Nonlinear and Convex Analysis, vol. 17, no.3, pp. , 2016

Symmetric cone monotone functions and symmetric cone convex functions

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March 18, 2013

Abstract. Symmetric cone (SC) monotone functions and SC-convex functions are real scalar valued functions which induce Löwner operators associated with a simple Euclidean Jordan algebra to preserve the monotone order and convex order, respectively. In this paper, for a general simple Euclidean Jordan algebra except for octonion case, we show that the SC-monotonicity (respectively, SC-convexity) of order r is implied by the matrix monotonicity (respectively, matrix convexity) of some fixed order r' ($\geq r$). As a consequence, we draw the conclusion that (except for octonion case) a function is

[1]Corresponding author. The author's work is supported by Ministry of Science and Technology, Taiwan.

[2]The author's work is supported by National Young Natural Science Foundation (No. 10901058), Guangdong Natural Science Foundation (No. 9251802902000001) and the Fundamental Research Funds for the Central Universities (SCUT).

SC-monotone (respectively, SC-convex) if and only if it is matrix monotone (respectively, matrix convex).

Key words: Euclidean Jordan algebra, Symmetric cone, matrix-monotone, Löwner operator, SC-monotone, SC-convex.

1 Introduction

A *Euclidean Jordan algebra* is a triple $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ where $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ is a finite dimensional inner product space over the real field $\mathbb{R}$, and $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a bilinear mapping satisfying the following conditions: for all $x, y, z \in \mathbb{V}$, (i) $x \circ y = y \circ x$; (ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ with $x^2 = x \circ x$; (iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$, in which $x \circ y$ is called the Jordan product of x and y. We assume that there exists an element $e \in \mathbb{V}$ (called the *unit element*) such that $x \circ e = x$ for all $x \in \mathbb{V}$. A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. For details regarding Euclidean Jordan algebras, we refer to the lecture note [14] and the monograph [9].

Let $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra. For any $x \in \mathbb{V}$, define

$$\zeta(x) := \min\left\{k : \{e, x, x^2, \cdots, x^k\} \text{ are linearly dependent}\right\}.$$

Then, the *rank* of $\mathbb{A}$ is well defined by $r := \max\{\zeta(x) : x \in \mathbb{V}\}$. Recall that an element $c \in \mathbb{V}$ is said to be *idempotent* if $c^2 = c$; and an idempotent is said to be *primitive* if it is nonzero and can not be written as the sum of two other nonzero idempotents. A finite set $\{c_1, c_2, \cdots, c_r\}$ of primitive idempotents in $\mathbb{V}$ is said to be a *Jordan frame* if

$$c_i \circ c_j = 0 \quad \text{when} \quad i \neq j \quad \text{and} \quad c_1 + c_2 + \cdots + c_r = e.$$

Then, we have the following important spectral decomposition theorem.

Theorem 1.1 *[9, Theorem III.1.2] Suppose $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra of rank r. Then, for every $x \in \mathbb{V}$, there exist a Jordan frame $\{c_1, \cdots, c_r\}$ and real numbers $\lambda_1(x), \cdots, \lambda_r(x)$, arranged in the decreasing order $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$, such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

The numbers $\lambda_1(x), \ldots, \lambda_r(x)$ (counting multiplicities), uniquely determined by x, are called the spectral values of x and $\sum_{j=1}^r \lambda_j(x)c_j$ the spectral decomposition of x.

Suppose that $\phi\colon J \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar valued function. Let $\mathbb{V}_J$ be a subset in $\mathbb{V}$ such that all $x \in \mathbb{V}_J$ have the spectral in J. Then, by the spectral decomposition $\sum_{j=1}^r \lambda_j(x)c_j$ of $x \in \mathbb{V}_J$, it is natural to define a vector valued function [4, 14] $\phi_{\mathbb{V}} : \mathbb{V}_J \to \mathbb{V}$ by

$$\phi_{\mathbb{V}}(x) := \phi(\lambda_1(x))c_1 + \phi(\lambda_2(x))c_2 + \cdots + \phi(\lambda_r(x))c_r. \tag{1}$$

In a seminal paper [19], Löwner initiated the study for $\phi_{\mathbb{V}}$ in the setting of $\mathbb{V} = \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the space of $n \times n$ real symmetric matrices, and for $X \in \mathbb{S}^n_J$, which is a subset of $\mathbb{S}^n$ such that all eigenvalues of $X \in \mathbb{S}^n_J$ belong to J, $\phi_{\mathbb{S}^n}(X)$ has the expression

$$\phi_{\mathbb{S}^n}(X) := P\text{diag}(\phi(\lambda_1(X)), \cdots, \phi(\lambda_n(X)))P^T,$$

where P is an $n \times n$ orthogonal matrix and $\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X)$ are real numbers arranged in the decreasing order, such that

$$X = P\text{diag}(\lambda_1(X), \ldots, \lambda_n(X))P^T.$$

The result of [19] on the monotonicity of $\phi_{\mathbb{S}^n}$ was later extended to $\phi_{\mathbb{V}}$ by Korányi [15]. In addition, Sun and Sun [24] studied the continuous differentiability and strong semismoothness of $\phi_{\mathbb{V}}$, and called $\phi_{\mathbb{V}}$ Löwner operator associated with $\mathbb{V}$ in recognition of Löwner's contribution.

