partial(t\omega)^2}\right)+\left(\frac{\partial^2 p}{\partial(x\omega)^2}-c_e^{-2}\frac{\partial^2 p}{\partial(t\omega)^2}+\alpha_e\frac{\partial^2 p^2}{\partial(t\omega)^2}\right)=0, \tag{2.10}$$

$$\alpha_f=\frac{1}{2V_0^2}\frac{d^2V_f}{dp^2}\bigg|_{p=p_0},\qquad \alpha_e=\frac{1}{2V_0^2}\frac{d^2V_e}{dp^2}\bigg|_{p=p_0},$$

and introduce new independent variables

$$T_0=t\omega,\quad X_0=x\omega,\quad T_{-2}=t\omega/\varepsilon^2,\quad X_{-2}=x\omega/\varepsilon^2.$$

It is precisely these variables that cause the equations, obtained within the framework of multiscale method[11,12]

$$O(\varepsilon^{+1}):\frac{\partial}{\partial T_0}\left(\frac{\partial^2 p}{\partial X_0^2}-c_f^{-2}\frac{\partial^2 p}{\partial T_0^2}+\alpha_f\frac{\partial^2 p^2}{\partial T_0^2}\right)=0,$$

$$O(\varepsilon^{0}):\frac{\partial^2 p}{\partial X_0^2}-c_e^{-2}\frac{\partial^2 p}{\partial T_0^2}+\alpha_e\frac{\partial^2 p^2}{\partial T_0^2}=0,$$

$$O(\varepsilon^{-1}):\left(\frac{\partial^3}{\partial X_0^2\partial T_{-2}}+2\frac{\partial^3}{\partial T_0\partial X_0\partial X_{-2}}\right)p-3c_f^{-2}\frac{\partial^3 p}{\partial T_0^2\partial T_{-2}}+3\alpha_f\frac{\partial^3 p^2}{\partial T_0^2\partial T_{-2}}=0,$$

$$O(\varepsilon^{-2}):\frac{\partial^2 p}{\partial X_0\partial X_{-2}}-c_e^{-2}\frac{\partial^2 p}{\partial T_0\partial T_{-2}}+\alpha_e\frac{\partial^2 p^2}{\partial T_0\partial T_{-2}}=0, \tag{2.11}$$

$$O(\varepsilon^{-3}):\left(\frac{\partial^3}{\partial T_0\partial X_{-2}^2}+2\frac{\partial^3}{\partial X_0\partial X_{-2}\partial T_{-2}}\right)p-3c_f^{-2}\frac{\partial^3 p}{\partial T_0\partial T_{-2}^2}+3\alpha_f\frac{\partial^3 p^2}{\partial T_0\partial T_{-2}^2}=0,$$

$$O(\varepsilon^{-4}):\ \frac{\partial^2 p}{\partial X_{-2}^2}-c_e^{-2}\frac{\partial^2 p}{\partial T_{-2}^2}+\alpha_e\frac{\partial^2 p^2}{\partial T_{-2}^2}=0,$$

$$O(\varepsilon^{-5}):\ \frac{\partial}{\partial T_{-2}}\left(\frac{\partial^2 p}{\partial X_{-2}^2}-c_f^{-2}\frac{\partial^2 p}{\partial T_{-2}^2}+\alpha_f\frac{\partial^2 p^2}{\partial T_{-2}^2}\right)=0,$$

to be partially uncoupled. The two leading equations depend on T_0 and X_0 only, while the last two equations include the independent variables T_{-2} and X_{-2} only. Thus, the low-frequency perturbations are described by the two leading equations, and the high-frequency perturbations are described by the last two equations. An interaction between these perturbations is described by the three center equations.

Let us write out the motion equations for low-frequency and high-frequency perturbations in the initial variables x and t. For low-frequency perturbations the main terms $\partial^2 p/\partial x^2$ and $c_e^{-2}\partial^2 p/\partial t^2$ appear in the second equation of the system (2.11), while for high-frequency perturbations the main terms $\partial^2 p/\partial x^2$ and $c_f^{-2}\partial^2 p/\partial t^2$ appear in the seventh equation of (2.11).

