Peristaltic Flow in a Deformable Channel

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The effects of wall contraction or expansion on the characteristics of the peristaltic flow have been considered in this paper. For that, we present a theoretical model of laminar incompressible viscous peristaltic flow in a deformable channel. The problem is modeled in terms of unsteady two-dimensional Navier Stokes equations and the solution is obtained using the perturbation method. The physical parameters appearing due to deformation and the peristaltic motion are the wall expansion ratio (α) and the wave number (δ), respectively. Analytic perturbation results are obtained for small wave number and small wall expansion ratio. Basically the study is undertaken to examine the peristaltic motion along with the deformation of the channel. This will enhance our understanding of deformation/squeezing and peristalsis phenomena independently and jointly. Deformation effects are shown on the otherwise peristaltic fluid flow. The results of peristaltic flow [Shapiro et al., J. Fluid Mech. Digit. Archive **37**, 799 (1969)] can be recovered for the limiting case of α equal to zero.

Key words: Deformable Channel; Peristaltic Pumping.

1. Introduction

Scientists, engineers, and mathematicians have shown great interest in peristaltic flows and many articles have been written [1 – 25] to explain the phenomenon. From biological and engineering stand points, the peristaltic fluid transport has a great significance. The phenomenon is responsible for physiological fluid transport in many biological systems such as muscle contractions that occur throughout the digestive system, the fluid secreted by kidneys into the bladder through tubular organs, and bitter digestive fluids from the liver into the duodenum.

Since the first investigation of Shapiro and Latham [1], many attempts have been made both experimentally and theoretically, Shapiro [2] obtained the exact solution for the problem of peristaltic pumping in a two-dimensional flexible tube under the conditions that (a) the appropriate Reynolds number is so small that flow may be considered inertia free and (b) the peristaltic wave is of large wavelength compared with the diameter of the tube. Sinusoidal [3] and arbitrary shapes [4] of these waves have been studied and some evaluating techniques are formed [5] to test and establish hydrodynamic systems. The numerical solution of two-dimensional peristaltic flows is given by [6] while a brief account of most experimental and theoretical inquiry reported until 1984 is presented by Srivastava and Srivastava [7].

Goto and Uchida [21], Dauenhauer and Majdalani [22, 23], Majdalani et. al. [24], and Naoko et al. [25] wrote quite a few articles on the viscous flow in deformable tube/channel with permeable walls. Majdalani [22] introduced the similarity variables to reduce the governing equations to a nonlinear ordinary differential equation by taking the wall expansion ratio (α) constant, and the solution is obtained both numerically and analytically.

To the best of author's knowledge, peristaltic flow in a deformable channel has not been discussed so far. Peristaltic flows in a deformable channel/tube induced by a travelling wave on its wall are known to have important relevance for fluid transport in many biological systems. In nature, the small deformations are likely to appear along with peristaltic motion to transport the physiological flows in human body. This model conveniently describes the human ureter, gastrointestinal tract, and mechanical roller pumps. The present study will not only shed light on the peristalsis but will take into account squeezing as well, thus giving credence to renewed interest in peristaltic squeezing [25]. The squeezing phenomenon is conjectured as the process to remove the uneven nuclei and cytoplasm debris from the back end of apyrene sperm, and similarly cytoplasm debris are discarded and preserve the nuclei in

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the case of eupyrene sperm. It is thus hoped that the study may give some mathematical insight to this important biological problem. The results obtained for the peristaltic flow in a deformable channel reduces to that of peristaltic flow by taking the wall expansion ratio equal to zero which provides a useful check.

2. Formulation of the Problem

Let us investigate the flow of an incompressible viscous fluid in a two-dimensional channel of width $2a$. A rectangular coordinate system is selected in such a way that $\bar{x}$ and $\bar{y}$ lie along and normal to the center line, respectively. The longitudinal and transverse velocity components are denoted by $\bar{u}$ and $\bar{v}$. An infinite train of sinusoidal waves of speed c travel in the $\bar{x}$-direction. The wall geometry is described as follows (see Fig. 1):

$$\bar{h}(\bar{x},\bar{t}) = a(\bar{t}) + b\sin\frac{2\pi}{\lambda}(\bar{x}-ct), \tag{1}$$

where b is the wave amplitude, λ is its wavelength, and $\bar{t}$ is the time.

The continuity and momentum equations are

$$\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}} = 0, \tag{2}$$

$$\frac{\partial\bar{u}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial\bar{x}} + \nu\left[\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{y}^2}\right], \tag{3}$$

$$\frac{\partial\bar{v}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{v}}{\partial\bar{y}} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial\bar{y}} + \nu\left[\frac{\partial^2\bar{v}}{\partial\bar{x}^2} + \frac{\partial^2\bar{v}}{\partial\bar{y}^2}\right], \tag{4}$$

in which $\bar{p}$ is the pressure, ρ is the density, and ν is the kinematic viscosity.

Note that the wall is expanding or contracting in the normal direction. This means that the distance between the two walls is a function of time ($a(\bar{t})$), and there is no motion of the wall in the longitudinal direction.

If $(\bar{u},\bar{v})$ and $(\hat{u},\hat{v})$ are the respective velocity components in the laboratory $(\bar{x},\bar{y})$ and wave $(\hat{x},\hat{y})$ frames

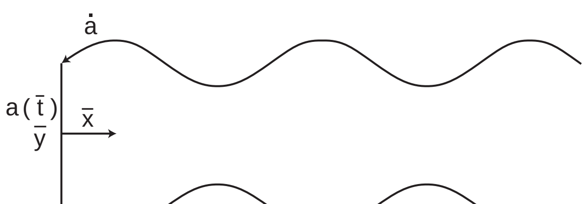

Fig. 1. Coordinate system and the channel under consideration.

then we define

$$\hat{x} = \bar{x} - c\bar{t}, \quad \hat{y} = \bar{y}, \quad \hat{t} = \bar{t}, \quad \hat{u} = \bar{u} - c, \quad \hat{v} = \bar{v}. \tag{5}$$