From [9, Theorem III.2.1] we know that the set of all squares $\mathcal{K} := \{x \in \mathbb{V} : x \circ x\}$ in $\mathbb{V}$ is a symmetric cone, i.e., a self-dual homogeneous closed convex cone. So, there is a natural partial order in $\mathbb{V}$. We write $x \succeq_{\mathcal{K}} y$ if $x - y \in \mathcal{K}$, and $x \succ_{\mathcal{K}} y$ if $x - y \in \textbf{int}\mathcal{K}$. For any $x, y \in \mathbb{V}_J$, let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ and $\lambda_1(y) \geq \lambda_2(y) \geq \cdots \geq \lambda_r(y)$ be the spectral values of x and y, respectively. From [3, Prop. 4.4] or [2, Theorem 23],

$$\sum_{i=1}^{r} (\lambda_i(x) - \lambda_i(y))^2 \leq \sum_{i=1}^{r} \lambda_i(x)^2 + \sum_{i=1}^{r} \lambda_i(y)^2 - 2\langle x, y \rangle = \|x - y\|^2.$$

By this, it is easy to verify that $\mathbb{V}_J$ is open in $\mathbb{V}$ if and only if J is open on $\mathbb{R}$. Also, since

$$\begin{aligned}\lambda_1(\alpha x + (1-\alpha)y) &\leq \alpha\lambda_1(x) + (1-\alpha)\lambda_1(y)\\ \lambda_r(\alpha x + (1-\alpha)y) &\geq \alpha\lambda_r(x) + (1-\alpha)\lambda_r(y)\end{aligned}$$

for any $\alpha \in [0, 1]$ (see [25, Lemma 14]), where $\lambda_1(\alpha x + (1-\alpha)y), \ldots, \lambda_r(\alpha x + (1-\alpha)y)$ are the spectral values of $\alpha x + (1-\alpha)y$, arranged in decreasing order, the set $\mathbb{V}_J$ is always convex. Now we introduce the concepts of SC-monotone and SC-convex functions.

Definition 1.1 *Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be a simple Euclidean Jordan algebra of rank r. For any given $\phi : J \subseteq \mathbb{R} \to \mathbb{R}$, let $\phi_{\mathbb{V}} : \mathbb{V}_J \to \mathbb{V}$ be defined as in (1). Then,*

(a) *ϕ is said to be SC-monotone of order r if for any $x, y \in \mathbb{V}_J$, it holds that*

$$x \succeq_{\mathcal{K}} y \implies \phi_{\mathbb{V}}(x) \succeq_{\mathcal{K}} \phi_{\mathbb{V}}(y).$$

(b) *ϕ is said to be SC-convex of order r if for any $x, y \in \mathbb{V}_J$ and $\alpha \in (0, 1)$, it holds that*

$$\phi_{\mathbb{V}}(\alpha x + (1-\alpha)y) \preceq_{\mathcal{K}} \alpha\phi_{\mathbb{V}}(x) + (1-\alpha)\phi_{\mathbb{V}}(y).$$

We call ϕ SC-monotone (SC-convex) if it is SC-monotone (SC-convex) of all orders.

When $\mathbb{V}$ is the algebra $\mathbb{S}^n$ of $n\times n$ real symmetric matrices, Def. 1.1 represents the concepts of matrix monotone and matrix convex functions of order n; when $\mathbb{V}$ is the Jordan spin algebra (see Example 2.4), it gives the concepts of SOC-monotone and SOC-convex functions [6, 7]. After the seminal paper [19], there are many research works about matrix monotone and matrix convex functions (see, e.g., [16, 8, 11, 12, 20, 5, 21, 22, 17, 26]). However, to our best of knowledge, there are few papers to study SC-monotone and SC-convex functions except that Korányi [15] gave a sufficient and necessary condition for differentiable SC-monotone functions, and furthermore, this condition is the same as the one for matrix monotone functions in [13, Theorem 6.6.36].

In this paper, we establish that the SC-monotonicity (respectively, SC-convexity) of order r of ϕ is implied by its matrix monotonicity (respectively, matrix convexity) of some fixed order r' $(\geq r)$. For example, ϕ is SC-monotone (respectively, SC-convex) of order r if it is matrix monotone (respectively, matrix convex) of order $4r$; see Theorem 3.1 As a consequence, we draw the conclusion that ϕ is SC-monotone (respectively, SC-convex) if and only if it is matrix monotone (respectively, matrix convex). These results are achieved by employing the connection between $\phi_{\mathbb{V}}$ and $\phi_{\mathbb{S}^n}$, the results of SOC-monotone (SOC-convex) functions [23], and the classification of simple Euclidean Jordan algebras.

2 Preliminaries

For any given $x \in \mathbb{V}$, we define the following linear operator $\mathcal{L}(x)$ of $\mathbb{V}$ by

$$\mathcal{L}(x)y := x \circ y \quad \text{for every } y \in \mathbb{V}.$$

Let $\{c_1, \cdots, c_r\}$ be a Jordan frame in a Euclidean Jordan algebra $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$. Then, from [9, Lemma IV.1.3], the operators $\mathcal{L}(c_j), j = 1, 2, \cdots, r$ commute and admit a simultaneous diagonalization. Besides, for $i, j \in \{1, 2, \cdots, r\}$, we denote the eigenspaces

$$\mathbb{V}_{ii} := \{x \in \mathbb{V} : x \circ c_i = x\} = \mathbb{R}c_i$$

and when $i \neq j$,

$$\mathbb{V}_{ij} := \left\{x \in \mathbb{V} : x \circ c_i = \frac{1}{2}x = x \circ c_j\right\}.$$

Then, from [9, Theorem IV.2.1], we have the following Peirce decomposition.

