Oscillatory solutions of singular equations arising in hydrodynamics*

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Abstract

We investigate the singular differential equation $(p(t)u'(t))' = p(t)f(u(t))$ on the half-line $[0,\infty)$, where f satisfies the local Lipschitz condition on $\mathbb{R}$ and has at least two simple zeros. The function p is continuous on $[0,\infty)$, has a positive continuous derivative on $(0,\infty)$ and $p(0) = 0$. We bring additional conditions for f and p under which the equation has oscillatory solutions with decreasing amplitudes.

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1 Introduction

We study the equation

$$(p(t)u'(t))' = p(t)f(u(t)) \tag{1}$$

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on the half–line $[0, \infty)$, where

$$f \in Lip_{\rm loc}(\mathbb{R}),\ p \in C^1(0,\infty) \cap C[0,\infty), \tag{2}$$

$$p(0) = 0,\ p'(t) > 0,\ t > 0,\ \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{3}$$

Equation (1) is singular at $t = 0$ because $p(0) = 0$. If f in (1) fulfils moreover assumptions

$$\text{there exists } L > 0 \text{ such that } f(x) = 0 \text{ for } x \geq L, \tag{4}$$

$$xf(x) < 0 \quad \text{for } x \in (-\infty, 0) \cup (0, L), \tag{5}$$

$$\text{there exists } \bar{B} < 0 \text{ such that } \int_{\bar{B}}^{L} f(z)\,\mathrm{d}z = 0, \tag{6}$$

then (1) generalizes equations which appear in hydrodynamics or in the nonlinear field theory, [4] – [7], [9].

Definition 1 A function $u \in C^1[0,\infty)$ which has continuous the second derivative on $(0,\infty)$ and satisfies equation (1) for all $t \in (0,\infty)$ is called a *solution* of (1).

Consider $B < 0$ and the initial conditions

$$u(0) = B, \quad u'(0) = 0. \tag{7}$$

The initial value problem (1), (7) has been investigated e. g. in [1] – [3], [8], [10] – [12]. In particular in [10] it was proved that for each negative B there exists a unique solution of problem (1), (7) under the assumptions (2) – (6). Consider such solution u and denote

$$u_{\rm sup} = \sup\{u(t) : t \in [0,\infty)\}.$$

Definition 2 If $u_{\rm sup} < L$ ($u_{\rm sup} = L$ or $u_{\rm sup} > L$), then u is called a *damped* solution (a *homoclinic* solution or an *escape* solution) of problem (1), (7).

In [10] and [12] these three types of solutions of problem (1), (7) have been studied and the existence of each type has been proved for sublinear or linear asymptotic behaviour of f near $-\infty$. In [11], f has been supposed to have a zero $L_0 < 0$. Here we generalize and extend the results of [10] – [12] concerning damped solutions. We prove their existence under weaker assumptions than in the above papers. Moreover, we bring conditions under which, each damped solution is oscillatory, that is, it has an unbounded set of isolated zeros.

We replace assumptions (4) – (6) by the following ones:
There exist $L_0 < 0$, $L > 0$, $C_L > 0$ such that

$$xf(x) < 0 \quad \text{for } x \in (L_0, 0) \cup (0, L), \tag{8}$$

$$0 \leq f(x) \leq C_L \quad \text{for } x \geq L \tag{9}$$

($L_0 = -\infty$ is possible).

2 Damped solutions

Theorem 3 *(Existence and uniqueness) Assume that* (2), (3), (8), (9) *hold and let* $B \in (L_0, 0)$. *Then problem* (1), (7) *has a unique solution* u *and moreover the solution* u *satisfies*

$$u(t) \geq B \quad \textit{for } t \in [0, \infty). \tag{10}$$

Proof. Step 1. Put

$$f_B(x) = \begin{cases} f(x) & \text{for} \quad x \geq B, \\ f(B) & \text{for} \quad x < B. \end{cases} \tag{11}$$

We will study the auxiliary differential equation

$$(p(t)u'(t))' = p(t) f_B(u(t)). \tag{12}$$

By virtue of (2) we find the Lipschitz constant $K > 0$ for f on $[B-1, |B|+1]$ and due to (2), (9) and (11), we find $M_B > 0$ such that

$$|f_B(x)| \leq M_B \quad \text{for } x \in \mathbb{R}. \tag{13}$$

Put $\varphi(t) = \int_0^t p(s)\, \mathrm{d}s / p(t)$ for $t > 0$. Having in mind (3), we see that p is increasing and so

$$0 < \varphi(t) \leq t \quad \text{for } t > 0, \qquad \lim_{t \to 0+} \varphi(t) = 0. \tag{14}$$