Invoking above transforms in (1) – (4) and then employing the following dimensionless variables

$$x = \frac{2\pi\hat{x}}{\lambda}, \quad y = \frac{\hat{y}}{a}, \quad u = \frac{\hat{u}}{c}, \quad v = \frac{\hat{v}}{c}, \quad h = \frac{\hat{h}}{a}, \quad p = \frac{2\pi a^2}{\lambda\mu c}\hat{p}, \tag{6}$$

we obtain

$$\frac{a^2}{\nu}\Psi_{y\hat{t}} - \frac{a\dot{a}}{\nu}y\Psi_{yy} + \mathrm{Re}\,\delta[\Psi_y\Psi_{yx} - \Psi_x\Psi_{yy}] = -\frac{\partial p}{\partial x} + [\delta^2\Psi_{yxx} + \Psi_{yyy}], \tag{7}$$

$$-\frac{a\dot{a}}{\nu}\Psi_x - \frac{a^2}{\nu}\Psi_{x\hat{t}} + \frac{a\dot{a}}{\nu}y\Psi_{xy} + \mathrm{Re}\,\delta[-\Psi_y\Psi_{xx} + \Psi_x\Psi_{xy}] = -\frac{1}{\delta^2}\frac{\partial p}{\partial y} - [\delta^2\Psi_{xxx} + \Psi_{xyy}], \tag{8}$$

$$h = 1 + \phi\sin x, \tag{9}$$

in which the subscript denote the partial derivative, $\phi = \frac{b}{a_0}$ is the amplitude ratio, $\delta = \frac{2\pi a_0}{\lambda}$ is the wave number, and $\mathrm{Re} = \frac{a_0 c}{\nu}$ is the Reynolds number. Where a_0 is the channel half spacing when $t = 0$.

It should be pointed out that in deriving (7) and (8) we have used $u = \frac{\partial\Psi}{\partial y}$, $v = -\delta\frac{\partial\Psi}{\partial x}$ and the continuity equation is automatically satisfied.

Eliminating the pressure p between (7) and (8) yields the following compatibility equation:

$$\alpha\nabla^2\Psi_{yy} - 2\delta^2\alpha\Psi_{xx} - \frac{a^2}{\nu}\nabla^2\Psi_{\hat{t}} + \alpha y\nabla^2\Psi_y + \mathrm{Re}\,\delta[-\Psi_y\nabla^2\Psi_x + \Psi_x\nabla^2\Psi_y] = -[\delta^2\nabla^2\Psi_{xx} + \nabla^2\Psi_{yy}], \tag{10}$$

$$\nabla^2 = \delta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{11}$$

where the wall expansion ratio α is given by

$$\alpha = \frac{a\dot{a}}{\nu}, \tag{12}$$

and α is positive for expansion. If the small parameter α remains constant and let the stream function Ψ

varies with α instead of t [23] then (7), (8), and (10) become

$$-\alpha y\Psi_{yy} + \mathrm{Re}\,\delta[\Psi_y\Psi_{yx} - \Psi_x\Psi_{yy}] = -\frac{\partial p}{\partial x} + [\delta^2\Psi_{yxx} + \Psi_{yyy}], \quad (13)$$

$$-\alpha\Psi_x + \alpha y\Psi_{xy} + \mathrm{Re}\,\delta[-\Psi_y\Psi_{xx} + \Psi_x\Psi_{xy}] = -\frac{1}{\delta^2}\frac{\partial p}{\partial y} - [\delta^2\Psi_{xxx} + \Psi_{xyy}], \quad (14)$$

$$\alpha[-\delta^2\Psi_{xx} + \Psi_{yy}] + \alpha y\nabla^2\Psi_y + \mathrm{Re}\,\delta[-\Psi_y\nabla^2\Psi_x + \Psi_x\nabla^2\Psi_y] = -[\delta^2\nabla^2\Psi_{xx} + \nabla^2\Psi_{yy}]. \quad (15)$$

The dimensionless boundary conditions are [4]

$$\Psi = 0, \quad \frac{\partial^2\Psi}{\partial y^2} = 0 \text{ at } y = 0 \text{ and } \Psi = F, \quad \frac{\partial\Psi}{\partial y} = -1 \text{ at } y = h. \quad (16)$$

In above boundary conditions

$$Q = F + 1, \quad (17)$$

where Q is the dimensionless time mean flow in the laboratory frame and is given by

$$F = \int_0^h \frac{\partial\Psi}{\partial y}\mathrm{d}y = \Psi(h) - \Psi(0). \quad (18)$$

3. Solution of the Problem

Here the problem consisting of (15) and the boundary conditions (16) will be solved by the perturbation technique. For that we expand

$$\Psi = \Psi_0 + \delta\Psi_1 + O(\delta^2), \quad (19)$$

$$p = p_0 + \delta p_1 + O(\delta^2), \quad (20)$$

$$F = F_0 + \delta F_1 + O(\delta^2). \quad (21)$$

Substituting above expressions into (13) to (16) we get the following differential systems.

3.1. Zeroth-Order System

$$\alpha\frac{\partial^2\Psi_0}{\partial y^2} + y\alpha\frac{\partial^3\Psi_0}{\partial y^3} + \frac{\partial^4\Psi_0}{\partial y^4} = 0, \quad (22)$$

$$\frac{\partial p_0}{\partial x} = \frac{\partial^3\Psi_0}{\partial y^3} + y\alpha\frac{\partial^2\Psi_0}{\partial y^2}, \quad (23)$$

$$\frac{\partial p_0}{\partial y} = 0, \quad (24)$$

$$y = h; \quad \frac{\partial\Psi_0}{\partial y} = -1, \quad \Psi_0 = F_0, \qquad y = 0; \quad \frac{\partial^2\Psi_0}{\partial y^2} = 0, \quad \Psi_0 = 0. \quad (25)$$

3.2. First-Order System

$$\alpha\frac{\partial^2\Psi_1}{\partial y^2} + y\alpha\frac{\partial^3\Psi_1}{\partial y^3} + \frac{\partial^4\Psi_1}{\partial y^4} + \mathrm{Re}\left(\frac{\partial^3\Psi_0}{\partial y^3}\frac{\partial\Psi_0}{\partial x} - \frac{\partial\Psi_0}{\partial y}\frac{\partial^3\Psi_0}{\partial x\partial y^2}\right) = 0, \quad (26)$$

$$\frac{\partial p_1}{\partial x} = \frac{\partial^3\Psi_1}{\partial y^3} + y\alpha\frac{\partial^2\Psi_1}{\partial y^2} + \mathrm{Re}\left(\frac{\partial^2\Psi_0}{\partial y^2}\frac{\partial\Psi_0}{\partial x} - \frac{\partial\Psi_0}{\partial y}\frac{\partial^2\Psi_0}{\partial x\partial y}\right), \quad (27)$$

$$\frac{\partial p_1}{\partial y} = 0, \quad (28)$$

$$y = h; \quad \frac{\partial\Psi_1}{\partial y} = 0, \quad \Psi_1 = F_1, \qquad y = 0; \quad \frac{\partial^2\Psi_1}{\partial y^2} = 0, \quad \Psi_1 = 0. \quad (29)$$