Proposition 2.1 *The space $\mathbb{V}$ is the orthogonal direct sum of spaces $\mathbb{V}_{ij}$ $(i \leq j)$. Also,*

$$\begin{aligned}
&\mathbb{V}_{ij} \circ \mathbb{V}_{ij} \subset \mathbb{V}_{ii} + \mathbb{V}_{jj};\\
&\mathbb{V}_{ij} \circ \mathbb{V}_{jk} \subset \mathbb{V}_{ik} \ \text{ if } \ i \neq k;\\
&\mathbb{V}_{ij} \circ \mathbb{V}_{kl} = \{0\} \ \text{ if } \ \{i, j\} \cap \{k, l\} = \emptyset.
\end{aligned}$$

Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x) c_j$, where $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ are the spectral eigenvalues of x and $\{c_1, c_2, \ldots, c_r\}$ is the corresponding Jordan frame. For all $i, j \in \{1, 2, \ldots, r\}$, let $\mathcal{C}_{ij}(x)$ be the orthogonal projection operator onto $\mathbb{V}_{ij}$, from [9, Theorem IV 2.1], it follows that for all $i, j = 1, 2, \ldots, r$,

$$\mathcal{C}_{jj}(x) = 2\mathcal{L}(c_j)^2 - \mathcal{L}(c_j) \text{ and } \mathcal{C}_{ij}(x) = 4\mathcal{L}(c_i)\mathcal{L}(c_j) = 4\mathcal{L}(c_j)\mathcal{L}(c_i) = \mathcal{C}_{ji}(x). \tag{2}$$

Moreover, the orthogonal projection operators $\{\mathcal{C}_{ij}(x) : \ i, j = 1, 2, \ldots, r\}$ satisfy

$$\mathcal{C}_{ij}(x) = \mathcal{C}_{ij}^*(x), \quad \mathcal{C}_{ij}^2(x) = \mathcal{C}_{ij}(x), \quad \mathcal{C}_{ij}(x)\mathcal{C}_{kl}(x) = 0 \text{ if } \{i, j\} \neq \{k, l\} \tag{3}$$

and

$$\sum_{1 \leq i \leq j \leq r} \mathcal{C}_{ij}(x) = \mathcal{I} \tag{4}$$

where $\mathcal{C}_{ij}^*(x)$ means the adjoint of $\mathcal{C}_{ij}(x)$, and $\mathcal{I}$ is the identity operator from $\mathbb{V}$ to $\mathbb{V}$.

The following lemma gives the spectral decomposition of the operator $\mathcal{L}(x)$, whose proof can be found in [14, Chapter V, Sec. 5 and Chapter VI, Sec. 4].

Lemma 2.1 *Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x) c_j$. Then, the linear symmetric operator $\mathcal{L}(x)$ has the spectral decomposition*

$$\mathcal{L}(x) = \sum_{j=1}^{r} \lambda_j(x)\mathcal{C}_{jj}(x) + \sum_{1 \leq j < l \leq r} \frac{1}{2}\left(\lambda_j(x) + \lambda_l(x)\right)\mathcal{C}_{jl}(x) \tag{5}$$

with the spectrum $\sigma(\mathcal{L}(x))$ consisting of all distinct numbers $\frac{1}{2}(\lambda_j(x) + \lambda_l(x))$.

Next, we introduce several examples of simple Euclidean Jordan algebras, and recall the classification theorem of simple Euclidean Jordan algebras.

Example 2.1. The algebra $\mathbb{H}^n$ of $n \times n$ complex Hermitian matrices. A square matrix A of complex entries is said to be *Hermitian* if $A^* := \bar{A}^T = A$, where 'bar' denotes the complex conjugate, and the superscript 'T' means the transpose. Let $\mathbb{H}^n$ be the set of all $n \times n$ complex Hermitian matrices. On $\mathbb{H}^n$, let define the Jordan product and inner product be $X \circ Y := \frac{1}{2}(XY + YX)$ and $\langle X, Y \rangle := \text{trace}(XY)$. Then, $\mathbb{H}^n$ is a Euclidean Jordan algebra of rank n and dimension n^2, with e being the $n \times n$ identity matrix I.

There exists an embedding from $\mathbb{H}^n$ to $\mathbb{S}^{2n}$ which is one-to-one and onto, and also preserves the Jordan algebra structures on the both sides by matrix block multiplication. As below, we present this embedding for $\mathbb{H}^2$. First, we know that $\mathbb{H}^2$ is the set which contains all

$$\begin{bmatrix} \alpha_1 & \beta \\ \bar{\beta} & \alpha_2 \end{bmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R} \text{ and } \beta \in \mathbb{C}.$$

We also know that each complex number $a+bi$ can be represented as a 2×2 real matrix:

$$a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ satisfies $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence, we can embed $\begin{bmatrix} \alpha_1 & \beta \\ \bar{\beta} & \alpha_2 \end{bmatrix}$ into an element in $\mathbb{S}^4$:

$$\mathbb{H}^2 \ni \begin{bmatrix} \alpha_1 & \beta \\ \bar{\beta} & \alpha_2 \end{bmatrix} \longmapsto \begin{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{bmatrix} & \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} & \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{bmatrix} \end{bmatrix} \in \mathbb{S}^4$$

where $\beta = a + ib$.

