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On Entropy Production in Adiabatic Flow in Turbomachines

In three recent papers, Smith [1], Marsh [2], and Novak [3][1] have given general equations for the flow in turbomachines and methods for their solution. Each author has worked from momentum equations for an inviscid fluid, but has introduced entropy increases in adiabatic flow. Such increases are inconsistent with the basic assumption of inviscid flow in the momentum equations, as the authors have stated. The purpose of this paper is to examine the nature of this inconsistency and to suggest modifications of the equations used by the three authors by the introduction of dissipative forces into the momentum equations. A particular example is calculated, with and without the dissipative force term, to illustrate the order of magnitude of the effect.

Introduction

The analysis presented by each author is first briefly reviewed. In referring to equations used, the notation S, M, and N is used for the Smith, Marsh, and Novak papers. The symbols used by the three papers are different, so the symbols used by each author are retained in discussion of his paper. A comparison between the three notations is given in Appendix 1. In general discussion, Smith's notation is followed.

Smith concentrates on the solution of the radial equilibrium equation for pressure variation,

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = f_1 \qquad \text{(S.29)}$$

The function f_1 is obtained after eliminating the term involving the variation in meridional velocity $\left(W_r \dfrac{DW_m}{Dm}\right)$ from the basic radial momentum equation. Smith presents two methods of achieving this elimination, one for reversible diabatic flow and one for irreversible adiabatic flow. In the latter he uses the streamwise energy equation and the Gibbs relation between entropy, enthalpy, density, and pressure; both these equations are of course valid in viscous, irreversible flow.

Novak prefers to write the radial equilibrium equation in terms of the mean absolute velocity in the meridional plane (C_m), retaining the term $\dfrac{\partial C_m}{\partial m}$ on the righthand side of the equation.

$$\frac{\partial C_m{}^2}{\partial r} = f_2 \qquad \text{(N.9)}$$

Novak's solution then proceeds in similar fashion to Smith's, streamlines being located radially at given axial stations.

Marsh develops the "through-flow" analysis of Wu [4], deriving a Poisson type of equation for the stream function,

$$\nabla^2\psi = f_3 \qquad \text{(M.24)}$$

and solving this by successive approximations, over the whole "mean streamline" surface. Equation (M.24) is obtained by substituting a form of the continuity equation into the radial momentum equation for inviscid flow.

Neither the "radial equilibrium" equation used by Novak nor the Wu-Marsh streamline equation contains the energy equation (1st Law) or the Gibbs entropy equation (2nd Law) explicitly, although the functions f_2 and f_3 (and Smith's f_1) contain gradients of stagnation enthalpy H (or relative stagnation enthalpy I) and entropy s. These gradients have to be determined from a previous solution of the master equation using both of the thermodynamic equations. Other terms in f_1, f_2, and f_3 are obtained from the previous solution for p, and the blade geometry (in the direct problem) or a design assumption for tangential velocity (in the indirect problem). In Novak's and Smith's solutions the radius of curvature of the meridional streamlines has also to be determined from the previous solution. Novak also has to estimate $\dfrac{\partial C_m}{\partial m}$ from the previous solution.

While Smith's equation has been obtained using the first and second laws of the thermodynamics in addition to the radial momentum and continuity equations, the radial momentum equation is initially written without viscous terms, although Smith expresses reservations about this inconsistency.

Smith, Marsh, and Novak all discuss the differences between the general equations they derive, which may include circumferential variations, and the equations for axisymmetric flow, in which the blade action is represented by body forces.

[1] Numbers in brackets designate References at end of paper.

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Introduction of Body Forces Into the Momentum Equations

The main point to be made here is that if an entropy variation is taken to be permitted in an adiabatic flow, then additional "dissipative" body forces should be introduced into the original momentum equation, which is the starting point for each of the three analyses, and the master equations for p, C_m^2, and ψ should be modified accordingly.

If a body force F per unit volume is introduced, the momentum equation in rotating coordinates becomes

$$\frac{\bar{F}}{\rho} - \frac{\nabla p}{\rho} = \frac{D\bar{W}}{Dt} - \omega^2 r + 2\bar{\omega} \times \bar{W} \tag{1}$$

where Smith's notation has been used.

Taking the scalar product of relative velocity $\bar{W}$ with this equation we obtain

$$-\frac{F_d W}{\rho} - \frac{W}{\rho}\frac{\partial p}{\partial x} = \frac{D}{Dt}\left(\frac{W^2}{2}\right) - \omega^2 r W_r \tag{2}$$

where F_d is the dissipative force, assumed to act in opposition to the direction of the streamline, x.

