Shape From Perspective Trihedral Angle Constraint †

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Abstract: *This paper defines and investigates a fundamental problem of determining the position and orientation of a 3D object using single perspective image view. The technique is based on the interpretation of trihedral angle constraint information. A new closed form solution to the problem is proposed. The method also provides a general analytic technique for dealing with a class of problem of shape from inverse perspective projection by using "Angle to Angle Correspondence Information". Simulation experiments show that our method is effective and robust for real application.*

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Key Words : *Shape from angle, Shape from perspective projection, Pose Estimation, 3D object recognition.*

1. Introduction

Shape from inverse perspective projection is an essential method for model-based 3D reconstruction. There are many applications of this approach in Robotics, Cartography and Computer Vision [8-13].

The formal definition of shape from inverse perspective projection can be stated as follows : Let perspective projection be the ideal model of a camera, then the camera imaging process is given by

$$\begin{bmatrix} \vec{p} \\ 1 \end{bmatrix} \equiv (\, KR \;\; K\vec{T}\,) \begin{bmatrix} \vec{P} \\ 1 \end{bmatrix}, \; K = \begin{bmatrix} k_u f & 0 & u_0 \\ 0 & k_v f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where, $\vec{P} = (\, x,\, y,\, z\,)^T$ is the description of a 3D point in an object coordinate system and $\vec{p} = (u, v)^T$ is the 2D projection of $\vec{P}$ on the image plane; rotation R and translation $\vec{T}$ form the transformation from the object coordinate system to the camera coordinate system; matrix K describes the intrinsic parameters of the camera. The problem of shape from inverse perspective projection is to determine the unknown rotation matrix R and the translation $\vec{T}$ from certain 3D geometric features of the object and their 2D image geometric features in a single perspective view.

Three types of situations are mostly discussed in the problem of shape from inverse perspective projection :

1. Perspective point to point correspondence problem. This problem is usually called **PnP** problem [8], when n pairs of corresponding points are known.

2. Perspective line to line correspondence problem. Like the case in the above, we call the problem as **LnL** problem, when n pairs of corresponding lines are specified.

3. Perspective angle to angle correspondence problem. We name this problem as **AnA** problem if n pairs of corresponding angles are given.

A closed form solution is the most desirable result for each of PnP, LnL and AnA problems for its simplicity, stability and speed. In this paper, a closed form solution is presented for the general problem of trihedral angle constraint, which is an A3A problem.

In recent years, trihedral angle constraint has been addressed by many authors from different viewpoints. The relevant work can be divided into the following two categories :

(a) Direct approach : In this category, angle information is employed directly. Kanade [6] proposes an analytic solution for the problem under orthographic projection. For perspective projection, algebraic solutions have been given by Kanatani [3], Shakunaga and Kaneko [5] for special cases when two or three space angles are right angles; in addition, some constructive algorithms are suggested by Horaud [7], Shakunaga and Kaneko [5] for solving the general problem .

(b) Indirect approach : Without using angles directly, the configuration of a trihedral angle can also be specified by four space points or by a junction of three 3D lines. In this sense, we can consider trihedral angle constraint as a special case of the P4P or L3L problem. Therefor, the methods for solving these two types of problems can be applied for trihedral angle constraint ([11], [12]).

Our new solution for trihedral angle constraint uses the direct approach. This method can be considered as a complete closed form solution for the general AnA problems in a minimal condition. The significant advantage of our approach over the methods of P4P [11] and L3L [12] is that the angle measure is independent of the coordinate system but the description of a point or a line varies when the related coordinate system is changed.

2. A new mathematical framework

2.1. Preliminary formulation

2.1.1 Canonical image structure

Assume that the intrinsic matrix K of a camera model are given. Then, we can derive from (1) that

$$\begin{bmatrix}\vec{p}' \\ 1\end{bmatrix} = K^{-1}\begin{bmatrix}\vec{p} \\ 1\end{bmatrix} \equiv (\; R\;\vec{T}\;)\begin{bmatrix}\vec{P} \\ 1\end{bmatrix} \qquad (2)$$

where $\vec{p}'$ is determined only by the extrinsic parameters of rotation R and translation $\vec{T}$. We call $\vec{p}'$ as Canonical Image and will consider only this representation in the following discussions for the development of our method.