For general n, it is also true that $\mathbb{H}^n$ is a Jordan sub-algebra of $\mathbb{S}^{2n}$. The general embedding map $T_{\mathbb{H}^n} : \mathbb{H}^n \hookrightarrow T(\mathbb{H}^n) \subset \mathbb{S}^{2n}$ is given by

$$\mathbb{H}^n \ni \begin{bmatrix} \alpha_1 & \beta & \cdots & \gamma \\ \bar{\beta} & \alpha_2 & \cdots & \delta \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\gamma} & \bar{\delta} & \cdots & \alpha_n \end{bmatrix} \longmapsto \begin{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{bmatrix} & \begin{bmatrix} a & b \\ -b & a \end{bmatrix} & \cdots & \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \\ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} & \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{bmatrix} & \cdots & \begin{bmatrix} e & f \\ -f & e \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} c & -d \\ d & c \end{bmatrix} & \begin{bmatrix} e & -f \\ f & e \end{bmatrix} & \cdots & \begin{bmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{bmatrix} \end{bmatrix} \in \mathbb{S}^{2n}$$

where $\beta = a + ib$, $\gamma = c + id$, $\delta = e + if$. By matrix block multiplication, it can be seen the embedding $T_{\mathbb{H}^n}$ preserves the Jordan algebra structures

$$T_{\mathbb{H}^n}(x \circ_{\mathbb{H}^n} y) = T_{\mathbb{H}^n}(x) \circ_{\mathbb{S}^{2n}} T_{\mathbb{H}^n}(y) \quad \forall \quad x, y \in \mathbb{H}^n.$$

Example 2.2. The algebra $\mathbb{Q}^n$ of $n \times n$ quaternion Hermitian matrices. The linear space of quaternions over $\mathbb{R}$, denoted by $\mathbb{Q}$, is 4-dimensional vector space [27] with a basis $\{1, i, j, k\}$. This space becomes an associated algebra via the multiplication table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

For any $x = x_0 1 + x_1 i + x_2 j + x_3 k \in \mathbb{Q}$, we define its *real part* by $\Re(x) := x_0$, its *conjugate* by $\bar{x} := x_0 1 - x_1 i - x_2 j - x_3 k$, and its norm by $|x| = \sqrt{x\bar{x}}$. A square matrix A with quaternion entries is called *Hermitian* if A coincides with its conjugate transpose. Let $\mathbb{Q}^n$ be the set of all $n \times n$ quaternion Hermitian matrices. For any $X, Y \in \mathbb{Q}^n$, let

$$X \circ Y := \frac{1}{2}(XY + YX) \quad \text{and} \quad \langle X, Y \rangle := \Re(\text{trace}(XY)).$$

Then, $\mathbb{Q}^n$ is a Euclidean Jordan algebra of rank n and dimension $n(2n-1)$ with e being the $n \times n$ identity matrix I. Analogous to complex number, each quaternion $x = a1 + bi + cj + dk \in \mathbb{Q}$ can be represented as a 4×4 real matrix $\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$ which is also equivalent to

$$a \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Following the same lines for $\mathbb{H}^n$, we can embed $\mathbb{Q}^n$ into $\mathbb{S}^{4n}$ such that $\mathbb{Q}^n$ can be viewed as a Jordan sub-algebra of $\mathbb{S}^{4n}$. Again, the embedding map under the case for $\mathbb{Q}^2$ is

$$\mathbb{Q}^2 \ni \begin{bmatrix} \alpha_1 & x \\ \bar{x} & \alpha_2 \end{bmatrix} \longmapsto \begin{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \end{bmatrix} & \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \\ \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} & \begin{bmatrix} \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} \end{bmatrix} \in \mathbb{S}^8$$

where $x = a1 + bi + cj + dk$.

Moreover, the general embedding map $T_{\mathbb{Q}^n} : \mathbb{Q}^n \hookrightarrow T(\mathbb{Q}^n) \subset \mathbb{S}^{4n}$ under this case is given by

$$\mathbb{Q}^n \ni \begin{bmatrix} \alpha_1 & x & \cdots & y \\ \bar{x} & \alpha_2 & \cdots & z \\ \vdots & \vdots & \ddots & \vdots \\ \bar{y} & \bar{z} & \cdots & \alpha_n \end{bmatrix} \longmapsto$$

$$\begin{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \end{bmatrix} & \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} & \cdots & \begin{bmatrix} e & f & g & h \\ -f & e & -h & g \\ -g & h & e & -f \\ -h & -g & f & e \end{bmatrix} \\ \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} & \begin{bmatrix} \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} & \cdots & \begin{bmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} e & -f & -g & -h \\ f & e & h & -g \\ g & -h & e & f \\ h & g & -f & e \end{bmatrix} & \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix} & \cdots & \begin{bmatrix} \alpha_n & 0 & 0 & 0 \\ 0 & \alpha_n & 0 & 0 \\ 0 & 0 & \alpha_n & 0 \\ 0 & 0 & 0 & \alpha_n \end{bmatrix} \end{bmatrix} \in \mathbb{S}^{4n}$$

where $x = a1 + bi + cj + dk$, $y = e1 + fi + gj + hk$ and $z = p1 + qi + rj + sk$.

In summary, we construct an embedding from $\mathbb{H}^n$ or $\mathbb{Q}^n$ to $\mathbb{S}^m$ respectively for certain m. Since the embedding is linear and preserves the Jordan algebra structures on both sides, it can be seen Löwner operator commutes with the embedding, which means that for all $x \in \mathbb{H}^n$ and $y \in \mathbb{Q}^n$, there have

$$\phi_{\mathbb{S}^{2n}}(T_{\mathbb{H}^n}(x)) = T_{\mathbb{H}^n}(\phi_{\mathbb{H}^n}(x)) \quad \text{and} \quad \phi_{\mathbb{S}^{4n}}(T_{\mathbb{Q}^n}(y)) = T_{\mathbb{Q}^n}(\phi_{\mathbb{Q}^n}(y)). \tag{6}$$

In the above, we present an embedding from a Jordan algebra $\mathbb{H}^n$ or $\mathbb{Q}^n$ to a Jordan sub-algebras of $\mathbb{S}^m$ respectively for certain m. Indeed, there is an alternative way to interpret this. For any $A = A_1 + A_2 j \in M_n(\mathbb{Q})$, its complex adjoint matrix, symbolized χ_A, is defined by [27] :

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

It is shown that if $A \in \mathbb{Q}^n$ then $\chi_A \in \mathbb{H}^{2n}$ [27, Theorem 4.2(6)]. This is an embedding and preserves operations. There is also an adjoint matrix $\pi_B \in M_{4n}(\mathbb{R})$ associated with $B \in M_{2n}(\mathbb{C})$. Then, we obtain that the composite $\pi \circ \chi(A) \in \mathbb{S}^{4n}$ for any $A \in \mathbb{Q}^n$. It is obvious to see that the composite $\pi \circ \chi$ is a Jordan algebra embedding from $\mathbb{Q}^n$ to $\mathbb{S}^{4n}$ as expected.

