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ANALYTICAL ROBUST DESIGN OF MECHANICAL SYSTEMS

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ABSTRACT
Based on general principles of robust design and axiomatic design, relationship among robustness, structural parameters, design parameters and uncontrollable factors has been established. Various factors that affect system robustness were analyzed mathematically to determine the relationship between robustness and structural characteristics of the linear system. The relations among functional requirements were also explored. Accordingly, an optimization model was established to determine design parameters. This new robust design approach can be used for linear mechanical system analysis.

Keywords: *Quality, Robust, Sensitivity*.

1 INTRODUCTION
Quality is a primary factor in determining whether a product is successful in the market place. It can be evaluated if the product performs the intended functions. The intended functionality may be deviated by variations resulted from raw materials, manufacturing processes, and/or operational environments. To minimize the effects of the variations on functions, the functions pf product and system should be made insensitive to those variations. Dr.Taguchi has proposed robust design in 1970s' and this method has been widely used in industry [1, 2]. When Taguchi method is used for early design stage, there are a few limitations:

1) Taguchi method usually requires experimental data. It is not very convenient or practical to obtain sufficient experimental data at early stage of design process. For example, for products with low production volume, it is impractical to obtain sufficient data for analysis.
2) The experimental based method normally relies on the experience of engineers. The engineers' experience and analysis based on a limited number of tests may lead to a group of partial optimal solutions.
3) The internal relationships between system robustness and various parameters may not be directly revealed by such experimental method. When an improvement is required for functions of product, it is difficult to maintain the original robustness.

Analytical robust design approach is a new method for design of robust mechanical systems. Through mapping from design parameters to functional requirements of system, this method aims to analyze the intrinsic relationships among structural characteristics, design parameters, uncontrollable factors and robustness of linear system. Based on Suh's Axiomatic Design and robust analysis of traditional robust design approach [3], this paper reports on studies of models of linear mechanical system for robust analysis, sensitivity index of system, and describes a optimal model of analytical robust design, reveals the primary factors on robustness. Examples were included to demonstrate this new approach.

2 ROBUST ANALYSIS

2.1 Models
The factors that influence functions of linear system could be divided into controllable factors and uncontrollable factors. It is important to distinguish these factors appropriately and establish analytical models accordingly.

Based on the previous work [3] and according to Axiomatic Design theory [4], for a product, if **Fr** represent function requirements, **Dp** represent design parameters, the performance function can be expressed as below:

$$\mathbf{Fr} = \mathbf{D} \cdot \mathbf{Dp} \tag{1}$$

where $\mathbf{Fr} = [\mathrm{Fr}_1, \mathrm{Fr}_2, \ldots, \mathrm{Fr}_n]^T$ $\mathbf{Dp} = [\mathrm{Dp}_1, \mathrm{Dp}_2, \ldots, \mathrm{Dp}_m]^T$, **D** is a $n \times m$ design matrix and $\mathrm{D}_{ij} = \partial \mathrm{Fr}_i / \partial \mathrm{Dp}_j$.

Here, design matrix **D** can be regarded as controllable factors reflecting structural characteristics. Means of design parameters $E(\mathbf{Dp})$ can be adjusted , if the following relationship is held:

$$\Delta\mathbf{Dp} = \mathbf{Dp} - E(\mathbf{Dp}) \tag{2}$$

where $\Delta\mathbf{Dp}$ denotes design deviation caused by environmental variations, in the following analysis it will be regarded as uncontrollable factors. Therefore, changes of function requirements caused by uncontrollable factors can be expressed below:

$$\Delta\mathbf{Fr} = \mathbf{D} \cdot \Delta\mathbf{Dp} \tag{3}$$

In this study, the following definitions will be made:

Means:

$$E(\Delta\mathbf{Fr}) = [E(\Delta\mathrm{Fr}_1), E(\Delta\mathrm{Fr}_2), \ldots, E(\Delta\mathrm{Fr}_n)]^T$$
$$E(\Delta\mathbf{Dp}) = [E(\Delta\mathrm{Dp}_1), E(\Delta\mathrm{Dp}_2), \ldots, E(\Delta\mathrm{Dp}_m)]^T$$

Variance:

$$Var(\Delta\mathbf{Fr}) = E\{[\Delta\mathbf{Fr} - E(\Delta\mathbf{Fr})]^2\}$$
$$Var(\Delta\mathbf{Dp}) = E\{[\Delta\mathbf{Dp} - E(\Delta\mathbf{Dp})]^2\}$$

Variance-Covariance:

$$VC(\Delta\mathbf{Fr}) = Cov(\Delta\mathbf{Fr}, \Delta\mathbf{Fr}) = E\{[\Delta\mathbf{Fr} - E(\Delta\mathbf{Fr})]\,[\Delta\mathbf{Fr} - E(\Delta\mathbf{Fr})]^T\}$$
$$VC(\Delta\mathbf{Dp}) = Cov(\Delta\mathbf{Dp}, \Delta\mathbf{Dp}) = E\{[\Delta\mathbf{Dp} - E(\Delta\mathbf{Dp})]\,[\Delta\mathbf{Dp} - E(\Delta\mathbf{Dp})]^T\}$$