Defining the relative stagnation enthalpy (rothalpy) as

$$I = h + \frac{W^2}{2} - \frac{\omega^2 r^2}{2} \tag{3}$$

and using the Gibbs relation

$$T\nabla s = \nabla h - \frac{\nabla p}{\rho} \tag{4}$$

it follows that

$$-\frac{\nabla p}{\rho} = -\nabla I + T\nabla s + \nabla\left(\frac{W^2}{2}\right) - \nabla\left(\frac{\omega^2 r^2}{2}\right) \tag{5}$$

Again taking the scalar product with $\bar{W}$, one obtains

$$-\frac{W}{\rho}\frac{\partial p}{\partial x} = WT\frac{\partial s}{\partial x} - W\frac{\partial I}{\partial x} + \frac{D}{Dt}\left(\frac{W^2}{2}\right) - \omega^2 r W_r \tag{6}$$

If the energy equation is now applied, then

$$\frac{DI}{Dt} = W\frac{\partial I}{\partial x} = 0 \tag{7}$$

it being assumed that partial derivatives with respect to time are zero.

Comparison of equations (2) and (6) then gives

$$\frac{F_d}{\rho} = T\frac{\partial s}{\partial x} \tag{8}$$

In the S, M, N analyses, assumptions are made for entropy variations—either directly (Novak), through the assumption of a total pressure loss coefficient (Smith), or through the assumption of a local polytropic efficiency (Marsh).

In the latter case, a pressure density relation of the form p/ρ^n is assumed, where

$$n = (\gamma\eta^{\pm 1})/(\gamma\eta^{\pm 1} - (\gamma - 1)) \tag{9}$$

γ being the ratio of specific heats and η the local polytropic efficiency. The positive sign is taken for pressure increasing and the negative sign for pressure decreasing. The differential form of the pressure-temperature relation is then

$$\left(\frac{n-1}{n}\right)\frac{dp}{p} = \frac{dT}{T} \tag{10}$$

It then follows that F_d/ρ may be written in several alternative ways which are listed below.

$$\frac{F_d}{\rho} = T\frac{\partial s}{\partial x} \tag{8}$$

$$= \frac{\gamma}{\gamma - 1}(1 - \eta^{\pm 1})R\frac{\partial T}{\partial x} \tag{8a}$$

$$= \frac{\gamma - n}{n(\gamma - 1)}\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{8b}$$

$$= -\frac{c_p T}{Q}\frac{\partial Q}{\partial x} \tag{8c}$$

$$= RT\frac{W_z}{W}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l} \tag{8d}$$

$$= kW^2 \tag{8e}$$

(8*a*) is essentially in the form most readily applicable to the Marsh equations (see M36). (8*c*) relates the dissipation force to Novak's Q function, $Q = e^{-s/c_p}$ (see N 6). (8*d*) relates the dissipation force to Smith's total pressure loss coefficient,

$$\bar{\omega}_z = -\frac{l}{P - p}\left[\frac{DP}{Dz} - \frac{DP_{\text{ideal}}}{Dz}\right]$$

where l is the axial length of a blade chord and P is the relative stagnation pressure. (The subscript "ideal" relates to an isentropic change in stagnation pressure, occurring as a result of a change in radius.)

Modifications of the S, M, N Equations

Smith. Smith has first used the inviscid momentum equation in the derivation of the right-hand side of his equation (e.g., S.29 or S.31). We follow here Smith's first approach of eliminating $\frac{DW_m}{Dm}$ from the radial momentum equation (S12), but using the modified form of the streamwise momentum equation, (1), instead of the inviscid form. Using also the Gibbs relation (4) we obtain for irreversible adiabatic flow two terms which replace the term

$$-\frac{W_r}{(1 - M_m^2)}\frac{Q'}{c_p T}$$

in Smith's equation (S.29), where Q' is the heat added per unit mass per unit time.