Example 2.3. The algebra $\mathbb{O}^3$ of 3×3 octonion Hermitian matrices. The space of octonion, denoted by $\mathbb{O}$, is a 8-dimensional real vector space with basis $\{1, e_1, \dots, e_7\}$. The space becomes a nonassociative algebra via the following multiplication table [1]: Note that $\mathbb{O}$ is a non-commutative and non-associative algebra. For an element $x = x_0 1 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \in \mathbb{O}$, we define its *real part* by

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

$\Re(x) := x_0$, its *conjugate* by $\bar{x} := x_0 1 - x_1 e_1 - x_2 e_2 - x_3 e_3 - x_4 e_4 - x_5 e_5 - x_6 e_6 - x_7 e_7$, and its norm by $|x| := \sqrt{x\bar{x}}$. As in the case of a quaternion Hermitian matrix, we may define an octonion Hermitian matrix. Suppose $\mathbb{O}^3$ is the set of all 3×3 octonion Hermitian matrices. On $\mathbb{O}^3$, let the Jordan product and inner product be

$$X \circ Y := \frac{1}{2}(XY + YX) \quad \text{and} \quad \langle X, Y \rangle := \Re(\text{trace}(XY)).$$

Then, $\mathbb{O}^3$ is a Euclidean Jordan algebra of rank 3 with e being the 3×3 identity matrix, and is a real vector space of dimension 27.

Example 2.4. The Jordan spin algebra $\mathbb{J}^n$. Consider $\mathbb{R}^n$ endowed with the usual inner product. For any $x \in \mathbb{R}^n$, write $x = \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix}$ with $x_0 \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^{n-1}$. Define

$$x \circ y = \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} \circ \begin{pmatrix} y_0 \\ \bar{y} \end{pmatrix} := \begin{pmatrix} \langle x, y \rangle \\ x_0\bar{y} + y_0\bar{x} \end{pmatrix}.$$

Then, $(\mathbb{R}^n, \circ, \langle \cdot, \cdot \rangle)$ is an Euclidean Jordan algebra, and we denote it by $\mathbb{J}^n$. The rank of the Euclidean Jordan algebra $\mathbb{J}^n$ is 2 and its unit element is given by $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In this algebra, the set of squares is also called the second-order cone or the Lorentz cone.

Theorem 2.1 *[9, Chapter V] Every simple Euclidean Jordan algebra is isomorphic to one of the following*

(i) *The Jordan spin algebra $\mathbb{J}^n$.*

(ii) *The algebra $\mathbb{S}^n$ of $n \times n$ real symmetric matrices.*

(iii) *The algebra $\mathbb{H}^n$ of all $n \times n$ complex Hermitian matrices.*

(iv) *The algebra $\mathbb{Q}^n$ of all $n \times n$ quaternion Hermitian matrices.*

(v) *The algebra $\mathbb{O}^3$ of all 3×3 octonion Hermitian matrices.*

3 Main result

For simplicity, we employ $\mathbb{S}^n_+, \mathbb{H}^n_+$ and $\mathbb{Q}^n_+$ to denote the corresponding symmetric cones in $\mathbb{S}^n, \mathbb{H}^n$ and $\mathbb{Q}^n$, respectively. In other words, they represent

$$\mathbb{S}^n_+ = \{x \circ x \,|\, x \in \mathbb{S}^n\}, \quad \mathbb{H}^n_+ = \{x \circ x \,|\, x \in \mathbb{H}^n\} \quad \text{and} \quad \mathbb{Q}^n_+ = \{x \circ x \,|\, x \in \mathbb{Q}^n\}.$$

To achieve our main result, we will show that the embeddings we construct in Examples 2.1-2.2 preserve their conic orders.

Lemma 3.1 *Suppose that $\mathbb{V}$ is the algebra $\mathbb{H}^n$ of $n \times n$ complex Hermitian matrices. The embedding $T_{\mathbb{H}^n}$ defined as in Example 2.1 keeps the conic order in the following sense:*

$$x \succeq_{\mathbb{H}^n_+} y \Longleftrightarrow T_{\mathbb{H}^n}(x) \succeq_{\mathbb{S}^{2n}_+} T_{\mathbb{H}^n}(y) \quad \forall x, y \in \mathbb{H}^n.$$

Proof. ($\Rightarrow$) Suppose that $x \succeq_{\mathbb{H}^n_+} y$. Then, there exists an $a \in \mathbb{H}^n$ such that $x - y = a^2$. Since $T_{\mathbb{H}^n}$ preserves Jordan algebra structure, we have

$$T_{\mathbb{H}^n}(x) - T_{\mathbb{H}^n}(y) = T_{\mathbb{H}^n}(x - y) = T_{\mathbb{H}^n}(a^2) = (T_{\mathbb{H}^n}(a))^2 \in \mathbb{S}^{2n}_+$$

which gives the desired result.
($\Leftarrow$) Suppose that $T_{\mathbb{H}^n}(x) \succeq_{\mathbb{S}^{2n}_+} T_{\mathbb{H}^n}(y)$. Then, there exists $X, Y \in \mathbb{S}^{2n}$ such that $T_{\mathbb{H}^n}(x) = X$ and $T_{\mathbb{H}^n}(y) = Y$. By assumption of $X \succeq_{\mathbb{S}^{2n}_+} Y$, there exists an $A \in \mathbb{S}^{2n}$ such that $X - Y = A^2$. Again, since $T_{\mathbb{H}^n}$ preserves Jordan algebra structure, we have

$$x - y = T^{-1}_{\mathbb{H}^n}(X) - T^{-1}_{\mathbb{H}^n}(Y) = T^{-1}_{\mathbb{H}^n}(X - Y) = T^{-1}_{\mathbb{H}^n}(A^2) = (T^{-1}_{\mathbb{H}^n}(A))^2 \in \mathbb{Q}^n_+$$

which gives the desired result. $\square$

Next we present three Lemmas which are needed to establish our main result.