Consequently we can choose $\eta > 0$ such that

$$\int_0^\eta \varphi(t)\, \mathrm{d}t \leq \min\left\{ \frac{1}{2K}, \frac{1}{M_B} \right\}. \tag{15}$$

Consider the Banach space $C[0, \eta]$ (with the maximum norm) and define an operator $\mathcal{F} : C[0, \eta] \to C[0, \eta]$ by

$$(\mathcal{F}u)(t) = B + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) f_B(u(\tau))\, \mathrm{d}\tau\, \mathrm{d}s.$$

Using (13) and (15), we have

$$\|\mathcal{F}u\|_{C[0,\eta]} \leq |B| + M_B \int_0^\eta \varphi(s)\, \mathrm{d}s \leq |B| + 1,$$

that is $\mathcal{F}$ maps the ball $\mathcal{B}(0, |B|+1) = \{u \in C[0, \eta] : \|u\|_{C[0,\eta]} \leq |B| + 1\}$ to itself. Due to (11) and the choice of K, we have for u_1, $u_2 \in \mathcal{B}(0, |B|+1)$

$$\begin{aligned} \|\mathcal{F}u_1 - \mathcal{F}u_2\|_{C[0,\eta]} &\leq \int_0^\eta \frac{1}{p(s)} \int_0^s p(\tau) |f_B(u_1(\tau)) - f_B(u_2(\tau))|\, \mathrm{d}\tau\, \mathrm{d}s \\ &\leq K \|u_1 - u_2\|_{C[0,\eta]} \int_0^\eta \varphi(s)\, \mathrm{d}s \leq \frac{1}{2} \|u_1 - u_2\|_{C[0,\eta]}. \end{aligned}$$

Hence $\mathcal{F}$ is a contraction on $\mathcal{B}(0, |B|+1)$ and the Banach fixed point theorem yields a unique fixed point $u \in \mathcal{B}(0, |B|+1)$ of $\mathcal{F}$.
Step 2. The fixed point u of Step 1 fulfils

$$u(0) = B \quad \text{and} \quad u'(t) = \frac{1}{p(t)} \int_0^t p(s) f_B(u(s))\,\mathrm{d}s, \ t \in (0, \eta]. \tag{16}$$

Hence u satisfies equation (12) on $(0, \eta]$. Finally, (13) and (14) yield

$$\lim_{t \to 0+} |u'(t)| \le M_B \lim_{t \to 0+} \varphi(t) = 0.$$

Consequently u fulfils (7). Choose an arbitrary $b > \eta$. Then, by (14) and (16),

$$|u'(t)| \le M_B b, \ |u(t)| \le |B| + M_B b^2, \ t \in [0, b].$$

Having in mind that $f_B \in Lip_{\mathrm{loc}}(\mathbb{R})$, u can be (uniquely) extended as a function satisfying equation (12) onto $[0, b]$. Since b is arbitrary, u can be extended onto $[0, \infty)$ as a solution of equation (12). We have proved that problem (12), (7) has a unique solution.
Step 3. According to Step 2 we have

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f_B(u(t)) \quad \text{for } t \in (0, \infty). \tag{17}$$

Multiplying (17) by u' and integrating between 0 and t, we get

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = \int_0^t f_B(u(s)) u'(s)\,\mathrm{d}s, \ t \in (0, \infty). \tag{18}$$

Put

$$F_B(x) = -\int_0^x f_B(z)\,\mathrm{d}z, \ x \in \mathbb{R}.$$

So, (18) has the form

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s + F_B(u(t)) = F_B(B), \ t \in (0, \infty). \tag{19}$$

Let $u(t_1) \in (L_0, B)$ for some $t_1 > 0$. Then (19) yields $F_B(u(t_1)) \le F_B(B)$ which is not possible because F_B is decreasing on $(L_0, 0)$ by (8) and (11). Therefore $u(t) \ge B$ for $t \in [0, \infty)$. Consequently, due to (11), u is a solution of equation (1).
Step 4. Assume that there exists another solution $\tilde{u}$ of problem (1), (7). Then we can prove similarly as in Step 3 that $\tilde{u}(t) \ge B$ for $t \in [0, \infty)$. This implies that $\tilde{u}$ is also a solution of problem (12), (7) and by Step 2, $\tilde{u} \equiv u$. We have proved that problem (1), (7) has a unique solution. $\square$

Lemma 4 *Let $C \in \{0, L\}$ and let u be a solution of equation* (1). *Assume that there exists $a > 0$ such that*

$$u(a) = C, \ \ u'(a) = 0. \tag{20}$$

Then $u(t) = C$ for all $t \in [0, \infty)$.