These two terms are

$$\left(\frac{W_r}{1 - M_m^2}\right)\left[-\frac{F_d}{\rho}\frac{W}{a^2} - \frac{W}{c_p}\frac{Ds}{Dx}\right]$$

$$= -\left(\frac{W_r}{1 - M_m^2}\right)\left[\frac{WT}{a^2}\frac{Ds}{Dx} + \frac{W}{c_p}\frac{Ds}{Dx}\right]$$

$$= -\left(\frac{W_r}{1 - M_m^2}\right)\left[\frac{W}{R}\frac{Ds}{Dx}\right]$$

$$= -\frac{W_r W_z}{(1 - M_m^2)}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l} \tag{9}$$

The final form of equation (9) is identical to (S.61), obtained by Smith in his second approach using the *energy* equation to eliminate $\frac{DW_m}{Dm}$. This is to be expected, since the modified momentum equation, when multiplied by the scalar of the velocity w, is identically the same as the energy equation used by Smith.

The modifications to equation (S.29) required for its use in irreversible adiabatic flow are, therefore

(1) The replacement of the term $\frac{-W_r Q'}{(1 - M_m^2)c_p T}$ by

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$$-\frac{W_r W_z}{(1 - M_m^2)}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l}$$ as suggested by Smith (S.61), and

(2) an additional dissipative body force term $\left(\frac{F_r}{\rho}\right)$ which should be placed on the lefthand side of the radial momentum equation (S.2) and should carry through to (S.29).

If it is logical to assume that the main dissipation force is in the x or streamwise direction, but opposing the motion, then

$$F_r = -F_d\left(\frac{W_r}{W}\right) \tag{10}$$

From Smith's Fig. 2, reproduced herein, it follows that

$$\frac{W_r}{W_z} = \tan\phi \tag{11a}$$

$$W = (W_r^2 + W_z^2\sec^2\beta)^{1/2}$$
$$= W_z(\tan^2\phi + \sec^2\beta)^{1/2} \tag{11b}$$

so that

$$\frac{W_r}{W} = \frac{1}{(1 + \sec^2\beta\cot^2\phi)^{1/2}} \tag{11c}$$

where β is the flow angle in the (z, θ) plane and ϕ is the flow angle in the (z, r) plane.

β, ϕ are related to the blade angle (β') in the (z, θ) plane by $\tan\beta' = \tan\beta + \tan\phi\tan\epsilon$, where ϵ is the blade lean, the angle between the blade line and a radial line in the (r, θ) plane.

From (10) and (11c) it follows that

$$\begin{aligned}\frac{F_r}{\rho} &= -\frac{F_d}{\rho}\,\frac{1}{(1 + \sec^2\beta\cot^2\phi)^{1/2}} \\ &= -T\,\frac{\partial s}{\partial x}\,\frac{1}{(1 + \sec^2\beta\cot^2\phi)^{1/2}}\end{aligned} \tag{12}$$

from equation (8). Alternatively the other forms (8a–8d) could have been used to eliminate F_d/ρ.

It is probably more convenient to add a term to the righthand side of (S.29). The equation, as modified by (S.61) and including an additional term for F_r/ρ becomes

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial r} &= \left(\frac{1 - M_z^2}{1 - M_m^2}\right)\left(\frac{C_u^2}{r} + \frac{W_m^2\sec\phi}{r_m}\right) \\ &\quad + \left(\frac{W_r}{1 - M_m^2}\right)\left[W_z\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tan\phi) + \frac{1}{r}\frac{\partial}{\partial\theta}(\tan\beta)\right) + \frac{W_u}{\rho a^2 r}\frac{\partial p}{\partial\theta}\right] \\ &\quad - \frac{W_r W_z}{(1 - M_m^2)}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l} \\ &\quad - RT\,\frac{W_z}{W}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l}\,\frac{1}{(1 + \sec^2\beta\cos^2\phi)^{1/2}}\end{aligned} \tag{13}$$

The last two terms may be written

$$\begin{aligned}&-\frac{W_r W_z}{(1 - M_m^2)}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l}\left[1 + \frac{RT}{WW_r}\,\frac{(1 - M_m^2)}{(1 + \sec^2\beta\cot^2\phi)}\right] \\ &= -\frac{W_r W_z}{(1 - M_m^2)}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l}\left[1 + \frac{(1 - M_m^2)}{\gamma M^2}\right]\end{aligned} \tag{14}$$

where $M = W/(\gamma RT)^{1/2}$ is the relative Mach number.

Solution of (14) should proceed in the normal way with iteration on the righthand side of the equation. (But see also Appendix 2.)