By using the above relationships, the following equation has been established:

$$\begin{aligned} VC(\Delta\mathbf{Fr}) &= VC(\mathbf{D} \cdot \Delta\mathbf{Dp}) \\ &= E[\,(\mathbf{D} \cdot \Delta\mathbf{Dp})(\,\mathbf{D} \cdot \Delta\mathbf{Dp})^T] \\ &= \mathbf{D} \cdot E[\Delta\mathbf{Dp} \cdot \Delta\mathbf{Dp}^T] \cdot \mathbf{D}^T \\ &= \mathbf{D} \cdot VC(\Delta\mathbf{Dp}) \cdot \mathbf{D}^T \end{aligned} \tag{4}$$

2.1 System sensitivity

Robust design aims to make functions of system insensitive to uncontrollable factors through selecting system structure and design parameters properly [2]. Accordingly, if a system is robust, its performance functions should be insensitive to uncontrollable factors, i.e. sensitivity index $S_v^2 = \sigma_D^2 / \sigma_F^2$ should be a small value; where σ_F^2 represents the variance of functional requirements and σ_D^2 represents the variance of uncontrollable factors [3].

In real engineering systems, the mathematical characteristics of uncontrollable factors can be obtained through mathematical analysis. In this study it is assumed that uncontrollable factors $\Delta\mathrm{Dp}_i$ $i = 1, 2, \ldots, m$ are mutually independent, and have normal distributions, furthermore, variance of each uncontrollable factor can be expressed as follows:

$$Var(\Delta\mathrm{Dp}_i) = \varepsilon_i^2\,\sigma_D^2 \tag{5}$$

where ε_i $i = 1, 2, ..., m$ are constants larger than 0.

From Equation (3), changes of functional requirements caused by uncontrollable factors can be represented as:

$$\begin{bmatrix} \Delta\mathrm{Fr}_1 \\ \Delta\mathrm{Fr}_2 \\ \vdots \\ \Delta\mathrm{Fr}_n \end{bmatrix} = \begin{bmatrix} \mathrm{D}_{11} & \mathrm{D}_{12} & \cdots & \mathrm{D}_{1m} \\ \mathrm{D}_{21} & \mathrm{D}_{22} & \cdots & \mathrm{D}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{D}_{n1} & \mathrm{D}_{n2} & \cdots & \mathrm{D}_{nm} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathrm{Dp}_1 \\ \Delta\mathrm{Dp}_2 \\ \vdots \\ \Delta\mathrm{Dp}_m \end{bmatrix} \tag{6}$$

According to Equation (5), the following transform can be made:

$$\begin{bmatrix} \Delta\mathrm{Fr}_1 \\ \Delta\mathrm{Fr}_2 \\ \vdots \\ \Delta\mathrm{Fr}_n \end{bmatrix} = \begin{bmatrix} \varepsilon_1\mathrm{D}_{11} & \varepsilon_2\mathrm{D}_{12} & \cdots & \varepsilon_m\mathrm{D}_{1m} \\ \varepsilon_1\mathrm{D}_{21} & \varepsilon_2\mathrm{D}_{22} & \cdots & \varepsilon_m\mathrm{D}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_1\mathrm{D}_{n1} & \varepsilon_2\mathrm{D}_{n2} & \cdots & \varepsilon_m\mathrm{D}_{nm} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathrm{Dp}_1/\varepsilon_1 \\ \Delta\mathrm{Dp}_2/\varepsilon_2 \\ \vdots \\ \Delta\mathrm{Dp}_m/\varepsilon_m \end{bmatrix} \tag{7}$$

Referring to Equation (3), we have:

$$\Delta\mathbf{Fr} = \mathbf{D'} \cdot \Delta\mathbf{Dp'} \tag{8}$$

where

$$\mathbf{D}' = \begin{bmatrix} \varepsilon_1\mathrm{D}_{11} & \varepsilon_2\mathrm{D}_{12} & \cdots & \varepsilon_m\mathrm{D}_{1m} \\ \varepsilon_1\mathrm{D}_{21} & \varepsilon_2\mathrm{D}_{22} & \cdots & \varepsilon_m\mathrm{D}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_1\mathrm{D}_{n1} & \varepsilon_2\mathrm{D}_{n2} & \cdots & \varepsilon_m\mathrm{D}_{nm} \end{bmatrix}$$

$$\Delta\mathbf{Dp'} = [\Delta\mathrm{Dp}_1 / \varepsilon_1, \Delta\mathrm{Dp}_2 / \varepsilon_2, \ldots, \Delta\mathrm{Dp}_m / \varepsilon_m]^T$$