Proof. We see that the constant function $\tilde{u} \equiv C$ is a solution of equation (1). Let u be a solution of equation (1) satisfying (20) and let $u(t) \neq \tilde{u}(t)$ for some $t \in [0, \infty)$. Then the regular initial problem (1), (20) has two different solutions u and $\tilde{u}$, which contradicts (2). □

Remark 5 Let us put

$$F(x) = -\int_0^x f(z)\,\mathrm{d}z \quad \text{for } x \in \mathbb{R}. \tag{21}$$

Due to (2) and (8) we see that F is continuous on $\mathbb{R}$, decreasing and positive on $(L_0, 0)$, increasing and positive on $(0, L)$. Therefore we can define $\bar{B} < 0$ by

$$\bar{B} = \inf\{B_0 \in (L_0, 0) : F(B) < F(L) \text{ for all } B \in (B_0, 0)\}, \tag{22}$$

($\bar{B} = -\infty$ is possible).

Theorem 6 *(Existence of damped solutions) Assume that* (2), (3), (8), (9) *hold. Let $\bar{B}$ be given by* (22) *and assume that u is a solution of problem* (1), (7) *with $B \in (\bar{B}, 0)$. Then u is a damped solution.*

Proof. Since $B \in (\bar{B}, 0)$, we can find $\epsilon > 0$ such that

$$F(B) \le F(L - \epsilon). \tag{23}$$

Assume on the contrary that u is not damped, that is

$$\sup\{u(t) : t \in [0, \infty)\} \ge L. \tag{24}$$

Then, according to Lemma 4, there exists $\theta \in (0, \infty)$ such that

$$u(\theta) = 0, \ \ u'(\theta) > 0, \ \ u(t) \in [B, 0) \text{ for } t \in [0, \theta).$$

By (1), (3) and (8) we have $(pu')' > 0$ on $(0, \theta]$. So, pu' is increasing and positive on $(0, \theta]$ and hence $u' > 0$ on $(0, \theta]$. Assumption (24) implies that there exists $b \in (\theta, \infty)$ such that

$$u(b) = L - \epsilon, \quad u(t) \in [B, L - \epsilon) \quad \text{for } t \in [0, b).$$

Since u fulfils (1), we have

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)) \text{ for } t \in (0, \infty). \tag{25}$$

Multiplying (25) by u' and integrating between 0 and b we get

$$0 < \frac{u'^2(b)}{2} + \int_0^b \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(B) - F(L - \epsilon). \tag{26}$$

This contradicts (23). □

3 Oscillatory solutions

In this section we assume that in addition to our basic assumptions (2), (3), (8) and (9), the following conditions are fulfilled:

$$\lim_{x\to 0-} \frac{f(x)}{x} < 0, \quad \lim_{x\to 0+} \frac{f(x)}{x} < 0, \tag{27}$$

$$p \in C^2(0,\infty), \quad \limsup_{t\to\infty} \left| \frac{p''(t)}{p'(t)} \right| < \infty. \tag{28}$$

Then the next lemmas can be proved.

Lemma 7 *Let u be a solution of problem* (1), (7) *with $B \in (L_0, 0)$. Then there exists $\theta > 0$ such that*

$$u(\theta) = 0, \quad u'(t) > 0 \text{ for } t \in (0,\theta]. \tag{29}$$

Proof. Step 1. Assume that such θ does not exist. Then

$$u(t) < 0 \quad \text{for } t \in [0,\infty). \tag{30}$$

Hence (1), (7) and (8) yield $(pu')' > 0$ and $u' > 0$ on $(0,\infty)$. Therefore u is increasing on $(0,\infty)$ and

$$\lim_{t\to\infty} u(t) = \ell \in (B, 0]. \tag{31}$$

Multiplying (25) by u' and integrating between 0 and t, we get due to (21)

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(B) - F(u(t)), \quad t \in (0,\infty). \tag{32}$$