Novak. Novak prefers the differential equation for C_m^2, and writes it outside the blade row. While it is arguable that streamwise variation in loss takes place within the blade rows and not within the clearance space, it would seem more logical to spread the whole loss out through the machine, using a polytropic $\left(\frac{p}{\rho^n} = \text{constant}\right)$ relation or a local efficiency. If this is accepted then Novak's statement of the Crocco equation (N.1) should be modified to

$$\nabla H - T\nabla s - \frac{\bar{F}}{\rho} = \bar{C}\times(\nabla\times\bar{C}) \tag{15}$$

(See Horlock [5].) His equation (11b) for C_m^2 then becomes

$$\begin{aligned}\frac{\partial C_m^2}{\partial r} &+ 2C_m^2\left[-\frac{\sin\phi}{C_m}\frac{\partial C_m}{\partial m} + \frac{\cos\phi}{r_m} + \frac{1}{2Q}\frac{\partial Q}{\partial r}\right] \\ &= 2\left[\frac{1}{Q}\frac{\partial}{\partial r}(HQ) - \frac{C_\theta}{r}\frac{\partial}{\partial r}(rC_\theta) - \frac{C_\theta^2}{2Q}\frac{\partial Q}{\partial r}\right] \\ &\quad - \frac{2c_pT}{Q}\frac{\partial Q}{\partial x}\,\frac{1}{(1 + \sec^2\beta\cot^2\phi)^{1/2}}\end{aligned} \tag{16}$$

using the equations (8c) and (12) in (15), and the notation of Fig. 1 for the angle β. If Novak's notation for angles is used, then the last term may be written

$$-\frac{2c_pT}{Q}\frac{\partial Q}{\partial x}\sin\phi\sin\bar{\beta}$$

where $\bar{\beta}$ is the angle between the velocity vector and the tangential direction,

$$\text{i.e.,}\quad \bar{\beta} = \tan^{-1}\left(\frac{W_m}{W_\theta}\right) = \tan^{-1}\left(\frac{W_x}{W_\theta\cos\phi}\right) = \tan^{-1}(\sec\phi\cot\beta)$$

The last term has been put in the form probably most convenient for solution of Novak's equation. Since Novak has not used the energy equation in the derivation of (15) no further modification is necessary. Note also, however, that the term $\frac{\partial Q}{\partial x}$ may be written

$$\frac{\partial Q}{\partial x} = \frac{C_m}{C}\frac{\partial Q}{\partial m} = \left[\frac{\operatorname{cosec}\phi}{1 + \sec^2\beta\cot^2\phi}\right]\frac{\partial Q}{\partial m} = \sin\bar{\beta}\,\frac{\partial Q}{\partial m}$$

Marsh. Marsh uses the equation (M.24) within the blade row

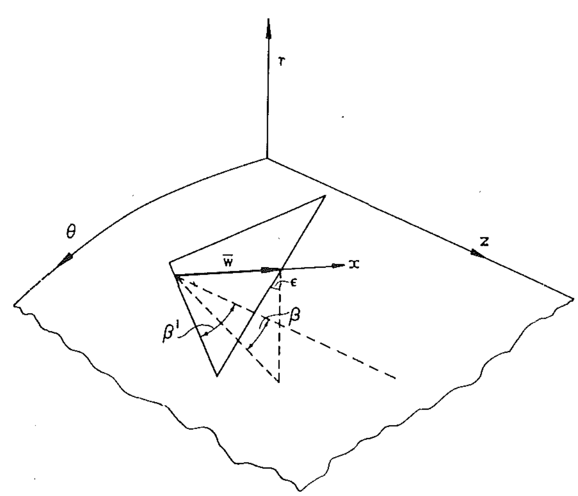

Fig. 1 Smith's coordinate system

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so there is a good argument for including the F_r term in the equation of motion.

Marsh already includes a "blade" body force term in his equation (M.24), determining this from his statement

$$F_b = -\frac{1}{\rho r n_u} \frac{\partial p}{\partial \phi} \bar{n}$$

and obtaining F_b from the second (tangential) momentum equation in inviscid flow (M.15). (Note that ϕ is now the circumferential coordinate.)

It is suggested here that the radial body force F_r in Marsh's equation (M.24) should be interpreted as the sum of a radial blade force F_{r_b} determined previously, plus a dissipative force F_{r_d},

$$\begin{aligned} F_r &= F_{r_b} + F_{r_d} \\ &= F_r - F_d \left(\frac{W_r}{W}\right) \\ &= -\frac{1}{\rho r} \frac{n_r}{n_u} \frac{\partial p}{\partial \phi} - \frac{\gamma R \rho}{(\gamma - 1)} (1 - \eta^{\pm 1}) \frac{\partial T}{\partial x} \left(\frac{W_r}{W}\right) \end{aligned} \quad (17)$$

The dissipative term has been written here in the form which is presumably most easily inserted into Marsh's program. Marsh has not used the streamwise momentum equation in deriving (M.24) so no further modification appears necessary.