For low-frequency perturbations ($\tau_p\omega\ll 1$) propagating in one direction, we obtain an evolution equation

$$\frac{\partial p}{\partial t}+c_e\frac{\partial p}{\partial x}+\alpha_e c_e^3 p\frac{\partial p}{\partial x}-\beta_e\frac{\partial^2 p}{\partial x^2}+\gamma_e\frac{\partial^3 p}{\partial x^3}=0,$$

$$\beta_e=\frac{c_e^2\tau_p}{2c_f^2}(c_f^2-c_e^2),\qquad \gamma_e=\frac{c_e^3\tau_p^2}{8c_f^4}(c_f^2-c_e^2)(c_f^2-5c_e^2). \tag{2.12}$$

This equation can be obtained in the following way. A dispersion relation for the linearized equation (2.10) can be written down with an accuracy $O(k^3)$ in the form $\omega=c_e k+i\beta_e k^2-\gamma_e k^3$, if the terms $c_e^{-1}\partial p/\partial t$ and $\partial p/\partial x$ are the main ones. For this dispersion relation we write a linear equation in which a nonlinear term is reconstructed in agreement with the initial equation.

The equation (2.12) is the well-known KdVB equation. It is encountered in many chapters of physics to describe nonlinear wave processes.[1] In Ref. 2 it was shown how hydrodynamic equations reduce to either the KdV or Burgers equation according to the choices for the state equation and the generalized force when analyzing the gasdynamical waves, waves in shallow water,[2] hydrodynamic waves in cold plasma,[13] or ion-acoustic waves in cold plasma.[14] The KdV equation ($\beta_e=0$) has stationary solutions (solitons). In the case of $\beta_e\neq 0$ the stationary solutions of the equation (2.12) are known also.[15]

For high-frequency perturbations ($\tau_p\omega\gg 1$), using the last two equations of the system (2.11), we get the following evolution equation:

$$\frac{\partial^2 p}{\partial x^2}-c_f^{-2}\frac{\partial^2 p}{\partial t^2}+\alpha_f c_f^2\frac{\partial^2 p^2}{\partial x^2}+\beta_f\frac{\partial p}{\partial x}+\gamma_f p=0,$$

$$\beta_f=\frac{c_f^2-c_e^2}{\tau_p c_e^2 c_f},\qquad \gamma_f=\frac{c_f^4-c_e^4}{2\tau_p^2 c_e^4 c_f^2}. \tag{2.13}$$

In addition to the nonlinear term with coefficient α_f, the equation has dissipative $\beta_f\,\partial p/\partial x$ and dispersive $\gamma_f p$ terms. If $\alpha_f=\beta_f=0$, this is a linear Klein–Gordon equation. There is a Green's function for this equation[16,17] that enables us to find the solution in quadrature, at least. Numerical solutions of the Klein–Gordon equation modeling the propagation of high-frequency perturbations in gas–liquid media have been presented in Ref. 17. Whitham's monograph[18] has also described a similar evolution equation for high-frequency perturbations, but its form coincides with that of Eq. (2.13) only when $\alpha_f=0$ and $\gamma_f=0$.

Landau and Lifshitz have shown that for high frequencies the dissipative term under high transport of heat agrees with corresponding term in the equation (2.13) (see section 79 and 81 in Ref. 7). Thus, the dynamic state equation (2.6) enable us to take into account the dissipative processes completely. But the form of the dissipative terms describing the inner exchange processes (transport of heat and momentum) are different for the high and low frequencies.

We call attention to the fact that the dispersion relations $\omega=\omega(k)$ for the linearized equations (2.12) and (2.13) have been restricted by the finite power series in k and in k^{-1}, respectively:

$$\omega=c_e k+i\beta_e k^2-\gamma_e k^3, \quad \tau_p\omega\ll 1,$$

$$c_f^{-2}\omega^2=k^2+i\beta_f k-\gamma_f, \quad \tau_p\omega\gg 1.$$

In the general case the equation (2.13) has been investigated insufficiently. It is likely that this is connected with the fact noted by Whitham[18] that the high-frequency perturbations attenuate very fast. However, in Ref. 18 the evolution equation without nonlinear and dispersive terms was considered. Certainly, the lack of such terms restricts the class of solutions. At least, there is no solution in the form of a solitary wave which is caused by nonlinearity and dispersion.

The studies of the equation (2.13) have some scientific interest both from the viewpoint of the investigation of the propagation of high-frequency perturbations and from the viewpoint of the existence of stable wave formations.

III. EVOLUTION EQUATION FOR HIGH-FREQUENCY PERTURBATIONS

The equation (2.13), which we are interested in, is written down in dimensionless form. Let us restrict our consideration to the propagation of high-frequency waves in positive direction x, then with necessary accuracy we can write the operator

$$\frac{\partial^2}{\partial x^2}-c_f^{-2}\frac{\partial^2}{\partial t^2}=\left(\frac{\partial}{\partial x}-c_f^{-1}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}+c_f^{-1}\frac{\partial}{\partial t}\right)\rightarrow 2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+c_f^{-1}\frac{\partial}{\partial t}\right)$$

(for example, see section 93 in Ref. 7). In the moving coordinates system with velocity c_f, the equation has the form in dimensionless variables $\tilde{x}=\sqrt{\gamma_f/2}(x-c_f t)$, $\tilde{t}=\sqrt{\gamma_f/2}c_f t$, $\tilde{u}=\alpha_f c_f^2 p$ (tilde over variables $\tilde{x},\tilde{t}$, $\tilde{u}$ is omitted):

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+\alpha\frac{\partial u}{\partial x}+u=0. \tag{3.1}$$

The constant $\alpha=\beta_f/\sqrt{2\gamma_f}$ is always positive.