Lemma 3.2 *Suppose that $\mathbb{V}$ is the algebra $\mathbb{H}^n$ of $n \times n$ complex Hermitian matrices. For any given $\phi : J \to \mathbb{R}$, let $\phi_{\mathbb{V}} : \mathbb{V}_J \to \mathbb{V}$ be defined as in (1). Then,*

(a) *ϕ is SC-monotone of order n associated with $\mathbb{H}^n$ if ϕ is matrix monotone of order $2n$.*

(b) *ϕ is SC-convex of order n associated with $\mathbb{H}^n$ if ϕ is matrix convex of order $2n$.*

Proof. (a) Suppose $x \succeq_{\mathbb{H}^n_+} y$ and ϕ is matrix monotone of order $2n$. First, Lemma 3.1 indicates $T_{\mathbb{H}^n}(x) \succeq_{\mathbb{S}^{2n}_+} T_{\mathbb{H}^n}(y)$. Then, from assumption of matrix monotonicity, we have

$$\phi_{\mathbb{S}^{2n}}(T_{\mathbb{H}^n}(x)) \succeq_{\mathbb{S}^{2n}_+} \phi_{\mathbb{S}^{2n}}(T_{\mathbb{H}^n}(y)).$$

This together with equation (6) implies $T_{\mathbb{H}^n}(\phi_{\mathbb{H}^n}(x)) \succeq_{\mathbb{S}^{2n}_+} T_{\mathbb{H}^n}(\phi_{\mathbb{H}^n}(y))$. Applying Lemma 3.1 again, we obtain $\phi_{\mathbb{H}^n}(x) \succeq_{\mathbb{H}^n_+} \phi_{\mathbb{H}^n}(y)$.

(b) Suppose ϕ is matrix convex of order $2n$. Then, for $0 \leq \alpha \leq 1$, we know

$$\phi_{\mathbb{S}^{2n}}\left(\alpha T_{\mathbb{H}^n}(x) + (1-\alpha) T_{\mathbb{H}^n}(y)\right) \preceq_{\mathbb{S}^{2n}_+} \alpha \phi_{\mathbb{S}^{2n}}(T_{\mathbb{H}^n}(x)) + (1-\alpha)\phi_{\mathbb{S}^{2n}}(T_{\mathbb{H}^n}(y)).$$

In addition, the linearity of $T_{\mathbb{H}^n}$ and equation (6) imply

$$\phi_{\mathbb{S}^{2n}}\left(T_{\mathbb{H}^n}(\alpha x + (1-\alpha)y)\right) \preceq_{\mathbb{S}^{2n}_+} \alpha T_{\mathbb{H}^n}(\phi_{\mathbb{H}^n}(x)) + (1-\alpha) T_{\mathbb{H}^n}(\phi_{\mathbb{H}^n}(y)).$$

Using equation (6) and linearity of $T_{\mathbb{H}^n}$ again, we have

$$T_{\mathbb{H}^n}\left(\phi_{\mathbb{H}^n}(\alpha x + (1-\alpha)y)\right) \preceq_{\mathbb{S}^{2n}_+} T_{\mathbb{H}^n}\left(\alpha \phi_{\mathbb{H}^n}(x) + (1-\alpha)\phi_{\mathbb{H}^n}(y)\right).$$

Then, applying Lemma 3.1 yields

$$\phi_{\mathbb{H}^n}(\alpha x + (1-\alpha)y) \preceq_{\mathbb{H}^n_+} \alpha \phi_{\mathbb{H}^n}(x) + (1-\alpha)\phi_{\mathbb{H}^n}(y)$$

which is the desired result. $\square$

Analogous to Lemma 3.1, there holds

$$x \succeq_{\mathbb{Q}^n_+} y \Longleftrightarrow T_{\mathbb{Q}^n}(x) \succeq_{\mathbb{S}^{4n}_+} T_{\mathbb{Q}^n}(y) \quad \forall x, y \in \mathbb{Q}^n$$

which also lead to the following lemma by similar arguments as in Lemma 3.2.

Lemma 3.3 *Suppose that $\mathbb{V}$ is the algebra $\mathbb{Q}^n$ of $n \times n$ complex Hermitian matrices. For any given $\phi : J \to \mathbb{R}$, let $\phi_{\mathbb{V}} : \mathbb{V}_J \to \mathbb{V}$ be defined as in (1). Then,*

(a) *ϕ is SC-monotone of order n associated with $\mathbb{Q}^n$ if ϕ is matrix monotone of order $4n$.*

(b) *ϕ is SC-convex of order n associated with $\mathbb{Q}^n$ if ϕ is matrix convex of order $4n$.*

Lemma 3.4 *[23, Theorem 3.1, Theorem 4.1] Suppose that $\mathbb{V}$ is the Jordan spin algebra $\mathbb{J}^n$. For any given $\phi : J \to \mathbb{R}$, let $\phi_{\mathbb{V}} : \mathbb{V}_J \to \mathbb{V}$ be defined as in (1). Then,*

(a) *ϕ is SOC-monotone if ϕ is matrix-monotone of order 2.*

(b) *ϕ is SOC-convex if ϕ is matrix-convex of order 2.*

The main idea here is that we employ embeddings $T_{\mathbb{H}^n}$ and $T_{\mathbb{Q}^n}$ to provide a sufficient condition for ϕ being SC-monotone (SC-convex) by its matrix monotonicity (matrix convexity). Now, together with some result in [23], we prsent our main result.