Equation (17) is correct since F_{r_b} has been interpreted as

$$-\frac{1}{\rho} \frac{\partial p}{\partial \phi} \frac{n_r}{n_u} = -\frac{1}{\rho} \frac{\partial p}{\partial \phi} \tan \lambda$$

where λ is now the blade lean. If it is required to eliminate the tangential pressure gradient in terms of the rate of change of tangential momentum, then the effect of viscous action in the second (tangential) momentum equation should also be considered (see Appendix 2). However, both λ and $\left(\frac{W_r}{W}\right)$ are usually small so that F_r should in general also be small.

Example of Introduction of Dissipative Force

To the author's knowledge, two streamline curvature programs for axial flow turbomachines (Wilkinson [6], Shaalan, and Daneshyar [7]) are being prepared with inclusion of a viscous dissipation term, along the lines described previously. The effects are likely to be small unless radial variations in loss are large, and the inclination of the steamline to the radial direction is large. In an attempt to assess the importance of the inclusion of the dissipative force F_{r_d}, a particular example is considered here, the irreversible flow through a vaneless diffuser for a centrifugal compressor (Fig. 2). (In the notation of Fig. 1, this is the limiting case of $\beta \to 90$ deg, with the blade "lean" ϵ zero. The axial velocity vanishes, and the projections of the streamlines on the meridional plane are radial, i.e., $\phi = 90$ deg.) Analytical solutions are obtained, with and without the force F_{r_d}, the inconsistency of neglecting F_{r_d} being illustrated together with the magnitude of the error involved.

Flow With Dissipative Forces Included. The flow is supposedly two dimensional in that there are no variations in the z direction (although velocity variations with z must exist associated with the viscous dissipation); it is also axisymmetric. The fluid moves spirally outward and its movement is opposed by a frictional force given by (8c)

$$F_d = kC^2 \quad (18)$$

i.e., it is similar to a pipe flow with $k = \frac{2f}{D}$, where f is a friction coefficient $\left(f = \frac{\tau}{{}^1/_2 \rho C^2}\text{, where } \tau \text{ is a wall shear stress}\right)$ and D is a hydraulic diameter. For flow between parallel walls, distance h

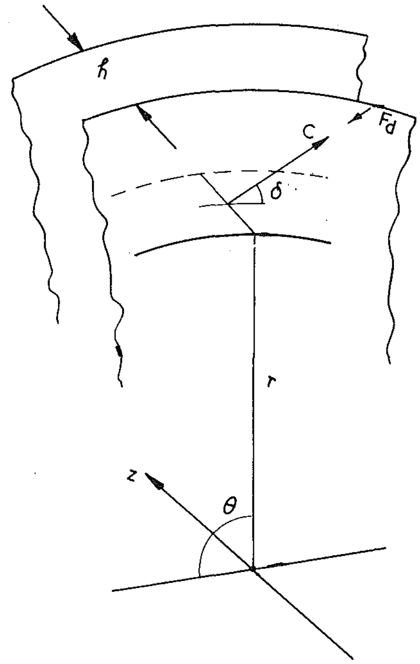

Fig. 2 Flow in vaneless diffuser

apart along the z axis, $D = 2h$ so that

$$F_d = \frac{fC^2}{h} \quad (18a)$$

Using Smith's notation, the Crocco equations become for incompressible flow,

$$\frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{1}{\rho} \frac{dP}{dr} = \frac{C_u}{r} \frac{d}{dr} (rC_u) + \frac{F_r}{\rho} \quad (19)$$

$$\frac{1}{\rho} \frac{\partial P}{\partial \theta} = 0 = \frac{F_u}{\rho} - \frac{C_r}{r} \frac{\partial}{\partial r} (rC_u) \quad (20)$$

where $C_u = C \cos \delta$ and $C_r = C \sin \delta$, if the streamline is at an angle δ to the tangential direction.