2.1.2 View orientation transformation

We define a view orientation transformation as a pure rotation upon a camera coordinate system. Because the corresponding relationship between the image points under such a transformation is uniquely determined, we can use this relation for facilitating the problem formulation of trihedral constraint [3].

Among the infinite view orientation transformations which can transform the view axis of a camera coordinate system from an old orientation to a new one, we consider the one which is formed by a rotation around the y-axis of camera, followed by a rotation around the x-axis of camera. Let a new view orientation be specified by an image point $(u, v)^T$, then the rotation matrix is as below :

$$R = \begin{bmatrix} 1/d_1 & 0 & -u/d_1 \\ -uv/d_1 d_2 & d_1/d_2 & -v/d_1 d_2 \\ u/d_2 & v/d_2 & 1/d_2 \end{bmatrix} \qquad (3)$$

where, $d_1 = \sqrt{u^2+1}$, $d_2 = \sqrt{u^2+v^2+1}$.

2.1.3 Trihedral angle constraint formulation

Let a trihedral angle be formed by $\vec{P}_i = (x_i, y_i, z_i)^T$, $i = 0, \ldots, 3$ with $\vec{P}_0$ as the angular point, and $\vec{p}_i = (u_i, v_i)^T$ be the perspective projection of $\vec{P}_i$. Without loss of generality, assume that $\vec{P}_0$ is on the view axis. Let β_i be the angle formed by $\vec{p}_i$ and the u-axis; $\vec{L}_i = \vec{P}_i - \vec{P}_0$ and $\vec{N}_i$ be the unit direction vector along $\vec{L}_i$; and γ_i be the angle formed by L_i and the view axis; then,

$$\sin\beta_i = \frac{v_i}{\sqrt{u_i^2+v_i^2}} \quad \cos\beta_i = \frac{u_i}{\sqrt{u_i^2+v_i^2}}$$
$$\vec{N}_i = (\,\sin\gamma_i\cos\beta_i,\ \sin\gamma_i\sin\beta_i,\ \cos\gamma_i\,)^T \qquad (4)$$

Denoting η_{ij} as the angle formed by $\vec{N}_i$ and $\vec{N}_j$, we have the angle constraints for a trihedral angle as below.

$$\begin{aligned}
\vec{N}_1\cdot\vec{N}_2 &= \sin\gamma_1\sin\gamma_2\cos(\beta_1-\beta_2)+\cos\gamma_1\cos\gamma_2 = \cos\eta_{12} \\
\vec{N}_1\cdot\vec{N}_3 &= \sin\gamma_1\sin\gamma_3\cos(\beta_1-\beta_3)+\cos\gamma_1\cos\gamma_3 = \cos\eta_{13} \quad (5) \\
\vec{N}_2\cdot\vec{N}_3 &= \sin\gamma_2\sin\gamma_3\cos(\beta_2-\beta_3)+\cos\gamma_2\cos\gamma_3 = \cos\eta_{23}
\end{aligned}$$

When the angles η_{12}, η_{13}, and η_{23} are given, we have a system of three equations with three unknowns. So we expect to solve γ_1, γ_2, γ_3 and then to determine the orientation of the trihedral angle in camera coordinate system.

Kanatani [3] first suggested the formulation for angle constraint and proposed a solution of (5) for the special case where at least two of η_{12}, η_{13}, and η_{23} are right angles. We will now present a complete solution for (5).

2.2. Analytical solution for trihedral angle constraint

2.2.1 Estimate the orientation

Our idea for solving (5) is straightforward. First, assume that $\vec{N}_3$ can be expressed by $\vec{N}_1$ and $\vec{N}_2$ as

$$\vec{N}_3 = a\vec{N}_1 + b\vec{N}_2 + c\vec{N}_1 \times \vec{N}_2 \qquad (6)$$

We have

$$\begin{aligned}
\vec{N}_1\cdot\vec{N}_3 &= a + b\cos\eta_{12} = \cos\eta_{13} \\
\vec{N}_2\cdot\vec{N}_3 &= a\cos\eta_{12} + b = \cos\eta_{23} \qquad (7) \\
\vec{N}_3\cdot\vec{N}_3 &= a\cos\eta_{13} + b\cos\eta_{23} + c^2\sin^2\eta_{12} = 1
\end{aligned}$$