As uncontrollable factors are expressed by $\Delta\mathrm{Dp}_i = \mathrm{Dp}_i - E(\mathrm{Dp}_i)$ $i = 1, 2, \ldots, m$ we know:

$$E(\Delta\mathbf{Dp'}) = [E(\Delta\mathrm{Dp}_1/\varepsilon_1), E(\Delta\mathrm{Dp}_2/\varepsilon_2), \ldots, E(\Delta\mathrm{Dp}_m/\varepsilon_m)]^T = 0 \tag{9}$$

$$E(\Delta\mathrm{Fr}_j) = E[\mathrm{D}_{j1}\Delta\mathrm{Dp}_1 + \mathrm{D}_{j2}\Delta\mathrm{Dp}_2 + \ldots + \mathrm{D}_{jm}\Delta\mathrm{Dp}_m] = 0 \tag{10}$$

where $j = 1, 2, \ldots, n$

As it is assumed that uncontrollable factors $\Delta\mathrm{Dp}_i$ $i = 1, 2, \ldots, m$ are mutually independent, and have normal distributions (for other distributions, corresponding methods can also be developed with similar approach). By bringing Equations (5) and (9) together, we obtain:

$$\begin{aligned} VC(\Delta\mathbf{Dp'}) &= E\{[\Delta\mathbf{Dp'} - E(\Delta\mathbf{Dp'})]\,[\Delta\mathbf{Dp'} - E(\Delta\mathbf{Dp'})]^T\} \\ &= \{[\Delta\mathbf{Dp'}]\,[\Delta\mathbf{Dp'}]^T\} \\ &= \sigma_D^2 \cdot \mathbf{I} \end{aligned} \tag{11}$$

According to Equation (4), it is found:

$$\begin{aligned} VC(\Delta\mathbf{Fr}) &= \mathbf{D'} \cdot VC(\Delta\mathbf{Dp'}) \cdot \mathbf{D'}^T \\ &= \mathbf{D'} \cdot \sigma_D^2 \cdot \mathbf{I} \cdot \mathbf{D'}^T \\ &= \sigma_D^2\,\mathbf{D'} \cdot \mathbf{D'}^T \end{aligned} \tag{12}$$

Left side of Equation (12)

$$VC(\Delta\mathbf{Fr}) =$$

$$\begin{bmatrix} E\{[\Delta Fr_1 - E(\Delta Fr_1)]^2\} & E\{[\Delta Fr_1][\Delta Fr_2]\} & \cdots & E\{[\Delta Fr_1][\Delta Fr_n]\} \\ E\{[\Delta Fr_1][\Delta Fr_2]\} & E\{[\Delta Fr_2 - E(\Delta Fr_2)]^2\} & \cdots & E\{[\Delta Fr_2][\Delta Fr_n]\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{[\Delta Fr_1][\Delta Fr_n]\} & E\{[\Delta Fr_2][\Delta Fr_n]\} & \cdots & E\{[\Delta Fr_n - E(\Delta Fr_n)]^2\} \end{bmatrix} = \begin{bmatrix} Var(\Delta Fr_1) & E\{[\Delta Fr_1][\Delta Fr_2]\} & \cdots & E\{[\Delta Fr_1][\Delta Fr_n]\} \\ E\{[\Delta Fr_1][\Delta Fr_2]\} & Var(\Delta Fr_2) & \cdots & E\{[\Delta Fr_2][\Delta Fr_n]\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{[\Delta Fr_1][\Delta Fr_n]\} & E\{[\Delta Fr_2][\Delta Fr_n]\} & \cdots & Var(\Delta Fr_n) \end{bmatrix} \quad (13)$$

Right side of Equation (12)

$$\sigma_D^2\, \mathbf{D}' \cdot \mathbf{D}'^{\mathrm{T}} = \sigma_D^2 \begin{bmatrix} \varepsilon_1 D_{11} & \varepsilon_2 D_{12} & \cdots & \varepsilon_m D_{1m} \\ \varepsilon_1 D_{21} & \varepsilon_2 D_{22} & \cdots & \varepsilon_m D_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_1 D_{n1} & \varepsilon_2 D_{n2} & \cdots & \varepsilon_m D_{nm} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 D_{11} & \varepsilon_1 D_{21} & \cdots & \varepsilon_1 D_{n1} \\ \varepsilon_2 D_{12} & \varepsilon_2 D_{22} & \cdots & \varepsilon_2 D_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_m D_{1m} & \varepsilon_m D_{2m} & \cdots & \varepsilon_m D_{nm} \end{bmatrix} \quad (14)$$

If consider elements on diagonal of the matrix only, we have,

$$\sigma_D^2\, \mathbf{D'} \cdot \mathbf{D'}^{\mathrm{T}} = \sigma_D^2 \begin{bmatrix} \varepsilon_1^2 D_{11}^2 + \varepsilon_2^2 D_{12}^2 + \ldots + \varepsilon_m^2 D_{1m}^2 & . & . & . \\ . & \varepsilon_1^2 D_{21}^2 + \varepsilon_2^2 D_{22}^2 + \ldots + \varepsilon_m^2 D_{2m}^2 & . & . \\ . & . & \ddots & . \\ . & . & . & \varepsilon_1^2 D_{n1}^2 + \varepsilon_2^2 D_{n2}^2 + \ldots + \varepsilon_m^2 D_{nm}^2 \end{bmatrix} \quad (15)$$

As we know,

$$Var(\Delta Fr_i) = Var[Fr_i - E(Fr_i)] = Var(Fr_i) \quad (16)$$

If σ_F^2 denotes variance of functional requirement, the following relations are held:

$$\sigma_F^2 = Var(Fr_i) = Var(\Delta Fr_i) \quad i = 1, 2, \ldots, n \quad (17)$$