Along the streamline

$$-\frac{1}{\rho} \frac{\partial P}{\partial x} = -\frac{1}{\rho} \frac{dP}{dr} \sin \delta = \frac{F_d}{\rho} \quad (21)$$

and

$$F_u = -F_d \cos \delta = -\frac{fC^2}{h} \cos \delta \quad (22)$$

$$F_r = -F_d \sin \delta = -\frac{fC^2}{h} \sin \delta \quad (23)$$

Equations (19)–(23) may be shown to be selfconsistent. From (19) and (20)

$$\frac{1}{\rho} \frac{dP}{dr} = \frac{C_u}{C_r} \frac{F_u}{\rho} + \frac{F_r}{\rho}$$

and from (22) and (23)

$$\frac{1}{\rho} \frac{dP}{dr} = \frac{1}{\tan \delta} \left(-\frac{fC^2}{h} \cos \delta\right) - \frac{fC^2}{h} \sin \delta = -\frac{fC^2}{h \sin \delta}$$

which is identical with (21). Thus it is not necessary to use all three equations (19)–(21)—two are sufficient.

From the continuity equation, since the density is constant,

$$rC_r = rC \sin \delta = \text{constant } (K) \quad (24)$$

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Hence, from (20), (22), and (24)

$$F_u = -\frac{f}{h}\frac{K^2 \cos\delta}{r^2 \sin^2\delta} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(\frac{K}{\tan\delta}\right) = -\frac{K^2}{r^2 \sin^2\delta}\frac{d\delta}{dr}$$

i.e.

$$\frac{d\delta}{dr} = \frac{f}{h}\cos\delta \tag{25}$$

Integration of this equation yields

$$\frac{\sec\delta + \tan\delta}{\sec\delta_1 + \tan\delta_1} = e^{(f/h)(r-r_1)} \tag{26}$$

if $\delta = \delta_1$ at $r = r_1$

Flow With Dissipation Forces Neglected. If the dissipative forces are not included in the momentum equations but the stagnation pressure is still supposed to change along a streamline (essentially as in the S, M, N analyses) then the (inconsistent) equations governing the motion become

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{C_u}{r}\frac{d}{dr}(rC_u) \tag{19a}$$

$$0 = C_r \frac{d}{dr}(rC_u) \tag{20a}$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{1}{\rho}\frac{dP}{dr}\sin\delta = \frac{fC^2}{h} \tag{21a}$$

$$rC_r = K \tag{24a}$$

Equations (19a) and (21a) illustrate the inconsistency in a rather extreme way. (19a) results from the inviscid, radial equation of motion (Novak's equation (N.9) simplifies to this form with $\phi = 90$ deg, and Smith's equation (S.29), with $\frac{\partial}{\partial t} = \frac{\partial}{\partial \theta} = 0$ and $\phi = 90$ deg, may also be written this way, eliminating p in favor of P). However, tracing the loss along the streamlines leads to (21a), which is inconsistent with (19a) and (20a).

S and N type solutions would presumably involve the use of (20a) and (24a), which would yield the inviscid result, rC_u = constant, rC_r = constant, $\delta = \tan^{-1}\left(\frac{C_r}{C_u}\right)$ = constant. The M solution would probably involve Marsh's alternative equation for the stream function (M.25) (obtained by substituting the continuity equation into the z momentum equation) and would also yield δ = constant. In all three cases, it would be logical to use (21a) for the change of stagnation pressure, and ignore the (incorrect) radial momentum equation (19a).

A third solution is obtained by looking at the flow in the (r, θ) plane, ignoring the (incorrect) tangential momentum equation (20a), and using (19a), (21a), and (24a). This is somewhat similar to the approach of Smith and Frost [8] in obtaining solutions in the Wu "S1" plane, in which the inviscid momentum equation across the streamlines is used with the continuity equation. It then follows that

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{K}{r^2 \tan\delta}\frac{d}{dr}\left(\frac{K}{\tan\delta}\right) = -\frac{f}{h\sin\delta}\left(\frac{K^2}{r^2\sin\delta}\right) \tag{27}$$

i.e.