Then, the coefficients a, b and c can be derived

$$\begin{aligned}
a &= (\,\cos\eta_{13} - \cos\eta_{12}\cos\eta_{23})\,/\,\sin^2\eta_{12} \\
b &= (\,\cos\eta_{23} - \cos\eta_{12}\cos\eta_{13})\,/\,\sin^2\eta_{12} \qquad (8) \\
c &= \pm\sqrt{(\,1 - a\cos\eta_{13} - b\cos\eta_{23})\,/\,\sin^2\eta_{12}}
\end{aligned}$$

By the values of a, b and c, we can rewrite (8) to get

$$\begin{aligned}
\sin\gamma_3 &= \frac{a\sin\gamma_1\cos\beta_1 + b\sin\gamma_2\cos\beta_2}{\cos\beta_3} + \frac{c(\sin\gamma_1\sin\beta_1\cos\gamma_2 - \sin\gamma_2\sin\beta_2\cos\gamma_1)}{\cos\beta_3} \\
\sin\gamma_3 &= \frac{a\sin\gamma_1\sin\beta_1 + b\sin\gamma_2\sin\beta_2}{\sin\beta_3} + \frac{c(\sin\gamma_2\cos\beta_2\cos\gamma_1 - \sin\gamma_1\cos\beta_1\cos\gamma_2)}{\sin\beta_3} \qquad (9) \\
\cos\gamma_3 &= a\cos\gamma_1 + b\cos\gamma_2 + c\sin\gamma_1\sin\gamma_2 sin(\beta_2-\beta_1)
\end{aligned}$$

Without loss of generality, suppose $\cos(\beta_1-\beta_2)\neq 0$. Then,

$$\sin\gamma_1\sin\gamma_2 = \frac{\cos\eta_{12} - \cos\gamma_1\cos\gamma_2}{\cos(\beta_1-\beta_2)} \qquad (10)$$

by the first equation of (5). Substituting (9) and (10) into the second and third equations of (5), we get

$$\begin{aligned} A_1\cos^2\gamma_1 + B_1\cos\gamma_1\cos\gamma_2 + C_1\cos^2\gamma_1\cos\gamma_2 \\ + D_1\cos\gamma_1 + E_1\cos\gamma_2 + F_1 = 0 \\ A_2\cos^2\gamma_2 + B_2\cos\gamma_2\cos\gamma_1 + C_2\cos^2\gamma_2\cos\gamma_1 \\ + D_2\cos\gamma_2 + E_2\cos\gamma_1 + F_2 = 0 \end{aligned} \quad (11)$$

Note that two equalities in (9) for sin γ_3 yield the same result described by (11), where

$$\begin{aligned} A_1 &= \text{a}\cos(\beta_1\text{-}\beta_2)\sin(\beta_1\text{-}\beta_2) \\ B_1 &= \text{b}\sin(\beta_2\text{-}\beta_3) \\ C_1 &= \text{c}\cos(\beta_2\text{-}\beta_1)\sin(\beta_3\text{-}\beta_1) \\ D_1 &= \text{-c}\cos(\beta_2\text{-}\beta_3)\cos\eta_{12} \\ E_1 &= \text{c}\cos(\beta_1\text{-}\beta_2)\cos(\beta_1\text{-}\beta_3) \\ F_1 &= \text{a}\cos(\beta_1\text{-}\beta_3)\sin(\beta_2\text{-}\beta_1) + \sin(\beta_3\text{-}\beta_2)\cos\eta_{13} \end{aligned} \quad (12)$$

$$\begin{aligned} A_2 &= \text{b}\cos(\beta_1\text{-}\beta_2)\sin(\beta_2\text{-}\beta_3) \\ B_2 &= \text{a}\sin(\beta_1\text{-}\beta_3) \\ C_2 &= \text{c}\cos(\beta_2\text{-}\beta_1)\sin(\beta_3\text{-}\beta_2) \\ D_2 &= \text{c}\cos(\beta_1\text{-}\beta_3)\cos\eta_{12} \\ E_2 &= \text{-c}\cos(\beta_1\text{-}\beta_2)\cos(\beta_2\text{-}\beta_3) \\ F_2 &= \text{b}\cos(\beta_2\text{-}\beta_3)\sin(\beta_1\text{-}\beta_2) + \sin(\beta_3\text{-}\beta_1)\cos\eta_{23} \end{aligned} \quad (13)$$

