The 4D, 4th Rank, Antisymmetric Tensor and the 4D Equivalent to the Cross Product
or
More Fun with Tensors!!!

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This document describes the four dimensional fourth rank antisymmetric tensor, ε4, and some of its uses. The ε4(i,j,k,l) tensor is a vary close relative of Mathcad's ε(i,j,k) tensor. First the tensor itself will be examined. After that its use to evaluate 4 by 4 determinants and to calculate the 4 dimensional equivalent of the cross product will be explored.

Below is the definition of ε4, the antisymmetric tensor of rank 4, for a 4-dimensional vector space. The component's value is 0 if any of the indices are equal, and
1 if the indices are (0,1,2,3). The tensor component is equal to -1 if the indices are an odd permutation of (0,1,2,3), and 1 if the indices are an even permutation of (0,1,2,3).

```
perm4(A) :=  | cnt ← 0
             | end ← last(A)
             | for  j    end − 1 .. 0
             |   for  i    0 .. j
             |     if  A_i > A_(i+1)
             |          | t ← A_i
             |          | A_i ← A_(i+1)
             |          | A_(i+1) ← t
             |          | cnt ← cnt + 1
             | cnt
```

If the indices are all different then ε4(i,j,k,l) uses perm4(A) to count the number of permutations needed to arrange the indices in ascending order, (0,1,2,3).

```
ε4(i, j, k, l) :=  | if  [(i≠j)·(j≠k)·(k≠l)·(i≠k)·(j≠l)·(i≠l)]
                   |    | A_0 ← mod(i, 4)
                   |    | A_1 ← mod(j, 4)
                   |    | A_2 ← mod(k, 4)
                   |    | A_3 ← mod(l, 4)
                   |    | cnt ← perm4(A)
                   |    | val ← 1   if  mod(cnt, 2) = 0
                   |    | val ← −1  otherwise
                   | val ← 0  otherwise
                   | val
```

If any of the indices are the same, then that component is equal to zero.
If they are all different, examine the permutation count. If the count is odd, then the component is equal to -1.
If the count was even, then that component of the tensor is equal to 1.

Below are a few of the 256 components of the fourth rank antisymmetric tensor.

$\varepsilon 4(0,1,2,3) = 1$	$\varepsilon 4(0,0,2,3) = 0$	$\varepsilon 4(3,2,1,0) = 1$
$\varepsilon 4(1,2,3,0) = -1$	$\varepsilon 4(0,1,1,3) = 0$	$\varepsilon 4(2,1,0,3) = -1$
$\varepsilon 4(2,3,0,1) = 1$	$\varepsilon 4(0,1,2,2) = 0$	$\varepsilon 4(1,0,3,2) = 1$
$\varepsilon 4(3,0,1,2) = -1$	$\varepsilon 4(3,1,2,3) = 0$	$\varepsilon 4(0,3,2,1) = -1$

The range variables; i,j,k,l range over zero to three for the indexes of the ε4 tensor.

$i := 0..3 \qquad j := 0..3 \qquad k := 0..3 \qquad l := 0..3$

Properties of a 4th rank, 4 dim. tensor

The number of components in a 4th rank, 4 dimensional tensor is: $$\sum_l \sum_k \sum_j \sum_i 1 = 256$$

from contraction of δ the dimension of our space is: $$\sum_i \delta(i,i) = 4$$

Coordinate Transformation Law is: $$T_{i,j,k,l} = \sum_{d=0}^{3} \sum_{c=0}^{3} \sum_{b=0}^{3} \sum_{a=0}^{3} T_{a,b,c,d} \cdot \lambda_{a,i} \cdot \lambda_{b,j} \cdot \lambda_{c,k} \cdot \lambda_{d,l}$$

Some of the properties of the ε4(i,j,k,l) tensor

The number of non-zero components of ε4 is: $$\sum_l \sum_k \sum_j \sum_i \left|\varepsilon 4(i,j,k,l)\right| = 24$$

The number of components of ε4 equal to 1 is: $$\sum_l \sum_k \sum_j \sum_i (\varepsilon 4(i,j,k,l) = 1) = 12$$

The number of components of ε4 equal to -1 is: $$\sum_l \sum_k \sum_j \sum_i (\varepsilon 4(i,j,k,l) = -1) = 12$$