The equation (3.1) without the dissipative term has the form of the nonlinear equation[3,4]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0. \tag{3.2}$$

These equations are related to that of Whitham[18] with the kernels $K(x)=\frac{1}{2}[\alpha(2\Theta(x)-1)+|x|]$ and $K(x)=\frac{1}{2}|x|$ (see Eq. (2) in Ref. 3) and are written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha u+\frac{1}{2}\int_{-\infty}^{\infty}|x-s|\frac{\partial u}{\partial s}ds=0, \tag{3.3}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{2}\int_{-\infty}^{\infty}|x-s|\frac{\partial u}{\partial s}ds=0, \tag{3.4}$$

where $\Theta(x)$ is a Heaviside function. There is no derivative in the dissipative term αu of Eq. (3.3).

Papers[3,4] are devoted to the analysis of the equation (3.2). In Ref. 4 it was named Vakhnenko's equation. The equation (3.2) has two families of traveling wave solutions.[3,4] In one case the solutions have looplike form (see Fig. 1 in Ref. 3). Only in this case is there a solitary wave solution. In Ref. 4 it is predicted that both families of solutions are stable to long wavelength perturbations. The existence of singular points, at which the derivatives tend to infinity, required the application of a nonstandard method.[4] The ambiguous structure is similar to the loop soliton solution to an equation that models a stretched rope.[19] The looplike solitons on a vortex filament were investigated by Hasimoto[20] and Lamb, Jr.[21]

The material described below deals with the ambiguous looplike solutions of the equation (3.1). From the mathematical point of view the ambiguous solution does not present difficulties while the physical interpretation of ambiguity always has some difficulties. In this connection the problem of ambiguous solutions is regarded to be important. The problem consists in whether the ambiguity has a physical nature or is related to the incompleteness of mathematical model, in particular to the lack of dissipation.

We will consider the problem related to the singular points when the dissipation takes place. At these points the dissipative term $\alpha\,\partial u/\partial x$ tends to infinity. The question arises: are there solutions of the equation (3.1) in a looplike form? That the dissipation is likely to destroy the looplike solutions can be associated with the following well-known fact.[1] For a simplest nonlinear equation without dispersion and dissipation,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0, \tag{3.5}$$

any initial smooth solution with boundary conditions

$$u|_{x\to+\infty}=0,\quad u|_{x\to-\infty}=u_0=\text{const}>0$$

becomes ambiguous in the final analysis. When the dissipation is considered, we have a Burgers equation[22]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}=0.$$

The dissipative terms of this equation and Eq. (2.13) for low frequency are coincided. The inclusion of the dissipative term transforms the solutions so that they cannot be ambiguous as a result of evolution. The wave parameters are always unambiguous. What happens in our case for the high frequency when the dissipative term has the form αu [Eq. (3.3)]? Will the inclusion of dissipation give rise to unambiguous solutions? It turns out that, and here this has been proved, the dissipative term, with a dissipation parameter less than some limit value, does not destroy the looplike solutions. A physical interpretation is given to ambiguous solutions.

IV. AMBIGUOUS SOLUTIONS

Let us pass to the coordinates in which the equation (3.2) has stationary periodic solutions [see Eq. (7) in Ref. 3]

$$\eta=x-vt,\quad \tau=t, \tag{4.1}$$

where v is a nonzero constant. Eq. (3.2) has looplike solutions when $v>0$. Parkes noted[4] that there are no stationary periodic solution of (3.2) when $v=0$. After substitution of

$$z=u-v$$

into Eq. (3.1) we get the evolution equation

$$z_{\tau\eta}+(zz_{\eta})_{\eta}+(z+v)+\alpha z_{\eta}=0. \tag{4.2}$$

We investigate the solution behavior within the neighborhood of singular points $z=0$ where $z_{\eta}\to\pm\infty$ and $z_{\tau}\ll z_{\eta}$. Therefore in the investigated equation (4.2) we neglect the term z in comparison with v, and also discard the term z_{τ}, to obtain