Letting $t \to \infty$, we get

$$\lim_{t\to\infty} \frac{u'^2(t)}{2} = -\lim_{t\to\infty} \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s + F(B) - F(\ell).$$

Since the function $\int_0^t p'(s)/p(s)u'^2(s)\,\mathrm{d}s$ is positive and increasing, it follows that there exists $\lim_{t\to\infty} u'(t) \ge 0$. If $\lim_{t\to\infty} u'(t) > 0$, then $\lim_{t\to\infty} u(t) = \infty$ contrary to (31). Consequently

$$\lim_{t\to\infty} u'(t) = 0. \tag{33}$$

Letting $t \to \infty$ in (25), we get by (3), (8) and (31),

$$\lim_{t\to\infty} u''(t) = f(\ell) \ge 0.$$

Due to (33), we conclude that $f(\ell) = 0$ and hence $\ell = 0$. We have proved that if $\theta > 0$ fulfilling (29) does not exist, then

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{34}$$

STEP 2. We define a function

$$v(t) = \sqrt{p(t)}u(t), \quad t \in [0, \infty). \tag{35}$$

By (3) and (28), we have $v \in C^2(0, \infty)$,

$$v'(t) = \frac{p'(t)u(t)}{2\sqrt{p(t)}} + \sqrt{p(t)}u'(t),$$

$$v''(t) = v(t)\left[\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 + \frac{f(u(t))}{u(t)}\right], \ t \in (0, \infty) \tag{36}$$

and

$$\lim_{t\to\infty} \frac{p''(t)}{p(t)} = \lim_{t\to\infty} \frac{p''(t)}{p'(t)}\frac{p'(t)}{p(t)} = 0.$$

Due to this fact, (3), (27) and (34) there exist $\omega > 0$ and $R > 0$ such that

$$\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 + \frac{f(u(t))}{u(t)} < -\omega \quad \text{for } t \geq R. \tag{37}$$

Due to (30), (35), (36) and (37), we get

$$v''(t) > -\omega v(t) > 0 \quad \text{for } t \geq R. \tag{38}$$

Thus, v' is increasing on $[R, \infty)$ and has the limit

$$\lim_{t\to\infty} v'(t) = V. \tag{39}$$

If $V > 0$, then $\lim_{t\to\infty} v(t) = \infty$, which contradicts (30) and (35). If $V \leq 0$, then $v' < 0$ on $[R, \infty)$ and

$$v(t) \leq v(R) < 0 \quad \text{for } t \in [R, \infty).$$

In view of (38) we can see that

$$0 < -\omega v(R) \leq -\omega v(t) < v''(t) \quad \text{for } t \in [R, \infty).$$

We get $\lim_{t\to\infty} v'(t) = \infty$ which contradicts $V \leq 0$. The obtained contradictions imply that (30) cannot occur and hence $\theta > 0$ satisfying (29) must exist. $\square$

Corollary 8 Let u be a solution of problem (1), (7) with $B \in (L_0, 0)$. Further assume that there exist $b_1 > 0$ and $B_1 \in (B, 0)$ such that

$$u(b_1) = B_1, \quad u'(b_1) = 0. \tag{40}$$

Then there exists $\theta_1 > b_1$ such that

$$u(\theta_1) = 0, \quad u'(t) > 0 \text{ for } t \in (b_1, \theta_1]. \tag{41}$$

Proof. We can argue as in the proof of Lemma 7 working with b_1 and B_1 instead of 0 and B. $\square$

Lemma 9 *Let u be a solution of problem (1), (7) with $B \in (L_0, 0)$. Further assume that there exist $a_1 > 0$ and $A_1 \in (0, L)$ such that*

$$u(a_1) = A_1, \quad u'(a_1) = 0. \tag{42}$$

Then there exists $\delta_1 > a_1$ such that

$$u(\delta_1) = 0, \quad u'(t) < 0 \quad \text{for } t \in (a_1, \delta_1]. \tag{43}$$