In this study it is assumed that variances of functional requirements are identical (if there is a significant difference among variations, the following analysis would be different), then, bringing Equations (12), (13), (15), (16) and (17) together, we obtain:

$$\begin{bmatrix} \sigma_F^2 & E\{[\Delta Fr_1][\Delta Fr_2]\} & \cdots & E\{[\Delta Fr_1][\Delta Fr_n]\} \\ E\{[\Delta Fr_1][\Delta Fr_2]\} & \sigma_F^2 & \cdots & E\{[\Delta Fr_2][\Delta Fr_n]\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{[\Delta Fr_1][\Delta Fr_n]\} & E\{[\Delta Fr_2][\Delta Fr_n]\} & \cdots & \sigma_F^2 \end{bmatrix}$$
$$= \sigma_D^2 \begin{bmatrix} \varepsilon_1^2 D_{11}^2 + \varepsilon_2^2 D_{12}^2 + \ldots + \varepsilon_m^2 D_{1m}^2 & . & . & . \\ . & \varepsilon_1^2 D_{21}^2 + \varepsilon_2^2 D_{22}^2 + \ldots + \varepsilon_m^2 D_{2m}^2 & . & . \\ . & . & \ddots & \vdots \\ . & . & . & \varepsilon_1^2 D_{n1}^2 + \varepsilon_2^2 D_{n2}^2 + \ldots + \varepsilon_m^2 D_{nm}^2 \end{bmatrix} \quad (18)$$

Comparing with diagonal elements of Equation (18), we have:

$$\begin{aligned} & \varepsilon_1^2 D_{11}^2 + \varepsilon_2^2 D_{12}^2 + \ldots + \varepsilon_m^2 D_{1m}^2 \\ = {} & \varepsilon_1^2 D_{21}^2 + \varepsilon_2^2 D_{22}^2 + \ldots + \varepsilon_m^2 D_{2m}^2 \\ & \ldots \\ = {} & \varepsilon_1^2 D_{n1}^2 + \varepsilon_2^2 D_{n2}^2 + \ldots + \varepsilon_m^2 D_{nm}^2 \end{aligned} \quad (19)$$

and sensitivity index obtains the following relation:

$$S_v^2 = \sigma_F^2 / \sigma_D^2 = \frac{\|\mathbf{D}'\|_{\mathrm{F}}^2}{n} \quad (20)$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the standard Frobenius norm.

3 ROBUST DESIGN

In addition to satisfy expected functional requirements, robust design needs to make deviations of functional requirements insensitive to uncontrollable factors resulted from materials, manufacturing, operational environment and so on. Sensitivity index is defined as follows:

$$S_v^2 = \sigma_F^2 / \sigma_D^2$$

$$= \varepsilon_1^2 \frac{Var(\Delta Fr_i)}{Var(\Delta Dp_1)} = \varepsilon_2^2 \frac{Var(\Delta Fr_i)}{Var(\Delta Dp_2)} = \ldots = \varepsilon_m^2 \frac{Var(\Delta Fr_i)}{Var(\Delta Dp_m)} \quad (21)$$

where $i = 1, 2, \ldots, n$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ are all proportional constants larger than 0. Thus, we obtain the same monotonicity.

$$\frac{Var(\Delta Fr_i)}{Var(\Delta Dp_1)}, \frac{Var(\Delta Fr_i)}{Var(\Delta Dp_2)}, \ldots, \frac{Var(\Delta Fr_i)}{Var(\Delta Dp_m)} \quad (22)$$

Because of

$$\frac{\sigma_F^2}{\sigma_D^2} = \varepsilon_j^2 \frac{Var(\Delta Fr_i)}{Var(\Delta Dp_j)} \quad (23)$$

where $i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, m$

If sensitivity index S_v^2 has the minimum value, influences caused by uncontrollable factors on variance of functional requirements will become minimum. According to basic idea of robust design [2], the lesser changes of functional requirements under influence of uncontrollable factors, the more robustness system achieved. From the perspective of probability and statistics, robust design means that performance function of system concentrates near an expected value with a high probability.

$E(\mathbf{Fr})$ represents the expected value of functional requirement **Fr**. According to the definition of variance, we know that σ_F^2 reveals the deviations of function from its expected value $E(\mathbf{Fr})$. If **Fr** concentrates near $E(\mathbf{Fr})$, σ_F^2 is small. Otherwise, it means a large dispersion, and σ_F^2 will be large. Therefore, σ_F^2 describes decentralization of functional requirements of system comparing to its expected value. It can be used to measure robustness of system.

