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The random field Ising model : algorithmic complexity and phase transition

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Résumé. — Le modèle d'Ising en champ aléatoire (RFIM) est étudié du point de vue de la complexité algorithmique. On montre que le problème d'optimisation de la recherche d'un état de base du modèle ferromagnétique est polynomial (P) en toutes dimensions. Un nouvel algorithme de rigidité pour l'étude de la morphologie des états de base est aussi donné. Par opposition, le problème associé au modèle antiferromagnétique est démontré être NP-complet. L'absence d'une dépendance explicite de la complexité du RFIM avec la dimensionnalité contraste avec les résultats connus pour le modèle de frustration des verres de spins. Nos résultats montrent en particulier l'absence d'une correspondance simple entre les propriétés : NP-complet et existence d'une transition de phase à T_c fini dans les modèles de mécanique statistique avec des interactions en compétition.

Abstract. — The random field Ising model (RFIM) is investigated from the complexity point of view. We prove that finding a ground state of the ferromagnetic RFIM is a polynomial (P) optimization problem in any dimension *d*. A new rigidity algorithm for the search of the ground state morphology is also given. In contrast, the problem associated to the antiferromagnetic RFIM is shown to be an NP-complete optimization problem. The absence of any sensivity to *d* contrasts sharply with the known results previously obtained for the frustration model of spin glasses. Our results show, in particular, the absence of a simple one to one correspondence between finite T_c phase transition and NP-completeness properties in statistical mechanics models with competing interactions.

1. — Recent studies [1-3] of spin-glass models suggest the possibility of some connections between the computational complexity of the problem of finding a ground state and the existence of a finite T_c phase transition in a given model (e.g. $\pm J$ Ising models). More precisely, it has been proved [3, 4] that the spin-glass problem defined as an optimization problem (i.e. find the ground states) is NP-complete in dimensions larger than 2. Though it is polynomial [1, 2] in 2D, it becomes NP-complete for two coupled planes [3] (i.e. at $d = 2 + \varepsilon$). Accordingly, it is tempting to link NP-completeness [5] with the occurrence of a sharp phase transition at finite temperature T_c

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in the thermodynamic (large size) limit. Indeed, according to current wisdom [6, 7], it is generally believed that there is a sharp spin-glass transition for $d = 3$ and above, and none in $d = 2$ which would agree with the preceding suggestion. It does seem first that NP-completeness is generally associated with some sort of ergodicity breaking, leading to non trivial configuration space landscapes (valley structure, large number of metastable states, ...). The purpose of this Letter is to show that such a direct connection between finite T_c and NP-completeness is incomplete and a more refined classification of these NP-complete problems is in order. More precisely, we shall show that the random field Ising models (RFIM) provide a simple example where algorithmic complexity and finite T_c phase transition can be compared and understood in simple terms.

In section 2, we show that the optimization problem associated with the ferromagnetic RFIM is a polynomial (P) problem in any dimension and can be solved using a polynomial-time flow algorithm. As a byproduct of this result, a new rigidity algorithm, outlined in section 3, is given for the search of ground state morphology. In section 4, the optimization problem associated with the antiferromagnetic RFIM is shown to be, in general, an NP-complete problem in sharp contrast with the ferromagnetic case. The extension of our results to the q-state Potts model is outlined and proposed to shed a new light on the connection between NP-completeness and finite T_c transition in random systems with competing interactions. In our conclusion we indicate some new directions for the search of such a connection.

2. Ferromagnetic RFIM.

Let us consider the random field Ising model, defined by the following Hamiltonian

$$\mathcal{H} = - \sum_{(ij)} J_{ij}\, \sigma_i\, \sigma_j - \sum_i h_i\, \sigma_i \tag{1}$$

where $\{ \sigma_i \}$ denote Ising spins ($\sigma_i = \pm 1$) located at the vertices of a given lattice and $\{ J_{ij} \}$ are the interaction constants. The first sum runs over all bonds (ij) of the considered lattice whereas the second runs over all nodes (i). The random magnetic fields $\{ h_i \}$ are taken according to a given probability distribution $p(\{ h_i \})$. For a given set of random fields, the search for a ground state of the model can be formulated as the following optimization problem.