Proof. We argue similarly as in the proof of Lemma 7.
STEP 1. Assume that such $\delta_1 > a_1$ does not exist. Then

$$u(t) > 0 \quad \text{for } t \in [a_1, \infty). \tag{44}$$

By (1), (7) and (8) we deduce $u' < 0$ on (a_1, ∞) and

$$\lim_{t\to\infty} u(t) = \ell_1 \in [0, A_1). \tag{45}$$

Multiplying (25) by u', integrating between a_1 and t and using (21), we obtain

$$\frac{u'^2(t)}{2} + \int_{a_1}^{t} \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(A_1) - F(u(t)), \quad t \in (a_1, \infty),$$

and derive as in the proof of Lemma 7 that (34) holds.
STEP 2. We define v by (35) and get (36) for $t \in (a_1, \infty)$. As in the proof of Lemma 7 we find $\omega > 0$ and $R > 0$ satisfying (37). Due to (44), (35), (36) and (37) we get

$$v''(t) < -\omega v(t) < 0 \quad \text{for } t \geq R. \tag{46}$$

So, v' is decreasing on $[R, \infty)$ and $\lim_{t\to\infty} v'(t) = V$. If $V < 0$, then $\lim_{t\to\infty} v(t) = -\infty$ which contradicts (44) and (35). If $V \geq 0$, then $v' > 0$ on $[R, \infty)$ and

$$v(t) \geq v(R) > 0 \quad \text{for } t \in [R, \infty).$$

In view of (46) we can see that

$$v''(t) < -\omega v(t) \leq -\omega v(R) < 0 \quad \text{for } t \in [R, \infty).$$

We get $\lim_{t\to\infty} v'(t) = -\infty$ contrary to $V \geq 0$. The obtained contradictions imply that (44) cannot occur and that $\delta_1 > a_1$ satisfying (43) must exist. $\square$

Theorem 10 *Assume that (2), (3), (8), (9), (27), (28) hold. Let u be a solution of problem (1), (7) with $B \in (L_0, 0)$. If u is a damped solution, then u is oscillatory and its amplitudes are decreasing.*

Proof. Let u be a damped solution. By (10) and Definition 2, we can find $L_1 \in (0, L)$ such that

$$B \le u(t) \le L_1 \quad \text{for } t \in [0, \infty). \tag{47}$$

STEP 1. Lemma 7 yields $\theta > 0$ satisfying (29). Hence there exists a maximal interval (θ, a_1) such that $u' > 0$ on (θ, a_1). Let $a_1 = \infty$. Then, by (47), we get $u \in (0, L)$, $u' > 0$ on (θ, ∞) and

$$\lim_{t\to\infty} u(t) = \ell_0 \in (0, L). \tag{48}$$

By (1), (3) and (8), we have $(pu')' < 0$ on (θ, ∞). So pu' and u' are decreasing on (θ, ∞) and, due to (48),

$$\lim_{t\to\infty} u'(t) = 0. \tag{49}$$

Letting $t \to \infty$ in (25) and using (3), (8) and (48), we get

$$\lim_{t\to\infty} u''(t) = f(\ell_0) < 0,$$

which contradicts (49). Therefore $a_1 < \infty$ and there exists $A_1 \in (0, L)$ such that (42) holds. Lemma 9 yields $\delta_1 > a_1$ satisfying (43). Therefore u has just one positive local maximum $A_1 = u(a_1)$ between its first zero θ and second zero δ_1.

STEP 2. By (43) there exists a maximal interval (δ_1, b_1), where $u' < 0$. Let $b_1 = \infty$. Then, by (47), we have $u \in [B, 0)$, $u' < 0$ on (δ_1, ∞), and

$$\lim_{t\to\infty} u(t) = \ell_1 \in [B, 0). \tag{50}$$

By (1), (3) and (8), we get $(pu')' > 0$ on (δ_1, ∞) and so pu' is increasing on (δ_1, ∞). Since $u' < 0$, we deduce that u' is increasing on (δ_1, ∞) and, by (50), we get (49). Letting $t \to \infty$ in (1) and using (3), (8) and (50), we get

$$\lim_{t\to\infty} u''(t) = f(\ell_1) > 0,$$

which contradicts (49). Therefore $b_1 < \infty$ and there exists $B_1 \in [B, 0)$ such that (40) holds. Corollary 8 yields $\theta_1 > b_1$ satisfying (41). Therefore u has just one negative minimum $B_1 = u(b_1)$ between its second zero δ_1 and third zero θ_1.