Theorem 3.1 *Suppose that* $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ *is a simple Euclidean Jordan algebra of rank* n *except for* $\mathbb{O}^3$*. For any given* $\phi : J \to \mathbb{R}$*, let* $\phi_{\mathbb{V}} : \mathbb{V}_J \to \mathbb{V}$ *be defined as in (1). Then,*

(a) ϕ *is matrix monotone (matrix convex) of order* n *if it is SC-monotone (SC-convex) of order* n*.*

(b) ϕ *is SC-montone (SC-convex) of order* n *associated with* $\mathbb{V}$ *if it is matrix monotone (matrix convex) of order* $4n$*.*

Proof. (a) When $n > 3$, Theorem 2.1 says $\mathbb{V}$ is isomorphic to the algebra $\mathbb{S}^n$, $\mathbb{H}^n$, or $\mathbb{Q}^n$. Note that a real number is a special complex number, which is also a special quaternion. The SC-monotonicity (SC-convexity) of order n of ϕ implies that ϕ is matrix monotone (matrix convex) of order n. When $n = 2$, the SC-monotonicity (SC-convexity) of order 2 of ϕ is equivalent to the SOC-monotonicity (SOC-convexity) (see [7]). Thus, from [23], it follows that ϕ is matrix monotone (matrix convex) of order 2.

(b) When $n > 3$, Theorem 2.1 says $\mathbb{V}$ is isomorphic to the algebra $\mathbb{S}^n$, $\mathbb{H}^n$, or $\mathbb{Q}^n$. Suppose ϕ is matrix monotone (matrix convex) of order $4n$. Then, we have that ϕ is also matrix monotone (matrix convex) of order $2n$ (order n). Thus, applying Theorem 2.1 and Lemmas 3.2-3.3, ϕ is SC-monotone (SC-convex) of order n. When $n = 2$, from [23] we know that ϕ is SOC-monotone (SOC-convex), which is equivalent to saying that ϕ SC-monotone (SC-convex) of order 2 due to Theorem 2.1. $\Box$

Remark 3.1 *It should be pointed out that for the SC-monotonicity of continuously differentiable* ϕ*, Korányi [15] showed that* ϕ *is SC-monotone of order* n *if and only if* ϕ *is matrix-monotone of order* n*. Thus, for the SC-monotonicity, the result of Theorem 3.1 is weaker than that of [15] obtained via direct analysis. However, for the SC-convexity, to our best knowledge, the result of Theorem 3.1 is new. For application in symmetric cone optimization it is very important to know which class of functions is SC-convex. Theorem 3.1 has good contribution in the literature in our opinion because it tells us that all matrix convex functions must be SC-convex.*

As a consequence of Theorem 3.1, we have the following corollary which builds a bridge between matrix monotonicity (matrix convexity) and SC-monotonicity (SC-convexity).

Corollary 3.1 *Let* $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ *be a simple Euclidean Jordan algebra except for* $\mathbb{O}^3$*. For any given* $\phi : J \to \mathbb{R}$*, let* $\phi_{\mathbb{V}} : \mathbb{V}_J \to \mathbb{V}$ *be defined as in (1). Then,* ϕ *is SC-monotone (respectively, SC-convex) associated with* $\mathbb{V}$ *if and only if it is matrix monotone (respectively, matrix convex).*

Unfortunately our method can not be applied to the only excetional case $\mathbb{O}^3$. There are two reasons to explain this. First, it seems imposible to embed $\mathbb{O}^3$ into some $\mathbb{S}^m$. Second, there exists a discrepancy between $\phi_{\mathbb{S}^m}(\mathcal{L}(x))$ and $\mathcal{L}(\phi_{\mathbb{O}^3}(x))$. For any $x \in \mathbb{O}^3_J$, suppose x has the spectral decomposition $x = \sum_{j=1}^3 \lambda_j(x)c_j$, where $\lambda_1(x) \geq \lambda_2(x) \geq \lambda_3(x)$ are the eigenvalues of x and $\{c_1, c_2, c_3\}$ (depending on x) is the corresponding Jordan frame. Let $\mathcal{L}(x), \mathcal{C}_{jl}(x)$ be defined as in Section 2. We have

$$\mathcal{L}(\phi_{\mathbb{O}^3}(x)) = \sum_{j=1}^{3} \phi(\lambda_j(x))\mathcal{C}_{jj}(x) + \sum_{1\leq j<l\leq 3} \frac{\phi(\lambda_j(x)) + \phi(\lambda_l(x))}{2}\mathcal{C}_{jl}(x) \qquad \forall x \in \mathbb{V}_J. \tag{7}$$

Note here that $\phi_{\mathbb{O}^3}(x) = \sum \phi(\lambda_j(x))c_j$. Let $\{u_1, u_2, \ldots, u_{27}\}$ be an orthonormal basis of $\mathbb{O}^3$. Let $L(x), C_{jl}(x)$ be the corresponding matrix representations of $\mathcal{L}(x), \mathcal{C}_{jl}(x)$ with respect to the basis $\{u_1, u_2, \ldots, u_{27}\}$. This means that for $1 \leq a, b \leq 27$

$$[L(x)]_{a,b} = \langle u_a, \mathcal{L}(x)u_b\rangle \quad \text{and} \quad [C_{jl}(x)]_{a,b} = \langle u_a, \mathcal{C}_{jl}(x)u_b\rangle.$$