From the first equation of (11), we have

$$\cos\gamma_2 = -\frac{A_1\cos^2\gamma_1 + D_1\cos\gamma_1 + F_1}{C_1\cos^2\gamma_1 + B_1\cos\gamma_1 + E_1} \quad (14)$$

Substituting (14) into the second equation of (11), we get

$$\sum_{i=0}^{5} s_i\cos^i\gamma_1 = 0 \quad (15)$$

Where,

$$\begin{aligned} s_5 &= C_2A_1^2 + E_2C_1^2 - B_2A_1C_1 \\ s_4 &= A_2A_1^2 + F_2C_1^2 - B_2A_1B_1 - B_2D_1C_1 \\ &\quad - D_2A_1C_1 + 2C_2A_1D_1 + 2E_2C_1B_1 \\ s_3 &= 2A_2A_1D_1 - D_2A_1B_1 - D_2C_1D_1 + 2E_2C_1E_1 \\ &\quad + 2C_2A_1F_1 + C_2D_1^2 - B_2A_1E_1 - B_2C_1F_1 \\ &\quad - B_2D_1B_1 + 2F_2C_1B_1 + E_2B_1^2 \\ s_2 &= 2A_2A_1F_1 + A_2D_1^2 - D_2A_1E_1 - D_2F_1C_1 \\ &\quad - D_2D_1B_1 + 2F_2C_1E_1 + F_2B_1^2 + 2C_2D_1F_1 \\ &\quad - B_2D_1E_1 - B_2B_1F_1 + 2E_2B_1E_1 \\ s_1 &= 2A_2D_1F_1 - D_2D_1E_1 - D_2F_1B_1 + 2F_2B_1E_1 \\ &\quad + C_2F_1^2 - B_2F_1E_1 + E_2E_1^2 \\ s_0 &= A_2F_1^2 + F_2E_1^2 - D_2F_1E_1 \end{aligned} \quad (16)$$

By (15), (14) and the third equality of (9), we can solve cos γ_1, cos γ_2 and cos γ_3 step by step. Then, a trihedral angle can be determined in camera coordinate system by :

$$\vec{P}_i = \vec{P}_0 + l_i\vec{N}_i \quad (i = 1,2,3) \quad (17)$$

where, $l_i > 0$ is the length of $\vec{L}_i$. Because so far only the three unit vectors $\vec{N}_i$ can be assigned by solving (5), we obtain only the orientation of a trihedral angle. To find its full position, more information is necessary.

2.2.2 Determine the full position of a trihedral angle

Let a trihedral angle in an object frame be given by

$$\vec{P}_i' = \vec{P}_0' + l_i'\vec{N}_i' \quad (i = 1,2,3) \quad (18)$$

Where, the corresponding relationship between (17) and (18) is specified by $\vec{P}_i$ to $\vec{P}_i'$, $\vec{N}_i$ to $\vec{N}_i'$ and $l_i = l_i'$. We can obtain the rotation R by the relation $\vec{N}_i = R\vec{N}_i'$. To find the translation $\vec{T} = (t_x, t_y, t_z)^T$, let $\vec{P}$ and $\vec{p}$ be a pair of matched object point and image point, $R = (r_{ij})_{3x3}$, we have

$$\begin{aligned} r_{11}x + r_{12}y + r_{13}z + t_x - u(r_{31}x + r_{32}y + r_{33}z + t_z) = 0 \\ r_{21}x + r_{22}y + r_{23}z + t_y - v(r_{31}x + r_{32}y + r_{33}z + t_z) = 0 \end{aligned} \quad (19)$$

If two pairs of matched points are available, the translation $\vec{T}$ can be obtained by solving (19). It follows that to get a full solution for trihedral angle constraint, we still need two pairs of matched object point and image point.