3.3. Second-Order System

$$\alpha\frac{\partial^2\Psi_2}{\partial y^2} + y\alpha\frac{\partial^3\Psi_2}{\partial y^3} + \frac{\partial^4\Psi_2}{\partial y^4} + \mathrm{Re}\left(\frac{\partial^3\Psi_1}{\partial y^3}\frac{\partial\Psi_0}{\partial x} + \frac{\partial^3\Psi_0}{\partial y^3}\frac{\partial\Psi_1}{\partial x} - \frac{\partial\Psi_1}{\partial y}\frac{\partial^3\Psi_0}{\partial x\partial y^2} - \frac{\partial\Psi_0}{\partial y}\frac{\partial^3\Psi_1}{\partial x\partial y^2}\right) - \alpha\frac{\partial^2\Psi_0}{\partial x^2} + y\alpha\frac{\partial^3\Psi_0}{\partial x^2\partial y} + 2\frac{\partial^3\Psi_0}{\partial x^2\partial y^2} = 0, \quad (30)$$

$$\frac{\partial p_2}{\partial x} = \frac{\partial^3\Psi_2}{\partial y^3} + y\alpha\frac{\partial^2\Psi_2}{\partial y^2} + \mathrm{Re}\left(\frac{\partial^2\Psi_1}{\partial y^2}\frac{\partial\Psi_0}{\partial x} + \frac{\partial^2\Psi_0}{\partial y^2}\frac{\partial\Psi_1}{\partial x} - \frac{\partial\Psi_1}{\partial y}\frac{\partial^2\Psi_0}{\partial x\partial y} - \frac{\partial\Psi_0}{\partial y}\frac{\partial^2\Psi_1}{\partial x\partial y}\right) - \frac{\partial^3\Psi_1}{\partial x^2\partial y}, \quad (31)$$

$$\frac{\partial p_2}{\partial y} = \alpha\frac{\partial\Psi_0}{\partial y} - \alpha y\frac{\partial^2\Psi_0}{\partial x\partial y} - \frac{\partial^3\Psi_0}{\partial x\partial y^2}, \quad (32)$$

$$y=h;\quad \frac{\partial \Psi_2}{\partial y}=0,\quad \Psi_2=F_2,\qquad y=0;\quad \frac{\partial^2 \Psi_2}{\partial y^2}=0,\quad \Psi_2=0. \tag{33}$$

For slowly expanding or contracting walls, the wall dilation ratio α is small in many biological applications. Hence, we write

$$\Psi_0=\Psi_{00}+\alpha\Psi_{01}+O(\alpha^2), \tag{34}$$

$$p_0=p_{00}+\alpha p_{01}+O(\alpha^2), \tag{35}$$

$$F_0=F_{00}+\alpha F_{01}+O(\alpha^2). \tag{36}$$

Invoking above equations into zeroth-order system consisting of (22)–(25) and equating the coefficient of α^0 and α we get

$$\frac{\partial^4 \Psi_{00}}{\partial y^4}=0,\quad \frac{\partial p_{00}}{\partial x}=\frac{\partial^3 \Psi_{00}}{\partial y^3},\quad \frac{\partial p_{00}}{\partial y}=0, \tag{37}$$

$$y=h;\quad \frac{\partial \Psi_{00}}{\partial y}=-1,\quad \Psi_{00}=F_{00},\qquad y=0;\quad \frac{\partial^2 \Psi_{00}}{\partial y^2}=0,\quad \Psi_{00}=0 \tag{38}$$

and

$$\frac{\partial^2 \Psi_{00}}{\partial y^2}+y\frac{\partial^3 \Psi_{00}}{\partial y^3}+\frac{\partial^4 \Psi_{01}}{\partial y^4}=0,\qquad \frac{\partial p_{01}}{\partial x}=\frac{\partial^3 \Psi_{01}}{\partial y^3}+y\frac{\partial^2 \Psi_{00}}{\partial y^2},\quad \frac{\partial p_{01}}{\partial y}=0, \tag{39}$$

$$y=h;\quad \frac{\partial \Psi_{01}}{\partial y}=0,\quad \Psi_{01}=F_{01},\qquad y=0;\quad \frac{\partial^2 \Psi_{01}}{\partial y^2}=0,\quad \Psi_{01}=0. \tag{40}$$

Adopting the procedure as used for Ψ_0 we obtain for Ψ_1 the following problem:

$$\frac{\partial^2 \Psi_{01}}{\partial y^2}+\mathrm{Re}\left(\frac{\partial^3 \Psi_{00}}{\partial y^3}\frac{\partial \Psi_{00}}{\partial x}-\frac{\partial \Psi_{00}}{\partial y}\frac{\partial^3 \Psi_{00}}{\partial x\partial y^2}\right)+\frac{\partial^4 \Psi_{10}}{\partial y^4}=0, \tag{41}$$

$$\frac{\partial p_{10}}{\partial x}=\frac{\partial^3 \Psi_{10}}{\partial y^3}+\mathrm{Re}\left(\frac{\partial^2 \Psi_{00}}{\partial y^2}\frac{\partial \Psi_{00}}{\partial x}-\frac{\partial \Psi_{00}}{\partial y}\frac{\partial \Psi_{00}}{\partial x\partial y}\right), \tag{42}$$

$$\frac{\partial p_{10}}{\partial y}=0, \tag{43}$$

$$y=h;\quad \frac{\partial \Psi_{10}}{\partial y}=0,\quad \Psi_{10}=F_{10},\qquad y=0;\quad \frac{\partial^2 \Psi_{10}}{\partial y^2}=0,\quad \Psi_{10}=0 \tag{44}$$

and

$$\frac{\partial^2 \Psi_{10}}{\partial y^2}+y\frac{\partial^3 \Psi_{10}}{\partial y^3}+\mathrm{Re}\left(\frac{\partial^3 \Psi_{01}}{\partial y^3}\frac{\partial \Psi_{00}}{\partial x}+\frac{\partial^3 \Psi_{00}}{\partial y^3}\frac{\partial \Psi_{01}}{\partial x}-\frac{\partial \Psi_{01}}{\partial y}\frac{\partial \Psi_{00}}{\partial x\partial y^2}-\frac{\partial \Psi_{00}}{\partial y}\frac{\partial \Psi_{01}}{\partial x\partial y^2}\right)+\frac{\partial^4 \Psi_{11}}{\partial y^4}=0, \tag{45}$$