According to Equation (20) we have:

$$\sigma_F^2 = \sigma_D^2 \frac{\left\|\mathbf{D}'\right\|_F^2}{n} \quad (24)$$

Therefore, according to the analysis above, we can establish a robust optimization model as follows:

$$\textit{Minimize} \quad \sigma_F^2 = \sigma_D^2 \frac{\left\|\mathbf{D}'\right\|_F^2}{n} \quad (25)$$

$$\textit{subject to:} \quad \varepsilon_1^2 D_{j1}^2 + \varepsilon_2^2 D_{j2}^2 + \ldots + \varepsilon_m^2 D_{jm}^2 = \varepsilon_1^2 D_{k1}^2 + \varepsilon_2^2 D_{k2}^2 + \ldots + \varepsilon_m^2 D_{km}^2 \quad (26)$$

where **D'** is a $n \times m$ transformed design matrix, D_{ij} are elements of design matrix **D**, and

$$\mathbf{D}' = \begin{bmatrix} \varepsilon_1 D_{11} & \varepsilon_2 D_{12} & \cdots & \varepsilon_m D_{1m} \\ \varepsilon_1 D_{21} & \varepsilon_2 D_{22} & \cdots & \varepsilon_m D_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_1 D_{n1} & \varepsilon_2 D_{n2} & \cdots & \varepsilon_m D_{nm} \end{bmatrix}$$

ε_i are constants larger than 0, $i = 1, 2, \ldots, m$ $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, n$; $j \neq k$

If $\boldsymbol{s} = [s_1, s_2, \ldots, s_p]^T$ represents system's structural characteristics, design matrix **D** will be a function of $\boldsymbol{s}$. Therefore, **D** reflects and determines structural characteristics of system. As ε_i $i = 1, 2, \ldots, m$ have direct relations with variance of uncontrollable factors, then transformed design matrix **D'** represents system's structural characteristics, also donates relationships of uncontrollable factors.

The analyses indicate that sensitivity index S_v^2 and variance of functional requirements σ_F^2 have direct relationships with system's structural characteristics and uncontrollable factors. However, no obvious relationships with means of design parameters have been determined. According to analysis of system sensitivity and the robust optimization model, it was found that system's robustness is mainly determined by system's structural characteristics as well as uncontrollable factors, but does not have obvious relationships with mean of design parameter $E(\mathbf{Dp})$ with under this system model.

4 DISCUSSION

Angeles [5] has qualitatively discussed the relationships between system robustness and $\mathbf{D} \cdot \mathbf{D}^T$ which relative to $\|\mathbf{D}'\|_F$ based on Suh's Axiomatic Design and the traditional robust design technique. Integrations of the independent analysis and robust analysis have been introduced in reference [3]. But all those discussions were base on the conditions that standard variations of both uncontrollable factors and changes on functional requirements have uniform values for σ_F and σ_D respectively, and their Variance and Covariance are simply considered to be isotropic. In this paper, those preconditions have been removed as they may be obstacles in the real engineering projects. We conducted an analysis on constitution of robustness. Blow is the discussion about relations of functional requirements which discussed in [3] on point of the Axiomatic Design.

For a system of multi-purpose functional requirements, as the functional changes can be expressed as:

$$\Delta Fr_i = Fr_i - E(Fr_i) \quad (27)$$

the following relationship can be established:

$$E\{[\Delta Fr_i][\Delta Fr_j]\} = E\{[Fr_i - E(Fr_i)][Fr_j - E(Fr_j)]\} = Cov(Fr_i, Fr_j) \quad (28)$$

where $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, n$, $i \neq j$

From Equation (13), the following relationships about functional requirements hold:

$$VC(\Delta\mathbf{Fr}) = \begin{bmatrix} Var(\Delta Fr_1) & E\{[\Delta Fr_1][\Delta Fr_2]\}\cdots & E\{[\Delta Fr_1][\Delta Fr_n]\} \\ E\{[\Delta Fr_1][\Delta Fr_2]\} & Var(\Delta Fr_2) \quad \cdots & E\{[\Delta Fr_2][\Delta Fr_n]\} \\ \vdots & \vdots \quad \ddots & \vdots \\ E\{[\Delta Fr_1][\Delta Fr_n]\} & E\{[\Delta Fr_2][\Delta Fr_n]\}\cdots & Var(\Delta Fr_n) \end{bmatrix}$$

$$= \begin{bmatrix} Var(Fr_1) & Cov(Fr_1, Fr_2)\cdots & Cov(Fr_1, Fr_n) \\ Cov(Fr_1, Fr_2) & Var(Fr_2) \quad \cdots & Cov(Fr_2, Fr_n) \\ \vdots & \vdots \quad \ddots & \vdots \\ Cov(Fr_1, Fr_n) & Cov(Fr_2, Fr_n)\cdots & Var(Fr_n) \end{bmatrix} \quad (29)$$

Through Equation (29), we know that Variance-Covariance matrix of changes of function requirements indicates mutual relationships of functional requirements. In order to fulfill requirements from real engineering project, more systematic and comprehensive analyses are required to develop mutual relationships of functional requirements, and provide more information necessary for the Axiomatic Design. For example, if in real engineering systems, it is acceptable to make the functional requirement Fr_i and Fr_j mutually independent, correlation coefficient ρ will be 0, that is equal to $Cov(Fr_i, Fr_j) = 0$ (as the following analysis in Case 2 of the Example Section of the paper); if functional requirements Fr_i and Fr_j follow a linear relationship, that means $Cov(Fr_i, Fr_j)$ equal to $k\sigma_F^2$, where k is coefficient of linear term, σ_F^2 represents variance of functional requirement.