Alternatively, if one of length l_i in (17) is known, the $\vec{P}_0$ can be simply determined by each of the following two equations, provided the denominator is not zero.

$$\begin{aligned} z_0 = l_i(\sin\gamma_i\cos\beta_i - u_i\cos\gamma_i) / u_i \\ z_0 = l_i(\sin\gamma_i\sin\beta_i - v_i\cos\gamma_i) / v_i \end{aligned} \quad (z_0 > 1) \quad (20)$$

Then the trihedral angle is completely determined in camera frame but does involve any object coordinate system.

2.2.3 The algorithm

The solution procedure for shape from trihedral angle constraint is summarized as follows :

Prerequisite : Suppose that the intrinsic parameters of the camera are given; then a 2D trihedral configuration is picked and the corresponding 3D angles are specified.

Step 1. Use (2) to get the canonical representation for the image features.

Step 2. Use the angular vertex of the 2D trihedral configuration to compute the matrix R (3); then, convert the original image features to their new coordinates.

Step 3. Match the 2D vs. 3D angles and determine the constraint equation system (5).

Step 4. Derive the fifth-order equation (15) and solve it to get cos γ_1 ; if there is no solution, go to step 8.

Step 5. Calculate cos γ_2 by (14); if there is no solution, go to step 8.

Step 6. Calculate cos γ_3 by the third equality of (9); if there is no solution, go to step 8;

Step 7. Check the solution against the original constraint equation system (5).

Step 8. If there is no solution but some other matching pattern exists for the 2D and 3D angles, adopt a new matching pattern, go to step 3; otherwise, terminate.

Step 9. If additional information is available for finding a full solution, find it using (19) or (20).

Step 10. Transform the final result (17) back to the original camera coordinate system using the inverse of the rotation matrix R defined in Step 2.

3. Solution analysis

3.1 The mirror solution

Notice that in (5), if γ_1, γ_2, γ_3 form a solution of the equation system, $\pi-\gamma_1$, $\pi-\gamma_2$, $\pi-\gamma_3$ must be another solution of the same system. These two solutions are symmetric to the plane which contains $\vec{P}_0$ and is parallel to the image plane. We call the two solutions as mirror solutions. Mirror solution is the mathematical formulation of the well-known "Necker's cube vision illusion".

If no further clue is available, two mirror solutions are all the possible solutions for a trihedral angle constraint. But we just need to execute the solution procedure once by taking an arbitrary sign for c in (8), and then use the mirror solution feature to find the other solutions. The mirror solution feature enables us to save half of the computation.

3.2 Comments on the general problems

The investigations on the closed form solution for the problems of PnP, LnL and AnA are listed in Table 3-1.

Table 3-1. A Comparison of the Existing Methods

	Constraint Relation	Linear Solution	Closed Form Solution
PnP	Linear	n ≥ 6	P4P [11]
	Nonlinear	No	P3P & P4P [8] P3P [9]
LnL	Linear	n ≥ 8	L3L [12, 13]
AnA	Nonlinear	No	A3A, This Paper

In this table, we divide the problems of PnP, LnL and AnA into two categories of linear and nonlinear constraints. The difference between the two categories is that the 3D features are defined in an object coordinate system for linear constraint, but they are given by a group of scalars for nonlinear constraint. LnL constraint belongs to linear category because it is necessary to refer to some object coordinate system for specifying a 3D line. Contrarily, AnA constraint is in nonlinear category because only n scalars are needed for specifying n spatial angles. An intersting fact is that PnP can be presented in both categories when n > 1 [8, 9, 11].

In mathematics, a linear formulation for the problem LnL or PnP may be changed to some nonlinear format of AnA or PnP. But we can not change a nonlinear formulation for the problem of AnA or PnP to some linear format of LnL or PnP because the nonlinear formulation is independent to object frame. So the nonlinear formulation may be more powerful than the linear formulation for certain applications.

3.3 Special configuration Cases

Some special configurations about trihedral angle constraint described below are commonly encountered in real applications. For these cases, The general fifth order equation (15) can be simplified to lower order to facilitate the solving procedures.