$$\frac{d\delta}{dr} = \frac{f}{h\cos\delta}$$

Integration of this equation gives

$$\sin\delta - \sin\delta_1 = \frac{f}{h}(r - r_1) \tag{28}$$

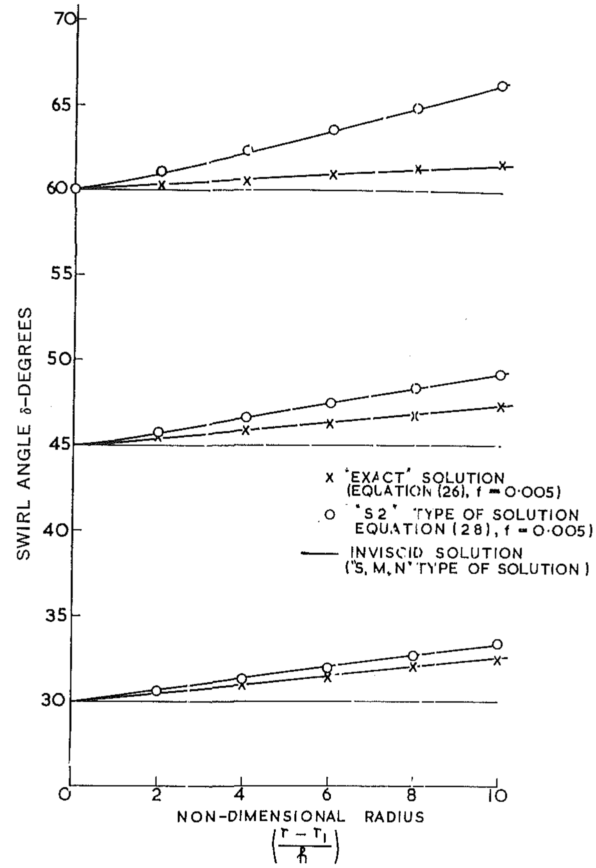

Fig. 3 Swirl angle in vaneless diffuser

Calculations. A comparison of streamline angles δ, calculated with and without the radial dissipative force, is shown in Fig. 3. δ is plotted against $(r - r_1)/h$ for $f = 0.005$ and for three starting values of δ, δ = 30 deg, 45 deg, and 60 deg, using the first "exact" solution (26), the second ("S, M, N") solution, δ = constant (also the inviscid solution) and the third solution (28).

Inclusion of the radial force term increases the flow angle with radius, in comparison with the equiangular spiral flow of the second "S, M, N" solution (or the inviscid solution). However, the third solution predicts an even greater change of flow angle with radius.

However, for the small values of loss taken, corresponding to a turbulent "flat plate" flow, the effects are small. For a flow with much larger local dissipation, such as occurs with local stalling in an axial turbomachine, the effects would be substantially bigger.

Conclusions. Equations given by Smith, Marsh, and Novak have been modified by the addition of a dissipative radial body force, to make the equations consistent with the assumption of varying entropy through the machine. The body force term may be expressed in terms of an assumed entropy change along the streamline (for Novak's equation) of an assumed efficiency and temperature change along the streamline (for Marsh's equation) or of an assumed total pressure loss coefficient along the streamlines (for Smith's equation).

It has been shown that the effect of this additional body force term is small when losses are small, i.e., near the design point operation of a turbomachine. However, when losses are large (i.e., at "off-design" operation) the flow may develop large radial components of velocity, and it is then that the "dissipative radial force" term may become significant.

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APPENDIX 1

Notation	Smith	Marsh	Novak
Speed of sound	a	c	a
Specific heat at constant pressure	c_p	c_p	c_p
Absolute velocity	$\bar{C}$	...	$\bar{C}$
Stagnation enthalpy	...	H	H
Static enthalpy	h	h	...
Stagnation rothalpy	I	I	I
Axial chord length	l	...	...
Distance in meridional direction	m	...	m
Relative Mach number	M	M	...
Absolute Mach number	...	...	M
Static pressure	p	p	P
Relative stagnation pressure	P	...	...
Heat added per unit mass per unit time	Q'	...	...
Radial coordinate	r	r	r
Radius of curvature of meridional stream line	r_m	...	r_m
Gas constant	R	R	...
Entropy	s	s	S
Time	t	t	t
Temperature	T	T	T
Blade speed	$\bar{U}$	$\bar{U}$	...
Relative velocity	$\bar{W}$	$\bar{W}$	$\bar{W}$
Distance in instantaneous flow direction	x	...	s
Axial coordinate	z	z	z
Absolute swirl angle	$\alpha = \tan^{-1} \frac{C_\theta}{C_z}$	...	$\bar{\alpha} = \tan^{-1} \left(\frac{C_z}{C_\theta}\right)$
Relative swirl angle	$\beta = \tan^{-1} \left(\frac{W_\theta}{W_z}\right)$	μ	$\bar{\beta} = \tan^{-1} \left(\frac{W_z}{W_\theta}\right)$
Blade angle	β'	...	...
Blade lean angle	ϵ	λ	...
Circumferential coordinate	θ(angle)	ϕ(angle)	θ(angle)
Loss coefficient	ω_z	...	...
Density	ρ	ρ	ρ
Meridional angle	ϕ	...	ϕ
Rotor angular velocity	$\bar{\omega}$	$\bar{\omega}$	...
Ratio of specific heats	...	γ	γ
Entropy function	...	...	$Q = e^{-s/c}$
Force vector	...	$\bar{F}$	...
Vector normal to stream surface	...	$\bar{n}$	...
Polytropic efficiency	...	η	...
Subscripts			
Meridional component	m	...	m
Radial component	r	r	r
Circumferential component	u	u	θ
Axial component	z	z	z
Streamline direction	x	...	s
Additional Subscripts			
Blade force	b		
Dissipative force	d		