5 EXAMPLES

Case 1: Figure 1 is a schematic drawing of a link with bolts under force F_0 and moment M. E is gravity center of the link, x_1 and x_2 represent distance of force F_0 from two bolts respectively, and y_1 and y_2 represent distance of the gravity center from two bolts respectively. Robust design of this simple component was carried out for the two bolts.

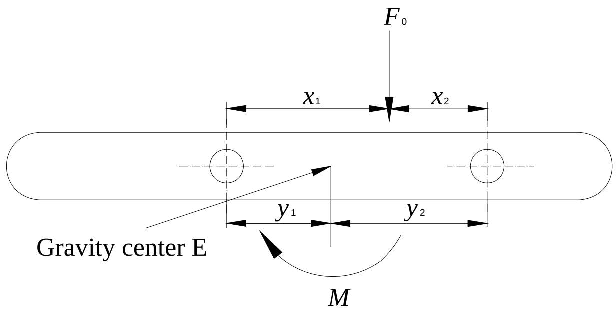

Figure 1. schematic drawing of link under F_0 and M (overlook)

In this case, it is assumed that two bolts under influence of force F_0 and moment M will respectively have two react forces F_1 and F_2 on link. For analysis purpose, it is considered the force F_1 is composed by F_{1F} and F_{1M} respectively, where F_{1F} is caused by F_0 and F_{1M} is cause by moment M. likewise, F_2 is composed by F_{2F} (caused by F_0)and F_{2M} (caused by M), as Figure 2 and Figure 3 show. Thus, under force F_0 (Figure 2), one can obtain:

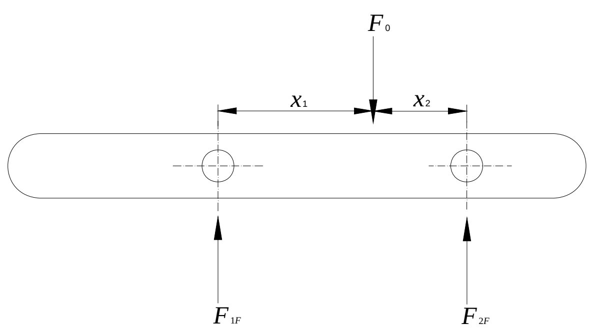

Figure 2. analysis under force F_0 (overlook)

$$F_0 = F_{1F} + F_{2F} \tag{30}$$

$$F_{1F}(x_1 + x_2) = F_0\, x_2 \tag{31}$$

By using Equations (30) and (31), we obtain:

$$F_{1F} = \frac{x_2 F_0}{x_1 + x_2} \tag{32}$$

$$F_{2F} = \frac{x_1 F_0}{x_1 + x_2} \tag{33}$$

Under moment M (Figure 3),

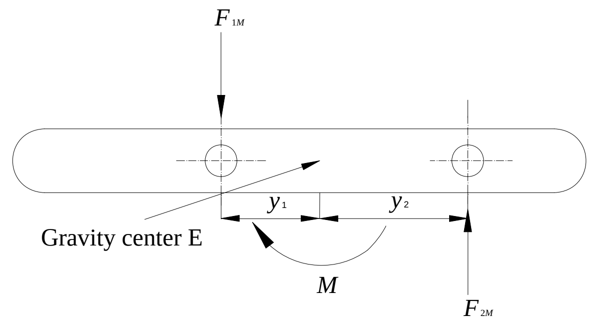

Figure 3. analysis under moment M (overlook)

If neglecting the friction forces, we obtain:

$$M = F_{1M}(y_1 + y_2) \tag{34}$$

$$F_{1M} = F_{2M} \tag{35}$$

By using Equations (34) and (35), we have:

$$F_{1M} = \frac{M}{y_1 + y_2} \tag{36}$$

$$F_{2M} = \frac{M}{y_1 + y_2} \tag{37}$$

According to force synthesis, the following relations will be hold,

$$F_1 = F_{1F} - F_{1M} = \frac{x_2 F_0}{x_1 + x_2} - \frac{M}{y_1 + y_2} \tag{38}$$

$$F_2 = F_{2F} + F_{2M} = \frac{x_1 F_0}{x_1 + x_2} + \frac{M}{y_1 + y_2} \tag{39}$$

then we have:

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2}{x_1 + x_2} & -\frac{1}{y_1 + y_2} \\ \frac{x_1}{x_1 + x_2} & \frac{1}{y_1 + y_2} \end{bmatrix} \cdot \begin{bmatrix} F_0 \\ M \end{bmatrix} \tag{40}$$

and,

$$\begin{bmatrix} \Delta F_1 \\ \Delta F_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2}{x_1 + x_2} & -\frac{1}{y_1 + y_2} \\ \frac{x_1}{x_1 + x_2} & \frac{1}{y_1 + y_2} \end{bmatrix} \cdot \begin{bmatrix} \Delta F_0 \\ \Delta M \end{bmatrix} \tag{41}$$

from Equation (26) we obtain:

$$\left(\frac{x_2}{x_1 + x_2}\right)^2 + \left(-\frac{1}{y_1 + y_2}\right)^2 = \left(\frac{x_1}{x_1 + x_2}\right)^2 + \left(\frac{1}{y_1 + y_2}\right)^2 \tag{42}$$

when $x_1 = x_2$, equation (42) is tenable. No matter to assume that $x_1 + x_2 = y_1 + y_2 = l$, so

$$0 < l \le L \tag{43}$$

where L is the whole length of link.