(a) Coplanar configuration

In this case, the three vectors $\vec{N}_1$, $\vec{N}_2$, $\vec{N}_3$ are located on a plane, equation (15) becomes

$$s_4\cos^4\gamma_1 + s_2\cos^2\gamma_1 + s_0 = 0$$

This is actually a quadratic equation on $\cos^2\gamma_1$

(b) The configuration with two or three right angles

In the case that there are at least two right angles in a trihedral angle, we can let $\vec{N}_3$ be normal to $\vec{N}_1$ and $\vec{N}_2$. Then, (15) can be rewritten as

$$s_5\cos^4\gamma_1 + s_3\cos^2\gamma_1 + s_1 = 0$$

As in case (a), we obtain a quadratic equation on $\cos^2\gamma_1$.

(c) Special image configurations

If one 2D right angle exists, say β_1-β_2=$\pi/2$, we have A_i=E_i=0 (i=1,2) in (12) and (13), so (15) becomes a cubic

$$s_4\cos^3\gamma_1 + s_3\cos^2\gamma_1 + s_2\cos\gamma_1 + s_1 = 0$$

If β_1-β_2=π, we have C_1=C_2= 0 in (12) and (13), so (15) becomes a quadrinomial as

$$s_4\cos^4\gamma_1 + s_3\cos^3\gamma_1 + s_2\cos^2\gamma_1 + s_1\cos\gamma_1 + s_0 = 0$$

4. Experimental validation

4.1 Experimental design

Regarding the performance of the new approach, we are mainly concerned about its effects on the following three aspects :

1. Subject to the following three inherent criteria :

   C-1 : Each solution obtained by (15), (14) and (9) must be in [-1, 1].

   C-2 : Each group of solutions should satisfy the primitive equation system (5).

C-3 : if additional information about the 3D length of the side of a trihedral angle is available, the solution of (20) should be bigger than 1.

we will investigate how many solutions can occur for an arbitrary trihedral angle constraint and whether the true solution is obtainable by our method.

2. We should inspect when a correctly matched trihedral angle constraint is derived, if the real solution is obtained by our method; or when an ill-matched trihedral angle constraint is presented, whether our method can identify the ill-condition.

3. Our next task is to study the presented approach for its sensitivity to noise.

To test the three questions in general, we arranged our experimental procedure as follows :

Data-1 : Randomly generate a set of ideal trihedral angle constraints in a camera coordinate system.

Test-1 : Use correct angle matching relationship on the ideal data to solve a trihedral angle constraint and then to investigate the solution pattern.

Test-2 : Use incorrect angle matching relationship on the ideal data to solve a trihedral angle constraint and then to check the solution results.

Data-2 : For a trihedral angle constraint, the net effects of noises can be simply considered as a noise acted on the β_1, β_2 and β_3 of (5). We choose an interval [-m, m] as the source of noise. Then, a noise triplet is randomly generated from the noise interval and the trihedral angle constraint generated by **Data-1** is added on the noise triplet to produce a noised data.

Table 4-1 The Solution Distribution of Equation (15)

Frequency		Number of Solutions 0	1	2	3	4	5
Ideal Data	Right Match	0	3	48	42	7	0
	Error Match	0	13	58	29	0	0
Noisy Data	Right Match	0	2	58	39	1	0
	Error Match	0	11	65	24	0	0

Table 4-2 The Reserved Solution Distribution

Frequency		Number of Solutions 0	1	2	3	4	5
Ideal Data	Right Match	0	45	49	5	1	0
	Error Match	85	10	5	0	0	0
Noisy Data	Right Match	2	52	44	2	0	0
	Error Match	86	8	6	0	0	0

Test-3 : Do Test-1 for Data-2.

Test-4 : Do Test-2 for Data-2.

The test results are given in the following paragraphs.

4.2 The solution distribution and patterns

According to the procedure depicted in 4.1, 100 groups of data are generated and the tested results are shown in the Table 4-1 and Table 4-2. In the tables, an entry represents the emerging frequency of the case specified by the corresponding row title and column title. For example, the entry 48 in the first row and the third column of Table 4-1 means that, when using a randomly generated ideal trihedral angle constraint and supposing that the correct match is employed, we got 48 times of the 2-solution cases in the 100 experiments.

By table 4-1, we see that the equation (15) usually has some solutions in the interval [-1, 1] no matter what experimental condition is assumed. But there is no significant difference to distinguish the ideal data from noised data or distinguish the correct match from error match by just referring to the solutions of (15).