Problem P1 : minimize the objective function

$$E = - \sum_{(ij)} J_{ij}\, \sigma_i\, \sigma_j - \sum_i h_i\, \sigma_i\,, \quad J_{ij} \geqslant 0$$

subject to $\sigma_i \in \{ -1, 1 \}$ for all i.

In what follows, we shall show that P1 can be converted into a Min-cut problem and then can be solved using a polynomial-time flow algorithm. In order to show this transformation some notations and basic concepts from graph theory are of some help (for more complete discussion, we direct the reader to classical textbooks [8]).

A graph $G = (\mathrm{V}, \mathrm{E})$ consists of a set of vertices V and a subset $\mathrm{E} \subset \mathrm{V} \times \mathrm{V}$ of pairs of vertices called arcs and denoted $e = (i, j)$, $i \in \mathrm{V}, j \in \mathrm{V}$. An undirected arc is called an edge and is denoted $\{ i, j \}$. The edge $\{ i, j \}$ is equivalent to the two arcs (i, j) and (j, i).

For a given subset of vertices $\mathrm{A} \subset \mathrm{V}$, $\omega^+(\mathrm{A})$ (resp. $\omega^-(\mathrm{A})$) denotes the set of arcs originating (resp. terminating) in A and terminating (resp. originating) in $\mathrm{V} - \mathrm{A}$.

A cutset is a set of arcs of the form $\omega^+(\mathrm{A})$, $\mathrm{A} \subset \mathrm{V}$. An (a, b)-cutset separating two given vertices a and b of G is a set of arcs of the form $\omega^+(\mathrm{A})$, where $a \in \mathrm{A}$ and $b \notin \mathrm{A}$. Such (a, b)-cutset can then be « viewed » as a partition of V into A and $\mathrm{V} - \mathrm{A}$, with $a \in \mathrm{A}$ and $b \in \mathrm{V} - \mathrm{A}$.

Suppose that each arc e of G has assigned to it a non negative number C_e, called capacity of e, the capacity of a given cutset is defined by

$$\mathcal{C}(\omega^+(\mathrm{A})) = \sum_{e \in \omega^+(\mathrm{A})} C_e\,.$$

To the problem P1, we shall associate the following graph structure $G = (V, E)$. To each node of the considered lattice is associated a vertex of G. To each bond of the lattice is associated an edge e and a capacity $C_e = J_e$ given by the ferromagnetic coupling constant. Two ghost vertices, denoted s and t, respectively, are added to the set of vertices. The first new vertex s is connected to all vertices where the local magnetic field h_i is negative, by new edges each having a capacity $C_i = | h_i | = - h_i$. Similarly, t is connected to each vertex where $h_i \geqslant 0$ by new edges of capacity $| h_i | = h_i$ (see Fig. 1). It is clear that any configuration of spins Γ can be viewed as a partition into two disjoint subsets : $\mathcal{D}$, the subset of spins down and $\mathcal{U}$ the subset of spins up. Let us denote $\mathcal{S} = \mathcal{D} \cup \{ s \}$ and $\mathcal{T} = \mathcal{U} \cup \{ t \}$. This construction defines a natural (s, t)-cutset γ, associated with $\mathcal{T} \cup \mathcal{S}$, and it is not difficult to show the following result.

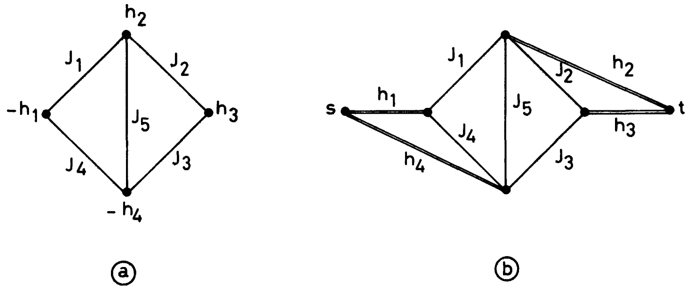

Fig. 1. — (a) A set of 4 Ising spins with the corresponding ferromagnetic interactions J_e and the local magnetic fields h_i ($h_i > 0$). (b) The corresponding graph is obtained by adding two ghost vertices s and t, and the associated edge capacities.