Bring with Equations (24), (41) and (42) together, we obtain:

$$\sigma_F{}^2 = \sigma_D{}^2 \left[\left(\frac{x_1}{x_1 + x_2}\right)^2 + \left(\frac{1}{y_1 + y_2}\right)^2 \right]$$

$$= \sigma_D^{\;2}\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{y_1 + y_2}\right)^2\right]$$

$$= \sigma_D^{\;2}\left(\frac{1}{4} + \frac{1}{l^2}\right) \tag{44}$$

if $l = L$, $S_v^2 = \sigma_D^2 / \sigma_F^2$ and σ_F^2 will obtain the minimum value, then:

$$x_1 = x_2 = y_1 = y_2 = L / 2 \tag{45}$$

thus the system will become robust as the Figure 4 shows.

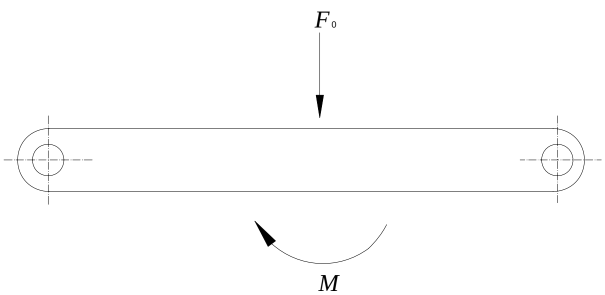

Figure 4. schematic drawing of link under robust condition (overlook)

Then, the design matrix is:

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{L} \\ \frac{1}{2} & \frac{1}{L} \end{bmatrix} \tag{46}$$

From the design matrix above, if our early robust approach introduced in [3] were used, it would be difficult to obtain the result of robust design.

For a linear system, system sensitivity should be defined in relation to deviations of design parameters. An analysis should be carried out to investigate relationships between sensitivity of system and structural characteristics and uncontrollable factors. In this case, when the system is robust, $l = L$. with Equation (44) we obtain:

$$\frac{\partial \sigma_F^{\;2}}{\partial l} = -\frac{2\sigma_D^{\;2}}{L^3} \tag{47}$$

$$\frac{\partial \sigma_F^{\;2}}{\partial \sigma_D^{\;2}} = \left(\frac{1}{4} + \frac{1}{L^2}\right) \tag{48}$$

Equations (47) and (48) directly reflect sensitivity of system robustness relative to structural characteristics as well as the uncontrollable factors. With Equation (44), it can be found that, when the link is longer, variance of functional requirements would be smaller, and more robustness of the system would be achieved. Furthermore, under these conditions, system sensitivity of robustness relative to structural characteristics as well as uncontrollable factors would have smaller value. At the same time, robustness of system would improve along with increasing of L as well as reducing of variance of uncontrollable factors. Otherwise, when L is of a smaller value, robustness of system would be lower along with reducing of L as well as increasing of variance of the uncontrollable factors.

Case 2: A rigid crossbeam under forces of F_{s1}, F_{s2} is shown in Figure 5. The whole length of the rigid crossbeam is L. In order to be balanced, there needs to exert support forces F_1 and F_2 in both sides of crossbeam separately. As we know, F_{s1} and F_{s2} are mutually independent and follow normal distributions, with values of $F_{s1} \sim N(1000, 10^2)$, $F_{s2} \sim N(1000, 10^2)$. The design was carried out to determine support forces F_1, F_2 for robust structure of this system.

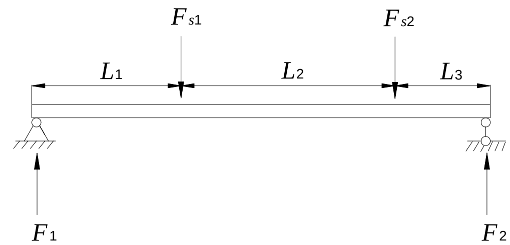

Figure 5. schematic drawing of rigid crossbeam under forces F_{s1} and F_{s2}

According to the system equilibrium condition, the following relationships hold:

$$F_1 L = F_{s1} (L_2 + L_3) + F_{s2} L_3 \tag{49}$$

$$F_2 L = F_{s2} (L_1 + L_2) + F_{s1} L_1 \tag{50}$$

By bringing Equations (49) and (50) together, we have:

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{L_2 + L_3}{L} & \frac{L_3}{L} \\ \frac{L_1}{L} & \frac{L_1 + L_2}{L} \end{bmatrix} \cdot \begin{bmatrix} F_{s1} \\ F_{s2} \end{bmatrix} \tag{51}$$

and

$$\begin{bmatrix} \Delta F_1 \\ \Delta F_2 \end{bmatrix} = \begin{bmatrix} \frac{L_2 + L_3}{L} & \frac{L_3}{L} \\ \frac{L_1}{L} & \frac{L_1 + L_2}{L} \end{bmatrix} \cdot \begin{bmatrix} \Delta F_{s1} \\ \Delta F_{s2} \end{bmatrix} \tag{52}$$