$$\frac{\partial p_{11}}{\partial x}=y\frac{\partial^2 \Psi_{10}}{\partial y^2}+\frac{\partial^3 \Psi_{11}}{\partial y^3}+\mathrm{Re}\left(\frac{\partial^2 \Psi_{01}}{\partial y^2}\frac{\partial \Psi_{00}}{\partial x}+\frac{\partial^2 \Psi_{00}}{\partial y^2}\frac{\partial \Psi_{01}}{\partial x}-\frac{\partial \Psi_{01}}{\partial y}\frac{\partial \Psi_{00}}{\partial x\partial y}-\frac{\partial \Psi_{00}}{\partial y}\frac{\partial \Psi_{01}}{\partial x\partial y}\right), \tag{46}$$

$$y=h;\quad \frac{\partial \Psi_{11}}{\partial y}=0,\quad \Psi_{11}=F_{11},\qquad y=0;\quad \frac{\partial^2 \Psi_{11}}{\partial y^2}=0,\quad \Psi_{11}=0. \tag{47}$$

Similarly, for Ψ_2 we obtain

$$2\frac{\partial^4 \Psi_{00}}{\partial x^2\partial y^2}+\mathrm{Re}\left(\frac{\partial^3 \Psi_{10}}{\partial y^3}\frac{\partial \Psi_{00}}{\partial x}+\frac{\partial^3 \Psi_{00}}{\partial y^3}\frac{\partial \Psi_{10}}{\partial x}-\frac{\partial \Psi_{10}}{\partial y}\frac{\partial \Psi_{00}}{\partial x\partial y^2}-\frac{\partial \Psi_{00}}{\partial y}\frac{\partial \Psi_{10}}{\partial x\partial y^2}\right)+\frac{\partial^4 \Psi_{20}}{\partial y^4}=0, \tag{48}$$

$$\frac{\partial p_{20}}{\partial x}=\frac{\partial^3 \Psi_{20}}{\partial y^3}+\frac{\partial^3 \Psi_{00}}{\partial x^2\partial y}+\mathrm{Re}\left(\frac{\partial^2 \Psi_{10}}{\partial y^2}\frac{\partial \Psi_{00}}{\partial x}+\frac{\partial^2 \Psi_{00}}{\partial y^2}\frac{\partial \Psi_{10}}{\partial x}-\frac{\partial \Psi_{10}}{\partial y}\frac{\partial \Psi_{00}}{\partial x\partial y}-\frac{\partial \Psi_{00}}{\partial y}\frac{\partial \Psi_{10}}{\partial x\partial y}\right), \tag{49}$$

$$\frac{\partial p_{20}}{\partial y}=-\frac{\partial^3 \Psi_{00}}{\partial x\partial y^2}, \tag{50}$$

$$y=h;\quad \frac{\partial \Psi_{20}}{\partial y}=0,\quad \Psi_{20}=F_{20},\qquad y=0;\quad \frac{\partial^2 \Psi_{20}}{\partial y^2}=0,\quad \Psi_{20}=0. \tag{51}$$

$$\frac{\partial^2 \Psi_{20}}{\partial y^2}+y\frac{\partial^3 \Psi_{20}}{\partial y^3}+2\frac{\partial^4 \Psi_{01}}{\partial x^2\partial y^2}-\frac{\partial^2 \Psi_{00}}{\partial x^2}+y\frac{\partial^3 \Psi_{00}}{\partial x^2\partial y}+\mathrm{Re}\left(\frac{\partial^3 \Psi_{11}}{\partial y^3}\frac{\partial \Psi_{00}}{\partial x}+\frac{\partial^3 \Psi_{10}}{\partial y^3}\frac{\partial \Psi_{01}}{\partial x}+\frac{\partial^3 \Psi_{01}}{\partial y^3}\frac{\partial \Psi_{10}}{\partial x}+\frac{\partial^3 \Psi_{00}}{\partial y^3}\frac{\partial \Psi_{11}}{\partial x}-\frac{\partial \Psi_{11}}{\partial y}\frac{\partial \Psi_{00}}{\partial x\partial y^2}-\frac{\partial \Psi_{10}}{\partial y}\frac{\partial \Psi_{01}}{\partial x\partial y^2}-\frac{\partial \Psi_{01}}{\partial y}\frac{\partial \Psi_{10}}{\partial x\partial y^2}-\frac{\partial \Psi_{00}}{\partial y}\frac{\partial \Psi_{11}}{\partial x\partial y^2}\right)+\frac{\partial^4 \Psi_{21}}{\partial y^4}=0, \tag{52}$$

$$\frac{\partial p_{21}}{\partial x}=y\frac{\partial^2\Psi_{20}}{\partial y^2}+\frac{\partial^3\Psi_{21}}{\partial y^3}+\frac{\partial^3\Psi_{01}}{\partial x^2\partial y}+\mathrm{Re}\left(\frac{\partial^2\Psi_{11}}{\partial y^2}\frac{\partial\Psi_{00}}{\partial x}+\frac{\partial^2\Psi_{10}}{\partial y^2}\frac{\partial\Psi_{01}}{\partial x}+\frac{\partial^2\Psi_{01}}{\partial y^2}\frac{\partial\Psi_{10}}{\partial x}+\frac{\partial^2\Psi_{00}}{\partial y^2}\frac{\partial\Psi_{11}}{\partial x}-\frac{\partial\Psi_{11}}{\partial y}\frac{\partial\Psi_{00}}{\partial x\partial y}-\frac{\partial\Psi_{10}}{\partial y}\frac{\partial\Psi_{01}}{\partial x\partial y}-\frac{\partial\Psi_{01}}{\partial y}\frac{\partial\Psi_{10}}{\partial x\partial y}-\frac{\partial\Psi_{00}}{\partial y}\frac{\partial\Psi_{11}}{\partial x\partial y}\right) \quad (53)$$

and

$$\frac{\partial p_{21}}{\partial y}=\frac{\partial\Psi_{00}}{\partial y}-y\frac{\partial^2\Psi_{00}}{\partial x\partial y}-\frac{\partial^3\Psi_{01}}{\partial x\partial y^2}, \quad (54)$$

$$y=h;\quad \frac{\partial\Psi_{21}}{\partial y}=0,\quad \Psi_{21}=F_{21},$$
$$y=0;\quad \frac{\partial^2\Psi_{21}}{\partial y^2}=0,\quad \Psi_{21}=0. \quad (55)$$