Proposition [9] : Let $E(\Gamma)$ be the energy associated with the configuration Γ and γ the corresponding (s, t)-cutset. Then

$$E(\Gamma) = - E^* + 2\, \mathcal{C}(\gamma)$$

where $E^* = \sum_{(ij)} J_{ij} + \sum_i | h_i |$, a constant quantity independent of Γ.

According to this result, problem P1 reduces to the search for an (s, t)-Min-cutset (i.e. of minimum capacity) separating the vertices s and t. This correspondence is one to one : to each Min-cutset is associated one ground state of the RFIM and *vice-versa*. Furthermore, the exhibited mapping between these two problems holds for any lattice, i.e. independently of the dimensionality d of the considered lattice, and does not involve the particular distribution $p(\{ h_i \})$ of random fields.

A classical result in combinatorial optimization [10] shows that the minimum capacity of a (s, t)-cutset is equal to the maximum value of a flow in the considered structure (Max-flow Min-cut theorem), where an extra arc (t, s) of capacity $C_{ts} = \infty$ has been added. However, the Max-flow problem is known to be a polynomial (P) problem [11], i.e. there is an algorithm giving a Max-flow and then a Min-cut in a polynomial-bounded number of steps, for any real number capacities.

In general, for an RFIM on a lattice consisting of L^d vertices, the number of steps increases as L^{3d}. Considered as a linear programming problem, the Max-flow is the dual to the Min-cut problem. Such a program has been implemented at $d = 2$ and 3 ($L \leqslant 128$ at $d = 2$, $L \leqslant 20$ at $d = 3$) for the investigation of the ground state structure. Further details and numerical results will be given elsewhere [12].

The above result calls for two comments. First, the problem of finding a ground state for the ferromagnetic RFIM is polynomial, whatever the dimensionality d of the lattice. Therefore, the algorithmic complexity of the associated optimization problem is not related to the existence of a finite T_c phase transition [13] : $T_c = 0$ at $d = 2$ and $T_c \neq 0$ at $d = 3$. Second, this situation contrasts with the $\pm J$ frustration model of spin glasses, where the algorithmic complexity of the associated optimization problem is very sensitive to d [1-4]. For the sake of clarity, some of the relevant results are summarized in table I.

Table I. — *Algorithmic complexity* (P : *polynomial*, NPC : NP-*complete*) *of the optimization problems associated to different spin models. Here* $d = 2 + \varepsilon$ *refers to two coupled planes.*

	$T_c = 0$	$T_c \neq 0$
P	$d = 2$ Spin glass	Ferromagnetic RFIM ($d \geqslant 3$)
NPC	$d = 2 + \varepsilon$ Spin glass	$d \geqslant 3$ Spin glass

3. Ground state rigidity.

In general, many spin configurations are associated to the ground state energy. In order to study the degeneracy of the ground state we introduce the notion of rigid spins and rigid bonds. A spin is rigid if its value ($+1$ or -1) is the same in all ground states. It is clear that if the edge of capacity C_i, associated to the local field h_i, belongs to all the (s, t)-Min-cutset, then the corresponding spin σ_i is in the opposite direction of its local field h_i in all ground states (i.e. $h_i \sigma_i < 0$). Inversely, if the associated edge does not belong to any (s, t)-Min-cutset, then $h_i \sigma_i > 0$ in all ground states. Similarly, a lattice edge e (bond) of capacity $C_e = J_e$ is rigid if it is either satisfied or unsatisfied in all ground states (i.e. e is associated to two solidary spins). It is clear that if e does not belong to any (s, t)-Min-cutset, then e is satisfied in all ground states. Inversely, if e belongs to all the (s, t)-Min-cutset then e is unsatisfied.