The number of components of ε4 equal to 0 is: $$\sum_l \sum_k \sum_j \sum_i (\varepsilon 4(i,j,k,l) = 0) = 232$$

Now that we have created the 4th rank totally anti-symmetric tensor, what can it be used for? The ε4 tensor can be used to evaluate the determinant of a 4 x 4 matrix, as shown below. To evaluate the determinant one converts the columns of the matrix to column vectors Next apply the tensor to all four vectors yielding a single number, the determinant. Below are some examples, using the ε4 tensor to calculate determinants and the built in Mathcad calculation for verification of the results.

$i := 0..3 \qquad j := 0..3 \qquad k := 0..3 \qquad l := 0..3$

$$D4(m) := \sum_{l}\sum_{k}\sum_{j}\sum_{i} \varepsilon4(i,j,k,l)\cdot\left(m^{<0>}\right)_i\cdot\left(m^{<1>}\right)_j\cdot\left(m^{<2>}\right)_k\cdot\left(m^{<3>}\right)_l$$

$$m := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D4(m) = 1 \qquad |m| = 1$$

$$m := \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \qquad D4(m) = 4 \qquad |m| = 4$$

$$m := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \end{bmatrix} \qquad D4(m) = 1 \qquad |m| = 1$$

$$m := \begin{bmatrix} 1 & 0 & 3 & 1 \\ 5 & 6 & 7 & 1 \\ 9 & 10 & 11 & 2 \\ 13 & 14 & 15 & 0 \end{bmatrix} \qquad D4(m) = 48 \qquad |m| = 48$$

Since the ε4 tensor can be used to define the 4 by 4 determinant, one should be able to define the ε4 tensor in terms of the 4 by 4 determinant and the δ function or δ tensor.

The 4x4 determinant of delta functions is used to define the eighth rank δ tensor with 4 contravariant indexes and 4 covariant indexes. The components of the ε4a tensor, defined from the 4 by 4 determinant , are a subset of the eighth rank δ tensor's components. If you are still using Mathcad-3.1 you would have to use the definition below for the 4D, 4th rank anti-symetric tensor.

$$\delta4(i,j,k,l,r,s,t,u) := \left|\begin{bmatrix} \delta(i,r) & \delta(i,s) & \delta(i,t) & \delta(i,u) \\ \delta(j,r) & \delta(j,s) & \delta(j,t) & \delta(j,u) \\ \delta(k,r) & \delta(k,s) & \delta(k,t) & \delta(k,u) \\ \delta(l,r) & \delta(l,s) & \delta(l,t) & \delta(l,u) \end{bmatrix}\right| \qquad \varepsilon4a(r,s,t,u) := \delta4(0,1,2,3,r,s,t,u)$$

Later in this paper both ε4 and ε4a will be used to calculate the same 4 dimensional cross product to verify their results.

The second interesting thing you can do with the 4th rank totally antisymmetric tensor is to perform the 4 dimensional equivalent of the cross product in Euclidean 3-space, R3. First let's review some of the properties of the cross product in R3 . The R3 cross product is geometrically defined as:

$$C = A \times B = |A| \cdot |B| \cdot \sin(\theta) \cdot \frac{E}{|E|}$$

E is perpendicular to A and B. The direction is given by the right hand rule in cartesianl coordinate systems.

$A \cdot E = 0$ and $B \cdot E = 0$

Vector C's magnitude is equal to the length of A times the length of B times the sin of the angle between A and B. The cross product of A and B gives a normal vector to both A and B, who's length is equal to the area of of the parallelogram constructed from A and B. A cross B is anti-symetric so:

$$A \times B = -B \times A$$

given e, the matrix of basis vectors: $e := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ The non-zero elements of the cross product of the basis vectors in R3 are:

$e^{<0>} \times e^{<1>} = e^{<2>}$ $\quad e^{<1>} \times e^{<2>} = e^{<0>}$ $\quad e^{<2>} \times e^{<0>} = e^{<1>}$