Solving system (37) – (40) and using (34) we get

$$\Psi_{00}=\beta_1 y-\beta_2 y^3, \quad (56)$$
$$\Psi_{01}=\beta_3 y+\beta_4 y^3+\beta_5 y^5, \quad (57)$$
$$u_{00}=\beta_1-3\beta_2 y^2, \quad (58)$$
$$u_{01}=\beta_3+3\beta_4 y^2+5\beta_5 y^4, \quad (59)$$
$$\frac{\mathrm{d}p_{00}}{\mathrm{d}x}=-6\beta_2, \quad (60)$$
$$\frac{\mathrm{d}p_{01}}{\mathrm{d}x}=6\beta_4, \quad (61)$$

where

$$\beta_1=\frac{3F_{00}+h}{2h},\quad \beta_2=-\frac{F_{00}+h}{2h^3},$$
$$\beta_3=-\frac{-15F_{01}+h^5\beta_2}{10h},\quad \beta_4=\frac{-5F_{01}+2h^5\beta_2}{10h^3}, \quad (62)$$

and $\beta_5=-\frac{\beta_2}{10}$.

We note from (37) and (39) that p_{00} and p_{01} are independent upon y. The pressure rise per wavelength at the zeroth-order is defined as

$$\Delta p_{\lambda 0}=\int_0^{2\pi}\frac{\mathrm{d}p_0}{\mathrm{d}x}\mathrm{d}x. \quad (63)$$

Using above definition and evaluating the involved integral we write

$$\Delta p_{\lambda_{00}}=-3\left[\frac{\pi(2+\phi^2)F_{00}}{(1-\phi^2)^{\frac{5}{2}}}+\frac{2\pi}{(1-\phi^2)^{\frac{3}{2}}}\right], \quad (64)$$

$$\Delta p_{\lambda_{01}}=-\frac{3}{5}\left[\frac{5\pi(2+\phi^2)F_{01}}{(1-\phi^2)^{\frac{5}{2}}}+\frac{2\pi F_{00}}{(1-\phi^2)^{\frac{1}{2}}}+2\pi\right]. \quad (65)$$

From (19) and (56) – (65) one has

$$\Psi_0=\beta_1 y-\beta_2 y^3+\alpha(\beta_3 y+\beta_4 y^3+\beta_5 y^5)+O(\alpha^2), \quad (66)$$
$$u_0=\beta_1-3\beta_2 y^2+\alpha(\beta_3+3\beta_4 y^2+5\beta_5 y^4)+O(\alpha^2), \quad (67)$$
$$\frac{\mathrm{d}p_0}{\mathrm{d}x}=-6\beta_2+6\alpha\beta_4, \quad (68)$$

$$\Delta p_0=-3\left[\frac{\pi(2+\phi^2)F_{00}}{(1-\phi^2)^{\frac{5}{2}}}+\frac{2\pi}{(1-\phi^2)^{\frac{3}{2}}}\right]-\frac{3\alpha}{5}\left[\frac{5\pi(2+\phi^2)F_{01}}{(1-\phi^2)^{\frac{5}{2}}}+\frac{2\pi F_{00}}{(1-\phi^2)^{\frac{1}{2}}}+2\pi\right]+O(\alpha^2). \quad (69)$$

It is interesting to note that the above expressions are identical to the results obtained by Shapiro et. al. [4] when $\alpha=0$ (for undeformed channel).

Solving (41) – (47) we have

$$\Psi_{10}=\beta_6 y+\beta_7 y^3+\beta_8 y^5+\beta_9 y^7, \quad (70)$$
$$\Psi_{11}(x,y)=\gamma_1 y+\gamma_2 y^3+\gamma_3 y^5+\gamma_4 y^7+\gamma_5 y^9, \quad (71)$$
$$u_{10}(x,y)=\beta_6+3\beta_7 y^2+5\beta_8 y^4+7\beta_9 y^6, \quad (72)$$
$$u_{11}(x,y)=\gamma_1+3\gamma_2 y^2+5\gamma_3 y^4+7\gamma_4 y^6+9\gamma_5 y^8, \quad (73)$$
$$\frac{\mathrm{d}p_{10}}{\mathrm{d}x}=6\beta_7, \quad (74)$$
$$\frac{\mathrm{d}p_{11}}{\mathrm{d}x}=6\gamma_2, \quad (75)$$
$$\Delta p_{\lambda_{10}}=-\frac{3\pi(2+\phi^2)F_{10}}{(1-\phi^2)^{\frac{5}{2}}}, \quad (76)$$
$$\Delta p_{\lambda_{11}}=-3\pi\left[\frac{2F_{10}}{5(1-\phi^2)^{\frac{1}{2}}}+\frac{(2+\phi^2)F_{11}}{(1-\phi^2)^{\frac{5}{2}}}\right], \quad (77)$$

where

$$\beta_6=\frac{3F_{10}+2h^5\beta_8+4h^7\beta_9}{2h},$$
$$\beta_7=-\frac{F_{10}+4h^5\beta_8+6h^7\beta_9}{2h^3},$$
$$\beta_8=-\frac{\mathrm{Re}(\beta_2\beta_1'-\beta_1\beta_2')}{20},$$
$$\beta_9=\frac{\mathrm{Re}\,\beta_2\beta_2'}{10},$$
$$\gamma_1=\frac{3F_{11}+2h^5\gamma_3+4h^7\gamma_4+6h^9\gamma_5}{2h},$$

$$\gamma_2 = -\frac{F_{11}+4h^5\gamma_3+6h^7\gamma_4+8h^9\gamma_5}{2h^3},$$
$$\gamma_3 = -\frac{2\beta_7+\mathrm{Re}(\beta_4\beta_1'-\beta_3\beta_2'+\beta_2\beta_3'-\beta_1\beta_4')}{20},$$
$$\gamma_4 = -\frac{20\beta_8+\mathrm{Re}(15\beta_5\beta_1'-3\beta_4\beta_2'-3\beta_2\beta_4'-5\beta_1\beta_5')}{210},$$
$$\gamma_5 = -\frac{42\beta_9+\mathrm{Re}(5\beta_5\beta_2'-9\beta_2\beta_5')}{504}.$$