In a given sample : $G = (\mathrm{V}, \mathrm{E})$, $\{ J_{ij} \}$ and $\{ h_i \}$ let us denote by $\mathrm{T} \subset \mathrm{E}$ the set of edges that belong to all the (s, t)-Min-cutsets, $\mathrm{A} \subset \mathrm{E}$ that of edges not belonging to any (s, t)-Min-cutset and $\mathrm{C} = \mathrm{E} - \mathrm{A} \cup \mathrm{T}$. Using these notations a bond is rigid and always satisfied if the associated edge $e \in \mathrm{A}$. This bond is rigid and never satisfied if $e \in \mathrm{T}$. Finally, this bond is non-rigid if $e \in \mathrm{C}$. Similarly, a spin σ_i is always in the same (resp. inverse) direction of the local field h_i if the associated edges $e \in \mathrm{C}$ (resp. $e \in \mathrm{T}$). Therefore, the search for the morphology of the ground states reduces to finding the sets A, T and C. This can be carried out efficiently with a post-optimality procedure [14]. In the following we outline such a procedure. For this we use the following notation.

Problem P : Given a graph $G = (\mathrm{V}, \mathrm{E})$ and a capacity function $C : \mathrm{E} \to \mathbb{R}^+$, find an (s, t)-Min-cutset separating s (source) and t (sink). v^* will be the value of the capacity of such a cutset.

Let $a \in \mathrm{E}$ and ε be a positive real number.

Problem P(a, ε) : For the same graph $G = (\mathrm{V}, \mathrm{E})$ and the capacity function C^ε defined by

$$C_a^\varepsilon = C_a + \varepsilon \quad \text{and} \quad C_e^\varepsilon = C_e \quad \text{for} \quad e \neq a$$

find an (s, t)-Min-cutset separating s and t. $v^*(a, \varepsilon)$ will be the value of the capacity of such a cutset.

Finally, for a given cutset K, let us denote :

$$\mathcal{C}(\mathrm{K}) = \sum_{e \in \mathrm{K}} C_e \quad \text{and} \quad \mathcal{C}_a^\varepsilon(\mathrm{K}) = \sum_{e \in \mathrm{K}} C_e^\varepsilon .$$

Using the previous notation, one can show, as in reference [2] :

Proposition : Given a cutset K, an edge a and $\varepsilon > 0$, we have

(i) $\mathcal{C}_a^{-\varepsilon}(\mathrm{K}) \leqslant \mathcal{C}(\mathrm{K}) \leqslant \mathcal{C}_a^{\varepsilon}(\mathrm{K})$

(ii) $v^*(a, -\varepsilon) \leqslant v^* \leqslant v^*(a, \varepsilon)$

(iii) if $a \notin \mathrm{K}$ then $\mathcal{C}_a^{-\varepsilon}(\mathrm{K}) = \mathcal{C}_a^{+\varepsilon}(\mathrm{K}) = \mathcal{C}(\mathrm{K})$

(iv) if $a \in \mathrm{K}$ then $\mathcal{C}_a^{-\varepsilon}(\mathrm{K}) + \varepsilon = \mathcal{C}_a^{+\varepsilon}(\mathrm{K}) - \varepsilon = \mathcal{C}(\mathrm{K})$.

This leads to the following theorem :

Theorem : Let $\Delta = \min\limits_{\{\mathrm{K} \mid \mathcal{C}(\mathrm{K}) \neq v^*\}} \{ \mathcal{C}(\mathrm{K}) - v^* \}$, then

(i) for $\varepsilon > 0$: $a \in \mathrm{T} \Leftrightarrow v^* < v^*(a, \varepsilon)$

(ii) for $0 < \varepsilon < \Delta$: $a \in \mathrm{A} \Leftrightarrow v^* = v^*(a, -\varepsilon)$.