$$(zz_{\eta})_{\eta}+v+\alpha z_{\eta}=0. \tag{4.3}$$

It is convenient to use the inverse function $\eta=\eta(z)$. Taking into account $z_{\eta}=1/\eta_z$ and $z_{\eta\eta}=-\eta_{zz}/\eta_z^3$, Eq. (4.3) is rewritten as

$$-z\,\eta_{zz}+v\,\eta_z^3+\alpha\,\eta_z^2+\eta_z=0.$$

Introducing the definition $q\equiv\eta_z$, this equation can be integrated to obtain

$$\int\frac{dq}{q(vq^2+\alpha q+1)}=\int\frac{dz}{z}.$$

Depending on the sign of the quantity $1-\alpha^2/4v$, the latter expression has two different forms. We have introduced the critical value α^* of the parameter α defined by

$$\alpha^*=2\sqrt{v}. \tag{4.4}$$

For $\alpha<\alpha^*$ (i.e., $1-\alpha^2/4v>0$), we get

$$\ln\left[\frac{z^2}{q^2}(vq^2+\alpha q+1)\right]=-\frac{2\alpha}{\sqrt{4v-\alpha^2}}\mathrm{tg}^{-1}\frac{2vq+\alpha}{\sqrt{4v-\alpha^2}}+\ln c_1, \tag{4.5}$$

and for $\alpha>\alpha^*$ (i.e., $1-\alpha^2/4v<0$), we have

$$\ln\left[\frac{z^2}{q^2}(vq^2+\alpha q+1)\right]=\frac{\alpha}{\sqrt{\alpha^2-4v}}\ln\left|\frac{2vq+\alpha+\sqrt{\alpha^2-4v}}{2vq+\alpha-\sqrt{\alpha^2-4v}}\right|+\ln c_2. \tag{4.6}$$

We analyze the expression (4.5). First let us verify the special case $\alpha=0$. We have

$$\frac{z^2}{q^2}(vq^2+1)=c_1,$$

or

$$vz^2+\frac{1}{4}(z^2)_{\eta}^2=c_1.$$

Hence in the vicinity of $z=0$,

$$\eta+\eta_0=\pm\frac{1}{2}\int\frac{dz^2}{\sqrt{c_1-vz^2}}=\mp\frac{\sqrt{c_1-vz^2}}{v}.$$

We arrive at the result given in Ref. 3, namely that with the lack of dissipation ($\alpha=0$) the integral curves pass over an ellipse at $z=0$.

Now we investigate the case $0<\alpha<\alpha^*$. It is easy to show that the r.h.s. of (4.5) is always bounded for any value $q\equiv z_\eta^{-1}$. In the neighborhood of $z=0$ the r.h.s. of (4.5) is close to value

$$-\frac{2\alpha}{\sqrt{4v-\alpha^2}}\,\mathrm{tg}^{-1}\frac{\alpha}{\sqrt{4v-\alpha^2}}+\ln c_1\equiv\ln c_3 .$$

Consequently, we arrive at the equation

$$\frac{z^2}{q^2}(vq^2+\alpha q+1)=c_3 .$$

Even not integrating this equation, it is easy to show that at $z=0$ we must have $q=0$ since in general $c_3\neq 0$. This means that at $z=0$ the derivatives have the values

$$\eta_z=0,\ z_\eta=\pm\infty.$$

At $z=0$ the solution becomes ambiguous.

In the case $\alpha>\alpha^*$ there is the solution

$$z=0,\quad q=\eta_z\neq 0,\qquad z_\eta\neq\pm\infty.$$

In fact, at $z=0$ we obtain from the r.h.s. of (4.6)

$$q=\eta_z=-\frac{\alpha}{2v}-\frac{\sqrt{\alpha^2-4v}}{2v}\neq 0. \tag{4.7}$$

Thus, the derivative z_η at $z=0$ is bounded by a finite value. The solution is always unambiguous.

Let us consider the solution behavior in the neighborhood of $z=0$ as $\alpha\to\alpha^*$. We first consider the case $\alpha\to\alpha^*-0$. According to (4.5) the r.h.s. of this equation tends to minus infinity, i.e., at $z\approx 0$ we have $q=\eta_z\neq 0$. Consequently, there is no looplike solution.

When $\alpha\to\alpha^*+0$ there is also a solution with $q=\eta_z\neq 0$ at $z=0$. The root $q=0$ at $z=0$ seems possible in this case since (4.6) transforms to

$$\ln\left[\frac{z^2}{q^2}(vq^2+\alpha q+1)\right]=\frac{2\alpha}{2vq+\alpha}+\ln c_2 . \tag{4.8}$$

However, as appears from (4.7), the r.h.s. of the equation (4.8) tends to minus infinity so that $q\neq 0$ at $z=0$. Therefore, in the case $\alpha\to\alpha^*$ the dissipation destroys the looplike solutions.