Since $\mathbb{O}^3$ is a Euclidean Jordan algebra, $\mathcal{L}(x)$ and $\mathcal{C}_{jl}(x)$ are self-adjoint. Thus, $L(x)$ and $C_{jl}(x)$ are real symmetric matrices in $\mathbb{S}^{27}_J$. It follows that

$$L(\phi_{\mathbb{O}^3}(x)) = \sum_{j=1}^{3} \phi(\lambda_j(x))C_{jj}(x) + \sum_{1\leq j<l\leq 3} \frac{\phi((\lambda_j(x)) + \phi(\lambda_l(x))}{2}C_{jl}(x), \qquad \forall x \in \mathbb{V}_J.$$

For any $h \in \mathbb{O}^3$, there exists a unique $\tilde{h} \in \mathbb{R}^{27}$ such that $h = \sum_{i=1}^{27} \tilde{h}_i u_i$. Then, it is obvious to check

$$\langle h, \phi_{\mathbb{O}^3}(x) \circ k\rangle_{\mathbb{O}^3} = \langle h, \mathcal{L}(\phi_{\mathbb{O}^3}(x))k\rangle_{\mathbb{O}^3} = \langle \tilde{h}, L(\phi_{\mathbb{O}^3}(x))\tilde{k}\rangle_{\mathbb{R}^{27}} \quad \forall \quad h, k \in \mathbb{O}^3,$$

which implies

$$\phi_{\mathbb{O}^3}(x) \succeq_{\mathbb{O}^3_+} \phi_{\mathbb{O}^3}(y) \Longleftrightarrow L(\phi_{\mathbb{O}^3}(x)) \succeq_{\mathbb{S}^{27}_+} L(\phi_{\mathbb{O}^3}(y)).$$

However, on the other hand, we know

$$\phi_{\mathbb{S}^{27}}(L(x)) = \sum_{j=1}^{3} \phi(\lambda_j(x))C_{jj}(x) + \sum_{1\leq j<l\leq 3} \phi\left(\frac{\lambda_j(x) + \lambda_l(x)}{2}\right) C_{jl}(x). \tag{8}$$

Note here that

$$L(x) = \sum_{j=1}^{3} \lambda_j(x)C_{jj}(x) + \sum_{1\leq j<l\leq 3} \frac{\lambda_j(x) + \lambda_l(x)}{2}C_{jl}(x).$$

Thus, the discrepency between $\phi_{\mathbb{S}^{27}}(L(x))$ and $L(\phi_{\mathbb{O}^3}(x))$ is

$$\phi_{\mathbb{S}^{27}}(L(x)) - L(\phi_{\mathbb{O}^3}(x)) = \sum_{1\leq j<l\leq 3} \left[\phi\left(\frac{\lambda_j(x) + \lambda_l(x)}{2}\right) - \frac{\phi((\lambda_j(x)) + \phi(\lambda_l(x))}{2}\right] C_{jl}(x),$$

which is complicated to handle. Therefore, we exclude this exceptional case $\mathbb{O}^3$ in the conclusion.

To close this section, we take a careful look at some examples of SC-monotone functions. By applying [26, Example 3] and Corollary 3.1, the following functions are SC-monotone.

Example 3.1 *For a general simple Euclidean Jordan algebra* $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ *except for* $\mathbb{O}^3$*,*

(i) $\phi(t) = t^q$ $(t \geq 0)$ *is SC-monotone associated with* $\mathbb{V}$ *if and only if* $0 \leq q \leq 1$*.*

(ii) $\phi(t) = -t^{-q}$ $(t > 0)$ *is SC-monotone associated with* $\mathbb{V}$ *if and only if* $0 \leq q \leq 1$*.*

(iii) $\phi(t) = -\cot(t)$ $(0 < t < \pi)$ *is SC-monotone associated with* $\mathbb{V}$*.*

(iv) $\phi(t) = \ln^q(x)$ $(t > 0)$ *with* $q \in (0, 1]$ *is SC-monotone associated with* $\mathbb{V}$*.*

Moreover, [26, Example 35] and Corollary 3.1 indicate that the following functions are SC-convex.

Example 3.2 *For a general simple Euclidean Jordan algebra* $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ *except for* $\mathbb{O}^3$*,*

(i) $\phi(t) = -\ln t$ $(t > 0)$ *is SC-convex associated with* $\mathbb{V}$*.*

(ii) $\phi(t) = -t^r$ $(t \geq 0)$ *with* $r \in [1, 2]$ *and* $\phi(t) = -t^r$ $(t > 0)$ *with* $r \in [-1, 0]$ *are SC-convex associated with* $\mathbb{V}$*.*

(iii) *the entropy function* $\phi(t) = t \ln t$ $(t \geq 0)$ *is SC-convex associated with* $\mathbb{V}$*.*

From the SC-monotonicity of the function in Example 3.1(i), we readily recover the results of [18, Corollary 9] and [10, Prop. 8]. Moreover, from the SC-monotonicity of the function in Example 3.1(ii), we have that $x \succeq_{\mathcal{K}} y \succ_{\mathcal{K}} 0$ if and only if $y^{-1} \succ_{\mathcal{K}} x^{-1} \succ_{\mathcal{K}} 0$. On the other hand, we show the SC-convexity of some well-known barrier functions: logarithmic barrier function $-\ln t$ $(t > 0)$ and the power function $-t^r$ $(t > 0)$ with $r \in [-1, 0)$, which can be employed in the interior point methods for for solving the symmetric cone optimization problems.

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