This theorem yields an efficient algorithm for the search of rigidity [12], the corresponding algorithm is actually polynomial as expected. Let us notice that the number Δ which appears in the above theorem has a simple physical meaning : it is nothing else other than the energy gap between the first excited state and the ground state energy. Note also that we can limit ourselves to the cases where $\{ J_{ij} \}$ and $\{ h_i \}$ are rational, i.e. J_{ij} and h_i are integers and then take $\varepsilon = 1$ in practice.

4. Antiferromagnetic RFIM.

Let us now consider the following Potts model [15] :

$$\mathcal{H} = - \sum_{(ij)} J_{ij}(q\delta_{\sigma_i\sigma_j} - 1) - \sum_i H_i(q\delta_{\varepsilon_i\sigma_i} - 1) \tag{2}$$

where σ_i is a Potts spin variable, located at vertex i of a given lattice, which can attain q different values ($\sigma_i = 1, 2, ..., q$). Here $J_{ij} \geqslant 0$ and $H_i \geqslant 0$ have the same meaning as defined previously. The random field variables ε_i favour the state $\sigma_i = \varepsilon_i$, and have the same probability $1/q$ to be along each one of the q available states. It is clear that for $q = 2$, equation (2) reproduces the RFIM discussed in the previous sections.

As for the RFI model, we use the concept of cutset in the investigation of the optimization problems associated with equation (2). Given a graph $G = (\mathrm{V}, \mathrm{E})$ and an integer $q \geqslant 2$, a q-cutset is a partition of V into q subsets $\mathrm{S}_1, \mathrm{S}_2, ..., \mathrm{S}_q \left(\bigcup_i \mathrm{S}_i = \mathrm{V}, \mathrm{S}_i \cap \mathrm{S}_j = \varnothing \text{ for } i \neq j \right)$. A q-cutset separating q given vertices $s_1, s_2, ..., s_q$ is a q-cutset such that : $s_i \in \mathrm{S}_i$ $(1 \leqslant i \leqslant q)$.

To the model defined by (2) we associate as above a graph structure, by adding q ghost vertices $s_1, s_2, ..., s_q$. Each s_i $(1 \leqslant i \leqslant q)$ is connected to the vertices having a local field i, using new edges of capacity H_i. The capacity of the lattice edges (bonds) are $\omega_e = J_e$, as before. Following the same line of ideas, one can show that there is one to one correspondence between the spin configurations, of energy E and the q-cutsets separating $s_1, s_2, ..., s_q$ of capacity Φ

$$E = -(q - 1) E^* + q\Phi ,$$

where

$$E^* = \sum_{(ij)} J_{ij} + \sum_i H_i .$$

In the limiting case $q = 2$ we recover the RFI model, and we have shown that the corresponding optimization problem (i.e. min E) is polynomial. For $q > 2$, the situation is more complicated and the search for max E can be carried out using the following results. Detailed proof will be given elsewhere [16].

Proposition 1 : The problem of finding a maximum q-cutset has the same algorithmic complexity as the problem of finding a maximum q-cutset separating q given vertices.

Proof : Based mainly on two transformations :

a) Max q-cutset into Max q-cutset separating q given vertices : add q new vertices $\{ s_i \}$ $(1 \leqslant i \leqslant q)$, use edges of zero capacities to connect them to the graph vertices and then find a Max q-cutset separating the new vertices $\{ s_i \}$.

b) Max q-cutset separating q given vertices into a Max q-cutset : for this we add $q(q-1)/2$ new edges each of capacity $1 + \sum_{e \in E} \omega_e$ between the vertices s_i $(1 \leqslant i \leqslant q)$.

Note that Proposition 1 holds even in the relevant case where the set $V - \{ s_1, s_2, ..., s_q \}$ is partitioned into V_i with $V_i = \{ a \in V \mid \{ a, s_i \} \in E \}$ $1 \leqslant i \leqslant q$ and for any capacities [16].

Proposition 2 : The problem of finding a Max 2-cutset separating two given vertices s_1 and s_2 is an NP-complete problem.