According to Equation (26), we have the following:

$$\left(\frac{L_2 + L_3}{L}\right)^2 + \left(\frac{L_3}{L}\right)^2 = \left(\frac{L_1}{L}\right)^2 + \left(\frac{L_1 + L_2}{L}\right)^2 \tag{53}$$

From Equation (53) and use $L_1 + L_2 + L_3 = L \;\; L > 0$, it is found that:

$$L_1 = L_3 \tag{54}$$

When F_1, F_2 are independent, all non-diagonal elements of Variance-Covariance matrix of functional changes should be 0. Then we have:

$$E\left[\left(\frac{L_2 + L_3}{L}\Delta F_{s1} + \frac{L_3}{L}\Delta F_{s2}\right)\left(\frac{L_1}{L}\Delta F_{s2} + \frac{L_1 + L_2}{L}\Delta F_{s2}\right)\right] = 0 \tag{55}$$

Since ΔF_{s1} and ΔF_{s2} are mutually independent, and $\Delta F_{s1} \sim N(0, 10^2)$、$\Delta F_{s2} \sim N(0, 10^2)$, so we have:

$$L_1 = L_3 = 0, \ \ L_2 = L \tag{56}$$

and

$$F_1 \sim N(1000, 10^2), \quad F_2 \sim N(1000, 10^2) \tag{57}$$

Figure 6 shows the structures.

Figure 6. schematic drawing when support forces are independent

If there is not any relationship restricted between F_1 and F_2 , by manipulating Equations (25) and (26), we have:

$$\sigma_F{}^2 = \sigma_D{}^2 \frac{\left(\frac{L_2+L_3}{L}\right)^2 + \left(\frac{L_3}{L}\right)^2 + \left(\frac{L_1}{L}\right)^2 + \left(\frac{L_1+L_2}{L}\right)^2}{2}$$

$$= \sigma_D{}^2 \frac{L_1{}^2 + L_2{}^2 + L_3{}^2 + L_1L_2 + L_2L_3}{L^2} \tag{58}$$

As $L_1 + L_2 + L_3 = L \ \ L > 0$, bring with Equation (58) together, we can determine that when $L_1 = L_3 = L / 2$ and $L_2 = 0$, variance of functional requirements reaches the minimum, thus:

$$\sigma_F{}^2 = \frac{\sigma_D{}^2}{2} \tag{59}$$

And according to $F_{s1} \sim N(1000, 10^2)$, $F_{s2} \sim N(1000, 10^2)$, the following relations will be hold:

$$F_1 = F_2 \sim N(1000, 7.1^2) \tag{60}$$

Figure 7 shows the structures.

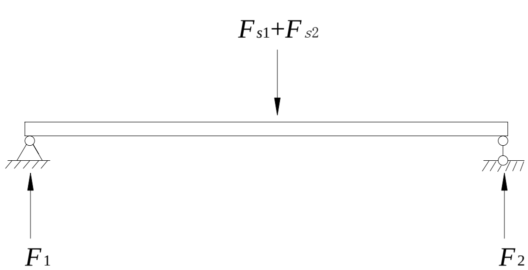

Figure 7. schematic drawing of robust structure

According to the discussion above, when support forces F_1, F_2 are mutually independent, $L_1 = L_3 = 0$, $L_2 = L$. Under the given independent conditions, structural characteristics of system directly influence system robustness. When there are not any restrictive relationships between F_1 and F_2, the analytical approach would lead to a robust design.

Under the condition as shown in Figure 7, when $L_1 = L_3 = L / 2$，$L_2 = 0$, from Equation (58), we obtain:

$$\frac{\partial \sigma_F{}^2}{\partial L_1} = \frac{\sigma_D{}^2}{L} \tag{61}$$

$$\frac{\partial \sigma_F{}^2}{\partial L_2} = \frac{\sigma_D{}^2}{2L} \tag{62}$$

$$\frac{\partial \sigma_F{}^2}{\partial L_3} = \frac{\sigma_D{}^2}{L} \tag{63}$$

$$\frac{\partial \sigma_F{}^2}{\partial \sigma_D{}^2} = \frac{1}{2} \tag{64}$$

Equations (61), (62), (63) and (64) show the system robustness acquired. When the rigid crossbeam is longer, and variance of forces F_{s1} and F_{s2} become smaller, system robustness would obtain a lower sensitivity to changes resulted from structural characteristics and uncontrollable factors. The system would not be disturbed by small changes of structural characteristics and uncontrollable factors.

6 CONCLUSION

The analyses carried out in this research showed that there are the intrinsic relationships among system robustness, structural characteristics, design parameters and uncontrollable factors of mechanical systems. By using the proposed analytical robust design approach, this research found that linear system robustness has a direct relationship with structural characteristics and uncontrollable factors. Furthermore, based on the analysis of linear system sensitivity to uncontrollable factors, a robust optimization model was established for determining robust design parameters. According to the optimization model (see Equations (25) and (26)) as well as the Axiomatic Design principles, structural characteristics of system and ranges for tolerance of uncontrollable factors should be properly selected for achieving system robustness. It should be mentioned that the model and the method represented in this paper were developed with assumptions. If an application does not satisfy these conditions, this method may not be applicable.

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