Similarly, solutions of (48) – (55) are given by

$$\Psi_{20}(x,y) = \gamma_6 y+\gamma_7 y^3+\gamma_8 y^5+\gamma_9 y^7 + \gamma_{10}y^9+\gamma_{11}y^{11}, \quad (78)$$
$$\Psi_{21}(x,y) = \chi_1 y+\chi_2 y^3+\chi_3 y^5+\chi_4 y^7 + \chi_5 y^9+\chi_7 y^{11}+\chi_8 y^{13}, \quad (79)$$
$$u_{20}(x,y) = \gamma_6+3\gamma_7 y^2+5\gamma_8 y^4+7\gamma_9 y^6 + 9\gamma_{10}y^8+11\gamma_{11}y^{10}, \quad (80)$$
$$u_{21}(x,y) = \chi_1+3\chi_2 y^2+5\chi_3 y^4+7\chi_4 y^6 + 9\chi_5 y^8+11\chi_7 y^{10}+13\chi_8 y^{12}, \quad (81)$$
$$\frac{\mathrm{d}p_{20}}{\mathrm{d}x} = 6\gamma_7, \quad (82)$$
$$\frac{\mathrm{d}p_{21}}{\mathrm{d}x} = 6\chi_2, \quad (83)$$
$$\nabla p_{\lambda_{21}} = \frac{\pi\mathrm{Re}^2\phi^2}{4000}\left\{\left(1+\frac{\phi^2}{4}\right)\left(1-\frac{13}{14}F_{00}\right) + \frac{3}{2}F_{00}^2+\frac{330}{7}F_{01}\right\}F_{00} + \frac{9\pi\mathrm{Re}^2}{140}\left(1-\sqrt{1-\phi^2}\right)F_{01}F_{00}^2, \quad (84)$$
$$\nabla p_{\lambda_{20}} = -\frac{3\pi(2+\phi^2)F_{20}}{(1-\phi^2)^{\frac{5}{2}}}+c_0+c_1F_{00} + c_2F_{00}^2+c_3F_{00}^3, \quad (85)$$

where

$$\gamma_6 = \frac{3F_{20}+2h^5\gamma_8+4h^7\gamma_9+6h^9\gamma_{10}+8h^{11}\gamma_{11}}{2h},$$
$$\gamma_7 = -\frac{F_{20}+4h^5\gamma_8+6h^7\gamma_9+8h^9\gamma_{10}+10h^{11}\gamma_{11}}{2h^3},$$
$$\gamma_8 = -\frac{2\beta_2''+\mathrm{Re}(\beta_7\beta_1'-\beta_6\beta_2'+\beta_2\beta_6'-\beta_1\beta_7')}{20},$$
$$\gamma_9 = -\frac{\mathrm{Re}(15\beta_8\beta_1'-3\beta_7\beta_2'-3\beta_2\beta_7'-5\beta_1\beta_8')}{210},$$
$$\gamma_{10} = -\frac{\mathrm{Re}(35\beta_9\beta_1'+5\beta_8\beta_2'-9\beta_2\beta_8'-7\beta_1\beta_9')}{504},$$
$$\gamma_{11} = -\frac{\mathrm{Re}(7\beta_9\beta_2'-5\beta_2\beta_9')}{330},$$
$$c_0 = \frac{12\pi}{5}\left(-4+\frac{4-3\phi^2}{(1-\phi^2)^{\frac{1}{2}}}\right)-\frac{197\pi\mathrm{Re}^2\phi^2}{40425},$$
$$c_1 = \frac{8\pi\mathrm{Re}^2(-1+\sqrt{1-\phi^2})}{175}+\frac{18\pi\phi^2}{5(1-\phi^2)^{\frac{5}{2}}},$$
$$c_2 = \frac{232}{2695}\pi\mathrm{Re}^2\left(1-\frac{1}{\sqrt{1-\phi^2}}\right),$$
$$c_3 = -\frac{312\pi\mathrm{Re}^2\phi^2}{13475(1-\phi^2)^{\frac{3}{2}}},$$
$$\chi_1 = \frac{1}{2h}[3F_{21}+2h^5\chi_3+4h^7\chi_4+6h^9\chi_5+8h^{11}\chi_6 +10h^{13}\chi_7],$$
$$\chi_2 = -\frac{1}{2h^3}[F_{21}+4h^5\chi_3+6h^7\chi_4+8h^9\chi_5 +10h^{11}\chi_6+12h^{13}\chi_7],$$
$$\chi_3 = -\frac{1}{20}[2\gamma_7+\mathrm{Re}(\gamma_2\beta_1'-\gamma_1\beta_2'+\beta_7\beta_3'-\beta_6\beta_4' +\beta_4\beta_6'-\beta_3\beta_7'+\beta_2\gamma_1'-\beta_1\gamma_2')+2\beta_4''],$$
$$\chi_4 = -\frac{1}{42}[4\gamma_8+\mathrm{Re}(3(\gamma_3\beta_1'+\beta_8\beta_3'+\beta_5\beta_6') -0.6(\gamma_2\beta_2'+(\beta_4\beta_7)'+\beta_2\gamma_2')-(\beta_6\beta_5'+\beta_3\beta_8'+\beta_1\gamma_3'))] +\frac{\beta_2''+20\beta_5''}{420},$$
$$\chi_5 = -\frac{1}{504}[42\gamma_9+\mathrm{Re}(5(7\gamma_4\beta_1'+\gamma_3\beta_2'+7\beta_9\beta_3'+\beta_8\beta_4' +\beta_5\beta_7')-9(\beta_7\beta_5'+\beta_4\beta_8'+\beta_2\gamma_3')-7(\beta_3\beta_9'+\beta_1\gamma_4'))],$$
$$\chi_6 = -\frac{1}{999}[72\gamma_{10}+\mathrm{Re}(63\gamma_5\beta_1'+21\gamma_4\beta_2'+21\beta_9\beta_4' -5(\beta_5\beta_8)'-15\beta_4\beta_9'-15\beta_2\gamma_4'-9\beta_1\gamma_5')],$$
$$\chi_7 = -\frac{1}{1716}[110\gamma_{11}+\mathrm{Re}(45\gamma_5\beta_2'+7\beta_9\beta_5' -15\beta_5\beta_9'-21\beta_2\gamma_5')].$$