Proof : Using Proposition 1, we reduce the so-called MAX-CUT problem ([ND 16], p. 210, Ref. [5]) which is known to be NP-complete, to our problem.

Proposition 3 : The problem of the Max q-cutset $(q > 2)$ separating q given vertices is NP-complete.

Proof : Using again Proposition 1, we have to find the complexity of the problem of a Max q-cutset. The answer is given by the following transformation of the so-called « K-colourability of a graph » problem ([GT 4], p. 191 of Ref. [5]): to each edge is assigned a capacity $+ 1$. If Φ denotes the value of a Max cutset, then $\Phi = | E |$ if and only if the considered graph is colourable with q colours.

The results of this section are summarized in table II. Two combinatorial optimization problems are associated with the model defined by equation (2). For $q \geqslant 2$ we have shown that finding the maximum of E is equivalent to the search for a Max q-cutset and therefore is an NP-complete problem. For $q = 2$ the problem of finding the minimum of E reduces to the search for a 2-cutset separating two given vertices and then a polynomial problem. The case $q > 2$ which corresponds to finding a ground state of the q-state Potts model remains unsolved.

Note that all the previous results are general and independent of the lattice dimensionality d. The result of Proposition 2 has a transparent physical signification. In fact, it is easy to see that for $q = 2$ the maximum of E is nothing else than the ground state energy of the antiferromagnetic RFI model (Eq. (1) with negative J_{ij}'s). This implies simply that the optimization problem (i.e. find a ground state) associated with the antiferromagnetic RFIM is an NP-complete problem, and hence probably intractable [5] in the sense that no algorithm requiring a time bounded by a

Table II. — *Algorithmic complexity of the problem of finding a Min- or Max-cutset separating q given vertices.*

	min E	max E
$q = 2$	P	NPC
$q > 2$	?	NPC

polynomial of the volume can exactly solve it. This result is somewhat surprising but probably related to the fact that the magnetic field is not conjugate to the magnetization in this case, already in the absence of disorder. This suggests that the ground states morphology is probably more rich in the antiferromagnetic case than the ferromagnetic one. It is important to notice that in some special cases the antiferromagnetic RFIM can lead to a polynomial problem. This is actually the case of lattices which can be viewed as bipartite graphs, i.e. can be decomposed into two independent sublattices. In such a case (for instance the square lattice), a simple gauge transformation reduces the actual problem to that of the ferromagnetic case, even in the presence of random fields. This particular example does not contradict our general statement holding for general graph structure.

5. Conclusion.

We conclude with two remarks.

a) It is not difficult to find trivial examples of spin models with random interactions, where the associated optimization problem is trivial. This is the case of random ferromagnetic interactions ($J_{ij} \geqslant 0$) : equation (1) with $h_i = 0$ for all i. On the other hand, the presence of the frustration in a planar antiferromagnetic Ising model under a uniform magnetic field (special case of Proposition 2, see also Ref. [3]) leads to NP-completeness. Therefore, frustration and competing interactions appear as the first ingredients for drawing the connection between statistical mechanics models and the combinatorial optimization problems. The concepts of dimensionality and symmetry (i.e. number of components of the spin variables) as well as the nature of interactions are certainly of basic importance for the search of a possible connection between the two fields. We believe that the valley structure (large or small number of valleys, dominant valleys, ...) is the key of the solution. In this respect, it would be very interesting to investigate the nature of the phase transition of the RFIM, for instance at $d \geqslant 3$: is it a spin-glass like transition (infinity of thermodynamic equilibrium states, fluctuation from sample to sample, ...) or rather of a more trivial character ?

b) The recent progress in solving the Travelling Salesman problem [17, 18] tells us the importance of using statistical mechanics methods (e.g. Monte Carlo simulation) in the investigation of NP-complete problems. We believe that, more generally, the probabilistic analysis of combinatorial optimization problems, as outlined in reference [19] will provide a natural framework for a cross fertilization of statistical mechanics and combinatorial problems and this will call for further extensions.

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