STEP 3. We can continue as in STEP 1 and STEP 2 and get the sequences $\{A_n\}_{n=1}^{\infty} \subset (0, L)$ and $\{B_n\}_{n=1}^{\infty} \subset [B, 0)$ of local maxima and local minima of u attained at a_n and b_n, respectively. Now, put $x_1(t) = u(t)$, $x_2(t) = u'(t)$ and write equation (1) as a system

$$x_1'(t) = x_2(t), \quad x_2'(t) = -\frac{p'(t)}{p(t)} x_2(t) + f(x_1(t)). \tag{51}$$

Consider F of (21) and define a Lyapunov function V by

$$V(x_1, x_2) = F(x_1) + \frac{x_2^2}{2} \quad \text{for } (x_1, x_2) \in D, \tag{52}$$

where $D = (L_0, L)\times\mathbb{R}$. By Remark 5, we see that $V(0,0) = 0$ and $V(x_1, x_2) > 0$ on $D \setminus \{(0,0)\}$. By (32) and (52), we have

$$V(u(t), u'(t)) = \frac{u'^2(t)}{2} + F(u(t)) = F(B) - \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s$$

and

$$\dot{V}(t) = \frac{\mathrm{d}V(u(t), u'(t))}{\mathrm{d}t} = -\frac{p'(t)}{p(t)} u'^2(t) \le 0 \quad \text{for } t \in (0,\infty).$$

Therefore

$$\dot{V}(t) < 0 \quad \text{for } t \in (0,\infty), \quad t \ne a_n, b_n,\ n \in \mathbb{N}. \tag{53}$$

By (47), $(u(t), u'(t)) \in D$ for $t \in [0,\infty)$. We see that $V(u(t), u'(t))$ is positive and decreasing (for the damped solution u) and hence

$$\lim_{t\to\infty} V(u(t), u'(t)) = c_B \ge 0. \tag{54}$$

So, sequences $\{F(A_n)\}_{n=1}^{\infty}$ and $\{F(B_n)\}_{n=1}^{\infty}$ are decreasing,

$$F(A_n) = V(u(a_n), u'(a_n)), \quad F(B_n) = V(u(b_n), u'(b_n))$$

for $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} F(A_n) = \lim_{n\to\infty} F(B_n) = c_B.$$

Further, due to Remark 5, the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and the sequence $\{B_n\}_{n=1}^{\infty}$ is increasing. Consequently

$$\lim_{n\to\infty} A_n \in [0, L), \quad \lim_{n\to\infty} B_n \in (B, 0].$$

□

Remark 11 There are two cases for the number c_B from the proof of Theorem 10: $c_B = 0$ and $c_B > 0$. Denote

$$\lim_{n\to\infty} A_n = A_\infty, \quad \lim_{n\to\infty} B_n = B_\infty.$$

If $c_B = 0$, then $F(A_\infty) = F(B_\infty) = 0$ and hence $A_\infty = B_\infty = 0$, that is $\lim_{t\to\infty} u(t) = 0$.
Let $c_B > 0$. Consider an arbitrary sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} t_n = \infty$. By (54) we have $\lim_{n\to\infty} V(u(t_n), u'(t_n)) = c_B$. By (47) and (32), the sequence $\{(u(t_n), u'(t_n)\}_{n=1}^{\infty}$ is bounded and so there exists a subsequence

$$\{(u(t_{m_n}), u'(t_{m_n})\}_{n=1}^{\infty}$$

such that $\lim_{n\to\infty}(u(t_{m_n}), u'(t_{m_n})) = (x_1^B, x_2^B)$, where (x_1^B, x_2^B) is a point of the level curve

$$F(x_1) + \frac{x_2^2}{2} = c_B.$$

Note that

$$c_B = 0 \quad \text{if and only if} \quad \int_0^\infty \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(B),$$

and

$$c_B > 0 \quad \text{if and only if} \quad \int_0^\infty \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s < F(B).$$

Theorem 12 *(Existence of oscillatory solutions) Assume that* (2)*,* (3)*,* (8)*,* (9)*,* (27) *and* (28) *hold. Let $\bar{B}$ be given by* (22) *and let u be a solution of problem* (1)*,* (7) *with $B \in (\bar{B}, 0)$. Then u is an oscillatory solution with decreasing amplitudes.*

Proof. The assertion follows from Theorems 6 and 10. □

Remark 13 The assumption (9) in Theorem 12 can be omitted, because it has no influence on the existence of oscillatory solutions. It follows from the fact, that (9) imposes conditions on the function values of the function f for arguments greater than L, however, the function values of oscillatory solutions are lower than this constant L. This condition (used only in the Theorem 3) guaranteed the existence of solution of each problem (1), (7) for each $B < 0$ on the whole half–line, which simplified the investigation of the problem.

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