Following the definition given in (19), then expressions of stream function, longitudinal velocity, longitudinal pressure gradient, and pressure rise per wavelength are

$$\Psi = \beta_1 y-\beta_2 y^3+\alpha(\beta_3 y+\beta_4 y^3+\beta_5 y^5)+\delta(\beta_6 y +\beta_7 y^3+\beta_8 y^5+\beta_9 y^7+\alpha(\gamma_1 y+\gamma_2 y^3+\gamma_3 y^5+\gamma_4 y^7 +\gamma_5 y^9))+\delta^2(\gamma_6 y+\gamma_7 y^3+\gamma_8 y^5+\gamma_9 y^7+\gamma_{10}y^9 +\gamma_{11}y^{11}+\alpha(\chi_1 y+\chi_2 y^3+\chi_3 y^5+\chi_4 y^7+\chi_5 y^9 +\chi_7 y^{11}+\chi_8 y^{13})), \quad (86)$$

$$u = \beta_1 - 3\beta_2 y^2 + \alpha(\beta_3 + 3\beta_4 y^2 + 5\beta_5 y^4) + \delta(\beta_6 + 3\beta_7 y^2 + 5\beta_8 y^4 + 7\beta_9 y^6 + \alpha(\gamma_1 + 3\gamma_2 y^2 + 5\gamma_3 y^4 + 7\gamma_4 y^6 + 9\gamma_5 y^8)) + \delta^2(\gamma_6 + 3\gamma_7 y^2 + 5\gamma_8 y^4 + 7\gamma_9 y^6 + 9\gamma_{10} y^8 + 11\gamma_{11} y^{10} + \alpha(\chi_1 + 3\chi_2 y^2 + 5\chi_3 y^4 + 7\chi_4 y^6 + 9\chi_5 y^8 + 11\chi_7 y^{10} + 13\chi_8 y^{12})), \quad (87)$$

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 6[-\beta_2 + \alpha\beta_4 + \delta(\beta_7 + \alpha\gamma_2) + \delta^2(\gamma_7 + \alpha\chi_2)], \quad (88)$$

$$\Delta p = -3\pi\left[\frac{(2+\phi^2)F_{00}}{(1-\phi^2)^{\frac{5}{2}}} + \frac{2}{(1-\phi^2)^{\frac{3}{2}}}\right] - \frac{6\pi\alpha}{5}\left[\frac{5(2+\phi^2)F_{10}}{2(1-\phi^2)^{\frac{5}{2}}} + \frac{F_{00}}{(1-\phi^2)^{\frac{1}{2}}} + 1\right] - \delta\frac{3\pi(2+\phi^2)F_{10}}{(1-\phi^2)^{\frac{5}{2}}} - \frac{3\alpha\delta\pi}{5}\left[\frac{5(2+\phi^2)F_{11}}{(1-\phi^2)^{\frac{5}{2}}} + \frac{2F_{10}}{(1-\phi^2)^{\frac{1}{2}}}\right] + \delta^2\left[-3F_{20}\frac{\pi(2+\phi^2)}{(1-\phi^2)^{\frac{5}{2}}} + \frac{6\pi}{5}\left(4 + \frac{4(\phi^2-1) - 3\phi^2 F_{00}}{(1-\phi^2)^{\frac{3}{2}}}\right) + 2\pi\mathrm{Re}^2(a_1 + a_2 F_{00} + a_3 F_{00}^2 + a_4 F_{00}^3) + 2\pi\alpha\left(\frac{69}{4}a_2 - \frac{9}{175}\phi^2\right)F_{00} + 2\pi\alpha\mathrm{Re}^2\left[-\frac{31}{95550}\phi^2 - \frac{31}{382200}\phi^4 - \frac{2713}{2627625}\phi^2 F_{00} - \frac{7717}{2627625}\phi^2 F_{00}^2 + \frac{9}{35}a_2 F_{00}^3 - 3F_{00}^2 F_{01} a_4 + 2F_{00}F_{01}a_3 + F_{10}a_2\right] - \frac{18\pi F_{10}}{5}\alpha\frac{\phi^2}{(1-\phi^2)^{\frac{3}{2}}} - 3\pi\alpha F_{21}\frac{(2+\phi^2)}{(1-\phi^2)^{\frac{5}{2}}} - \frac{6\pi\alpha}{5}F_{20}\frac{1}{(1-\phi^2)^{\frac{1}{2}}}\right], \quad (89)$$

where

$$a_1 = \frac{-197\phi^2}{80850},$$

$$a_2 = \frac{-4}{175}\left(1 - (1-\phi^2)^{\frac{1}{2}}\right),$$

$$a_3 = \frac{-116}{2695}\left(-1 + \frac{1}{(1-\phi^2)^{\frac{1}{2}}}\right),$$

$$a_4 = \frac{-156\phi^2}{13475(1-\phi^2)^{\frac{3}{2}}}.$$

Defining $F_1 = F_{00} + \alpha F_{01}$, $F_2 = F_{10} + \alpha F_{11}$, $F_3 = F_{20} + \alpha F_{21}$, $F^{(2)} = F_1 + \delta F_2 + \delta^2 F_3$, then $F_{00} = F_1 - \alpha F_{01}$, $F_{10} = F_2 - \alpha F_{11}$, $F_{20} = F_3 - \alpha F_{21}$, $F_1 = F^{(2)} - \delta F_2 - \delta^2 F_3$, $\alpha F_{01} = F_1 - F_{00}$, $\alpha F_{11} = F_2 - F_{10}$, $\alpha F_{21} = F_3 - F_{20}$, $\delta F_2 = F^{(2)} - F_1 - \delta^2 F_3$, $\delta^2 F_3 = F^{(2)} - F_1 - \delta F_2$, and substituting these expressions into (89) and then retaining only the terms up to $O(\alpha)$ and $O(\delta^2)$ we have

$$\Delta p = -3\pi\left[\frac{(2+\phi^2)F^{(2)}}{(1-\phi^2)^{\frac{5}{2}}} + \frac{2}{(1-\phi^2)^{\frac{3}{2}}}\right] - \alpha\frac{6\pi}{5}\left(1 + \frac{F^{(2)}}{(1-\phi^2)^{\frac{1}{2}}}\right) + 2\pi\mathrm{Re}^2\delta^2[a_1 + a_2F^{(2)} + a_3(F^{(2)})^2 + a_4(F^{(2)})^3] + \frac{2\pi}{35}\mathrm{Re}^2\delta^2\alpha\left[\frac{31\phi^2(\phi^2-4)}{10920} - \frac{2713\phi^2}{75075}F^{(2)} - \frac{7717\phi^2}{75075}(F^{(2)})^2 + 9a_2(F^{(2)})^3\right] + 2\pi\delta^2\alpha\left[-\frac{9\phi^2}{175} + \frac{69}{4}a_2F^{(2)}\right] + \delta^2\frac{6\pi}{5}\left[4 + \frac{4(\phi^2-1) - 3\phi^2F^{(2)}}{(1-\phi^2)^{\frac{3}{2}}}\right]. \quad (90)$$

The above expression reduces to the result presented in the study [4] when $\alpha \to 0$.