We have proved the following statement. For values of $\alpha<\alpha^*$ the inclusion of the dissipative term does not change the looplike solutions of equation (3.1), while for $\alpha\geqslant\alpha^*$ there is no solution with an infinite gradient.

The common form of the dissipative term for high-frequency perturbations αu (which does not depend on the nature of the exchange processes) cannot preclude the possibility of a formation of a multi-valued solution from an initial single-valued profile. In this case there are the infinite gradients in contrast to the profiles of a wave for the low frequencies when the dissipative term has the form $\beta\,\partial^2u/\partial x^2$.

The problem of a multi-valued solution can be forestalled in a following way. The equation (3.2) can be rewritten into new independent variables $y=y(x,t)$ and $t_1=t$ so that the dependent variable $u=u(y,t_1)$ will be a single-valued function of y. These variables have been defined by the relationships

$$\varphi dy = dx - u dt, \quad t_1 = t$$

in which the equation (3.2) has been reduced to a system of the equations in the unknowns u and φ:

$$\frac{\partial \varphi}{\partial t_1} = \frac{\partial u}{\partial y}, \quad \frac{\partial^2 \varphi}{\partial t_1^2} + u\varphi = 0.$$

For example, for a one-soliton solution we have $\varphi = 1 - u/v$ [see Eqs. (12) and (14) in Ref. 3]. In the space of new variables y and t_1 a solution is a single-valued function [Eq. (13) from Ref. 3]. Each state has been uniquely defined by the variable y at any time t.

Considering the dependent variable u and the coordinate x as the functions of new variable y we solve the problem of the ambiguous solution. A number of the states with their thermodynamic parameters can occupy one microvolume, but these states are distinguished by the coordinate y. It is assumed that the interaction between the separated states occupying one microvolume can be neglected in comparison with the interaction between the particles of one thermodynamic state. Even if we shall take into account the interaction between the separated states in accord with the dynamic state equation (2.6), then for high frequencies the dissipative term arises which is similar to the corresponding term in Eq. (2.13), but with the other relaxation time. In this sense the separated terms are distributed in space, but describing the wave process we consider them as interpenetratable. The similar situation, when several components with different hydrodynamic parameters occupied one microvolume, has been assumed in the mixture theory (see, for instance Refs. 23 and 24). Such a fundamental assumption in the theory of mixtures is physically impossible (see Ref. 23, p. 7), but it is appropriate in the sense that separated components are multi-velocity interpenetratable continua.

Thus in the frameworks of this model approach, the high-frequency perturbation can be described by the multi-valued functions.

V. CONCLUSIONS

The KdV and KdVB equations are employed to describe a number of evolution processes when the low-frequency approach turns out to be adequate. In these cases thermodynamic parameters of a medium are close to the equilibrium values, the microvolume state is defined by one set of thermodynamic values, and the disturbance from the equilibrium is taken into account by means of expansion in gradients.[25] If the low-order expansions within the framework of such an approach give rise to an inadequate description, we could take into account the terms of higher order and as a result consider higher frequencies. For example, if Eq. (3.5) has an ambiguous solution (or discontinuous solution), the improvement of models by means of adding higher degree derivatives excludes the ambiguous solutions. So, in the low-frequency approach, an ambiguity is connected with the incompleteness of the mathematical model.

In contrast to this, in models for the propagation of high-frequency perturbations the disturbance from the ''frozen'' state is taken into account by means of expansion in integral terms [see Eq. (3.3) and (3.4)]. The integral terms contain the prehistory of the process. We have just established that a higher order of expansion (in particular, the dissipative term) for the high-frequency evolution equation still allows ambiguous solutions. Consequently, the ambiguity of solution does not relate to the incompleteness of the mathematical model, in particular to the lack of dissipation. In addition there is the space of new independent variables where the solution is the single-valued function.

The following three circumstances show that in the framework of the approach considered here there are the multi-valued solutions when we model the high-frequency wave processes: (1) All parts of looplike solutions are stable to perturbations; this was proved by Parkes.[4] (2) The dissipation does not destroy the looplike solutions (the result of this work). (3) The investigation regarding the interaction of the solitons has shown that it is necessary to take into account the whole ambiguous solution, and not just the separate parts.

It is necessary to note that the substantiation of the nonlinear evolution equation (3.1) within the framework of statistic physics remains an important problem. At present this problem is too difficult since it is connected with the description of high nonequilibrium systems.

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