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APPENDIX 2

The Effect of Viscosity in the Tangential Equations of Motion. An expression has been derived (17) for the radial force to be inserted in Marsh's equation (M.24). The same expression should be used if working from Smith's axisymmetric form of the radial equilibrium equation (S.51).

This appendix makes the rather fine point that the second, or tangential equation of motion, should include a dissipative force, i.e.

$$F_u = F_{u_b} + F_{u_d}$$

$$= \frac{W}{r}\frac{\partial}{\partial x}(rC_u) \qquad (18)$$

where

$$F_{u_b} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

(see Marsh (M.15) or Smith (S.26), but following Smith's notation)

and $$F_{u_d} = -F_d \frac{W_u}{W}$$

The radial body force is

$$F_r = F_{r_b} + F_{r_d}$$

$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}\frac{n_r}{n_u} - F_d\frac{W_r}{W} \qquad (19)$$

where n_r and n_u are the radial and tangential normals to the blade surface.

From (18) and (19),

$$F_r = \left\{\frac{W}{r}\frac{\partial}{\partial x}(rC_u) + F_d\frac{W_u}{W}\right\}\delta \tan\epsilon - F_d\frac{W_r}{W}$$

$$= \frac{W}{r}\frac{\partial}{\partial x}(rC_u)\tan\epsilon + \frac{F_d}{W}(W_u \tan\epsilon - W_r) \qquad (20)$$

But

$$\left.\begin{aligned} W_r &= W_z \tan\phi \\ W_u &= W_z \tan\beta \\ W_z &= W_m \cos\phi \\ W &= W_z(\tan^2\phi + \sec^2\beta)^{1/2} \end{aligned}\right\} \qquad (21)$$

Hence

$$\frac{W_u}{W}\tan\epsilon = \tan\epsilon\left(\frac{W_r}{W_z}\right)\left(\frac{W_z}{W}\right)$$

$$= \frac{\tan\epsilon\tan\beta}{(\tan^2\phi + \sec^2\beta)^{1/2}} \qquad (22)$$

$$\frac{W_r}{W} = \left(\frac{W_r}{W_z}\right)\left(\frac{W_z}{W}\right)$$

$$= \frac{\tan\phi}{(\tan^2\phi + \sec^2\beta)^{1/2}}$$

and from (20)

$$F_r = \frac{W}{r}\tan\epsilon\frac{\partial}{\partial x}(rC_u) + T\frac{\partial s}{\partial x}\frac{(\tan\epsilon\tan\beta - \tan\phi)}{(\tan^2\phi + \sec^2\beta)^{1/2}}$$

This equation is an alternative to equation (17). If $\epsilon = 0$, there is no blade "lean," so $F_{r_b} = 0$ as there is no radial component of the pressure forces on the blade. If, on the other hand, there is no dissipation then $T\frac{\partial s}{\partial x} = 0$ and

$$F_r = F_r = \frac{W}{r}\tan\epsilon\frac{\partial}{\partial x}(rC_u), \qquad (25)$$

Strictly the tangential force F_u appearing in Smith's (S.51) should be interpreted as

$$\frac{W}{r}\frac{\partial}{\partial x}(rC_u) - F_d\frac{W_u}{W}$$

$$= \frac{W}{r}\frac{\partial}{\partial x}(rC_u) - T\frac{\partial s}{\partial x}\frac{\tan\beta}{(\tan^2\phi + \sec^2\beta)^{1/2}} \qquad (26)$$

The term $\frac{\partial s}{\partial x}$ in (26) and in (24) may be replaced by the alternative forms given in equation (8) of the main text.

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