4. Results and Discussion

In Figures 2 and 3 we have shown the results for dimensionless pressure rise versus the modified Reynolds number $\mathrm{Re}^* \equiv \mathrm{Re}\,\delta$ calculated from (90) for the zero pumping case ($Q = 0$). The other parameters are: $\phi = 0.4$, $\delta = 1.2566$ and $\phi = 0.6$, $\delta = 0.6283$, respectively. We also used different values of the wall expansion ratio α. These cases were chosen to match the results of Takabatake and Ayukawa [6] for non-deformable walls. Note that the volume flow rate (Q) is related to F by $Q = F + 1$.

Figures 4 and 5 present the graph of Δp_λ versus Q for $\phi = 0.6$, $\delta = 0.06283$, $\mathrm{Re}^* = 10$ and $\phi = 0.4$, $\delta = 1.2566$, $\mathrm{Re}^* = 10$, respectivly. Seven different values are chosen for α which represent three physical situations. Negative values of α show the case of contracting walls while the positive values represent the expanding walls and $\alpha = 0$ correspond to a non-deformable channel. For this range of the physical parameters, the pressure rise is effected by the wall contraction and expansion. There is a little difference between the contracting and non-contracting walls for small values of the deformation parameter.

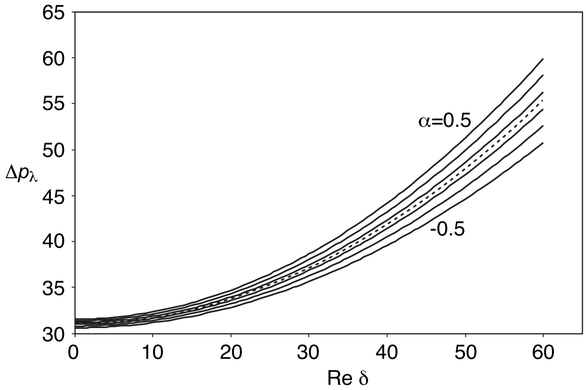

Fig. 2. Pressure rise per wavelength plotted against $\mathrm{Re}^* = \mathrm{Re}\,\delta$, where $\phi = 0.4$, $\delta = 1.2566$, $Q = 0$, and the values of α are: −0.5, −0.3, −0.1, 0, 0.1, 0.3, 0.5.

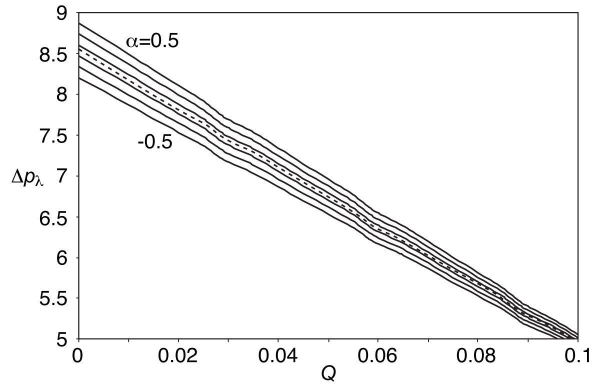

Fig. 4. Pressure rise per wavelength plotted against Q, where $\phi = 0.4$, $\delta = 1.2566$, $Re = 10$, and the values of α are: −0.5, −0.3, −0.1, 0, 0.1, 0.3, 0.5.

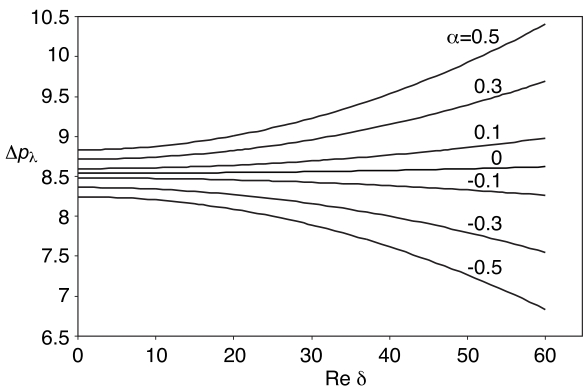

Fig. 3. Pressure rise per wavelength plotted against $\mathrm{Re}^* = \mathrm{Re}\,\delta$, where $\phi = 0.6$, $\delta = 0.06283$, $Q = 0$, and the values of α are: −0.5, −0.3, −0.1, 0, 0.1, 0.3, 0.5.

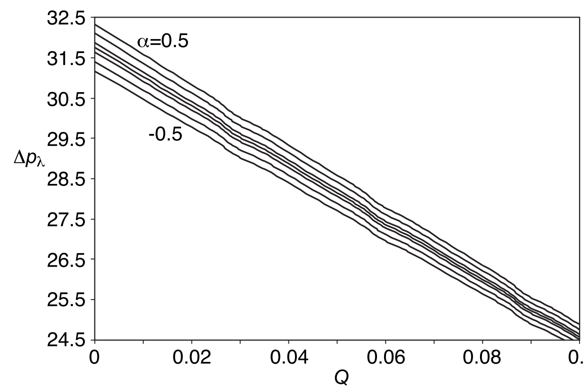

Fig. 5. Pressure rise per wavelength plotted against Q, where $\phi = 0.6$, $\delta = 0.06283$, $Re = 10$, and the values of α are: −0.5, −0.3, −0.1, 0, 0.1, 0.3, 0.5.

5. Closing Remarks

In this study we have shown the effect of wall deformation on the peristaltic flow of a viscous fluid in a planar deformable channel. Although wall deformation effects on channel flows are first studied by Majdalani et. al. [24] the peristaltic mechanism in a planar deformable channel has not been discussed so far. After a careful modelling and solution, it is found that the results obtained for flow quantities reduce to the undeformed channel case [4] when $\alpha = 0$. It is further noted that small effects of expansion and contraction give rise to appreciable influence on the flow quantities such as pressure gradient and volume flow rate.

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