When imposing the criteria **C-1**, **C-2** and **C-3** for the formal solutions of (15), (14) and (9), Table 4-2 shows that the reserved solutions have a very different distribution compared to Table 4-1 (where we identify a pair of mirror solutions as one solution). This time, we see that the overwhelming majority of the error matched trihedral angle constraints have no solution. So they can be effectively identified by our method. For each correctly matched case, we always find that true solution is included for ideal data and an approximate solution for the true value exists for noised data; and most correctly matched cases have just one or two reserved solutions. More details about the experiments are presented in [14].

4.3 Noise sensitivity analysis

For a trihedral angle constraint (5), the net effect of noise can be represented by a disturbance on the 2D angles β_i, i=1, 2, 3. Denote $\hat{\beta}_i$ as the noised β_i; let γ_i and $\hat{\gamma}_i$ be the correct solution of (5) corresponding to β_i and $\hat{\beta}_i$. Then, we consider the covariant relationships for the corresponding pairs ($\Delta\gamma_i$, $\Delta\beta_i$) and ($\hat{\gamma}_i$, γ_i) by following two linear regression models :

$$\Delta\gamma_i = a_{i0} + a_{i1}\Delta\beta_i + \varepsilon_i$$
$$\hat{\gamma}_i = a'_{i0} + a'_{i1}\gamma_i + \varepsilon_i$$

where, $\Delta\gamma_i = \hat{\gamma}_i - \gamma_i$, $\Delta\beta_i = \hat{\beta}_i - \beta_i$, (i = 1, 2, 3).

Our intention is to test the statistical hypotheses :

$$H_0\colon a_{i1} = 0 \qquad H_0\colon a'_{i1} = 0 \qquad (\mathrm{i} = 1,2,3)$$

by using the analysis of variance (ANOVA) to check the data fitness for the linear regression models. According to the above procedure, the synthetic test data were generated for regression analysis; where, the noises were selected from the noise interval $[-5^o, 5^o]$; and for multiple solution cases, we chose the best approximation to the correct value γ_i as $\hat{\gamma}_i$.

The results of the regression analysis are presented by Table 4-3 (refer [14] for more details). We see that there is no definite relationship between $\Delta\gamma_i$ and $\Delta\beta_i$; but very strong linear relationship exist between γ_i and $\hat{\gamma}_i$. Therefore, we can conclude that the solution of our method for trihedral angle constraint is stable under a noisy environment. So the method is robust in real application situations.

Table 4-3 Hypothesis Tests for the Regression Analysis

Models	P_value	Acceptance
$\Delta\gamma_1 = a_{10} + a_{11}\Delta\beta_1$	0.1910	Accept
$\Delta\gamma_2 = a_{20} + a_{21}\Delta\beta_2$	0.4728	Accept
$\Delta\gamma_3 = a_{30} + a_{31}\Delta\beta_3$	0.9821	Accept
$\hat{\gamma}_1 = a'_{10} + a'_{11}\gamma_1$	0.0001	Reject
$\hat{\gamma}_2 = a'_{20} + a'_{21}\gamma_2$	0.0001	Reject
$\hat{\gamma}_3 = a'_{30} + a'_{31}\gamma_3$	0.0001	Reject

5. Conclusion

Methods for solving the orientation and position of an object from a single perspective projection view are important for their wide applications and powers. The method presented in this paper permits us to find an analytic solution of a trihedral angle constraint by directly using angle information. Angle is a very common feature for characterizing a variety of objects. The knowledge about the angles of an object provides a strong clue for estimating the orientation and position of the object. Our method gives the first closed form solution for the problem of trihedral angle constraint in perspective projection. Trihedral angle is the simplest but also the most encountered angle constraint in 3D computer vision. This method also provides a basic approach for dealing with the general AnA problems, provided that the number of constraint equations on AnA problem is greater than or equal to the number of unknowns. The results of simulation experiments show that the new method is not only a real time technique of shape from angle constraint, but also powerful enough to cope with noisy environments in real applications. With the new developments, we present a general analysis on the essential characteristics of PnP, LnL and AnA techniques. The combination of the three techniques certainly is a very promising tool to deal with various situations of shape from inverse perspective projection. To design a sound algorithm for this unified approach is a topic for our further research.

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