$e^{<1>} \times e^{<0>} = -\left(e^{<2>}\right)$ $\quad e^{<2>} \times e^{<1>} = -\left(e^{<0>}\right)$ $\quad e^{<0>} \times e^{<2>} = -\left(e^{<1>}\right)$

$$\begin{array}{c|c} & \begin{matrix} e_0 & e_1 & e_2 \end{matrix} \\ \begin{matrix} e_0 \\ e_1 \\ e_2 \end{matrix} & \begin{bmatrix} 0 & e_2 & -e_1 \\ -e_2 & 0 & e_0 \\ e_1 & -e_0 & 0 \end{bmatrix} \end{array}$$

The multiplication table to the left summarizes the action of the cross product on the orthonormal basis vector of Euclidean 3-space. From this table one can see that the 3rd rank, 3 dimensional antisymmetric tensor, which is Mathcad's built in ε(i,j,k,) tensor, applied to two independent 3-vectors, can be used to evaluate the cross product.

The cross product implemented with the 3 dimensional anti-symetric tensor is:

$i3 := 0..2 \quad j3 := 0..2 \quad k3 := 0..2$

$$CP3(A,B) := \sum_{k3}\sum_{j3}\sum_{i3} e^{<k3>} \cdot \left(\varepsilon(i3,j3,k3) \cdot A_{i3} \cdot B_{j3}\right)$$

The k3 summation is used to collect the components, from the application of the ε tensor to vectors A and B, into the results vector. Note mathcad's order of operations is used to always force a vector result.

$A := [1 \ \ 2 \ \ 3]^T \quad B := [4 \ \ 5 \ \ 6]^T$ Two independent 3-vectors for the example below.

$Cp := CP3(A,B) \quad Cp^T = [-3 \ \ 6 \ \ -3] \qquad (A \times B)^T = [-3 \ \ 6 \ \ -3]$

$Cn := CP3(B,A) \quad Cn^T = [3 \ \ -6 \ \ 3] \qquad (B \times A)^T = [3 \ \ -6 \ \ 3]$ it is anti-communitive

$A \cdot Cp = 0 \quad B \cdot Cp = 0$ The cross product vector is orthogonal to both A and B

$CP3(A,A)^T = [0 \ \ 0 \ \ 0] \qquad (A \times A)^T = [0 \ \ 0 \ \ 0]$ The Cross product of dependent vectors is zero

The ε4 tensor will now be used to generate the four-dimensional equivalent of the cross product. This function takes three linearly independent 4 dimensional vector that span a 3D hyper-plane and returns a 4-vector perpendicular to all the input vectors. Note: A fourth rank tensor must be applied to three vector in order to produce a single vector as its output.

$i4 := 0..3 \quad j4 := 0..3 \quad k4 := 0..3 \quad l4 := 0..3$

$$id4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The id4 matrix has the four dimensional orthonormal basis vectors as its columns.

The four dimensional extension of the Cross product using the ε4 tensor and three 4-vectors is:

$$CP4(V1, V2, V3) := \sum_{l4} id4^{\langle l4 \rangle} \cdot \sum_{k4} \sum_{j4} \sum_{i4} \left(\varepsilon4(i4, j4, k4, l4) \cdot V1_{i4} \cdot V2_{j4} \cdot V3_{k4} \right)$$

Again, Mathcad's order of operations is used to force CP4 to always return a vector. The zero 4-vector is returned instead of just scalar zero.

$$CP4a(V1, V2, V3) := \sum_{l4} id4^{\langle l4 \rangle} \cdot \sum_{k4} \sum_{j4} \sum_{i4} \left(\varepsilon4a(i4, j4, k4, l4) \cdot V1_{i4} \cdot V2_{j4} \cdot V3_{k4} \right)$$

CP4a is base on the 4th rank, 4 dimensional antisymmetric tensor generated from the determinant and delta function definition. This had better yield the same results as CP4.

$T1 := id4^{\langle 0 \rangle} \quad T2 := id4^{\langle 1 \rangle} \quad T3 := id4^{\langle 2 \rangle} \quad T4 := id4^{\langle 3 \rangle}$

T1,T2,T3,T4 are set to the four orthonormal bais vectors

Below are the results of the four dimensional cross product applied to sets of three basis vectors at a time.

$O4 := CP4(T4, T3, T2) \qquad O4^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$O4 := CP4(T1, T3, T4) \qquad O4^T = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$O4 := CP4(T2, T1, T4) \qquad O4^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

$O4 := CP4(T1, T2, T3) \qquad O4^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

The 4D cross product of three of the four basis vectors yields the fourth basis vector that's perpendicular to the other three.

$O4 := CP4(T2, T3, T4) \qquad O4^T = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}$

$O4 := CP4(T2, T4, T3) \qquad O4^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$O4 := CP4(T3, T2, T4) \qquad O4^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$O4 := CP4(T2, T3, T4) \qquad O4^T = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}$

The 4D cross product is anti-communitive just like the 3 dimensional cross product

$O4 := CP4(T1, T2, T2) \qquad O4^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

$O4 := CP4(T2, T2, T3) \qquad O4^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

$O4 := CP4(T4, T4, T4) \qquad O4^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

The 4D cross product returns the zero vector if any of the input vectors are linearly dependent.

Below are three linearly independent test vectors and their four-dimensional cross product calculated by both ε4(i,j,k,l) and ε4a(i,j,k,l). This was done to check the results of both the programmed version of the ε tensor and the version generated from the 8th rank δ tensor. We get the same result from both calculations.

3 independent test 4-vectors

$$A1 := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad A2 := \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \qquad A3 := \begin{bmatrix} -9 \\ 10 \\ 11 \\ 12 \end{bmatrix}$$

Results of the 4 dimensional cross products are equal

$$CP4(A1, A2, A3) = \begin{bmatrix} 0 \\ 72 \\ -144 \\ 72 \end{bmatrix} \qquad CP4a(A1, A2, A3) = \begin{bmatrix} 0 \\ 72 \\ -144 \\ 72 \end{bmatrix}$$

The results of the 4D cross product, R4, is orthogonal to A1, A2, and A3 as expected.

$$R4 := CP4(A1, A2, A3)$$

$$R4^T = \begin{bmatrix} 0 & 72 & -144 & 72 \end{bmatrix}$$

$$R4 \cdot A1 = 0 \qquad R4 \cdot A2 = 0 \qquad R4 \cdot A3 = 0$$

Closing Notes:

The Author would like to point out that the results of the four dimensional cross product are exactly correct only for Euclidean vector spaces with orthonormal basis vectors. In these spaces contravariant vectors, or just plain "vectors", and covariant vectors, or one-forms have identical components. This is because the metric is equal to the identity matrix.

The ε4 tensor defined in this paper is of type (0,4), or it operates on four (contravariant) vectors and returns a real number. if this tensor operates on three vectors, as in the four dimensional cross product, the result is a (covariant vector) one-form. Only in a Euclidean space with orthonormal basis vectors will the one-form's components, returned by ε4, be equal to its dual vector's components.

To turn an one-form into a vector, must apply the inverse (contravariant) metric tensor to it. The inverse metric maps one-forms into vectors and the (covariant) metric tensors maps vectors into one-forms. The metric allows the mapping of vectors into one-forms and visa-versa. This establishes the duality between vectors and one-forms in general metric spaces.

A more general definition of the vector cross product with non-orthogonal basis vectors would be:

$$\overrightarrow{CP4} = G^{-1} \cdot \varepsilon 4\left(\vec{a1}, \vec{a2}, \vec{a3}\right)$$

G^{-1} is inverse of the metric

ε4 is the 4 dimensional anti-symmetric tensor

a1, a2, a3, and CP4 are contravariant vectors

Although the one form result of $\varepsilon 4\left(\vec{a1}, \vec{a2}, \vec{a3}\right)$ is just as useful.

References

1. M.A. Akivis & V.V. Goldberg, "An Introduction to linear Algebra & Tensors", Dover Publications, Mineola, N.Y., 1977
2. A.I.Borisenko & I.E. Tarapov, "Vector and Tensor Analysis with Applications", Dover Publications, Mineola, N.Y., 1979
3. Richard Shiffman, "The Cross Product, Dot Product and fun with Tensors", 1995, (URL:= http://www.mathsoft.com/pub/apps/crossdot.mcd)
4. Alfred Gray, Modern Differential Geometry of Curves and Surfaces, CRC Press, 1993
5. Bernard F. Schultz, "A First Coarse in General Relativity", Cambvridge University Press, 1985