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LETTER TO THE EDITOR

Formation of soliton trains in Bose–Einstein condensates by temporal Talbot effect

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Abstract
We study the formation of matter-wave soliton trains in Bose–Einstein condensates confined in a box-like potential. We find that the generation of 'real' solitons understood as multipeak structures undergoing elastic collisions is possible if the condensate is released from the box into the harmonic trap only within well-defined time intervals. When the box-like potential is switched off outside the existing time windows, the number of peaks in a train changes resembling missing solitons observed in recent experiment (Strecker *et al* 2002 *Nature* **417** 150). Our findings indicate that a new way of generating soliton trains in condensates through the temporal, matter-wave Talbot effect is possible.

The notion of soliton belongs to the most popular concepts in physics. Solitons are formed because of the existence of nonlinear interaction in the system which cancels the dispersion and hence allows for the propagation of shape preserving objects. In the case of dilute atomic quantum gases, the nonlinearity is determined by the effective interaction between atoms that can be both repulsive or attractive. For repulsive uniform condensates, the appropriate nonlinear equation (i.e. the Gross–Pitaevskii equation) predicts dark soliton (a hole in the density associated with a phase jump) as a solution [1]. Such excitations of Bose–Einstein condensates have already been observed in experiments with trapped alkali atoms [2].

Generating bright solitons in atomic quantum gases is a more difficult task because it requires working with attractive condensates. In such samples, the number of atoms is limited and small (see [3]). This obstacle was overcome in two ways. In the first one, large repulsive condensate is formed and then the interactions are changed from repulsive to attractive by using the Feshbach resonance technique [4, 5], whereas in the second attempt [6] the optical lattice and the notion of negative effective mass are utilized.

In this letter, we study the influence of the box-like confinement, as used in the experimental set-up of [4], on the production of bright soliton trains in attractive condensate

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of ^{7}Li atoms. Existing theoretical analysis of experiment of [4] involves quantum phase fluctuations [7] or modulational instability [8, 9] as the main mechanism leading to the observed structures. None of these papers, nor the experimental one, however, investigates the role of the box-like potential. In our letter, we show that a regime exists where multiple scattering of gas from the walls of the box might result in the formation of real solitons, i.e. multipeak structures that undergo elastic collisions. We suggest a new version of the experiment and specify physical conditions supporting the formation of 'genuine' soliton trains. The underlying physics of our proposition can be understood with the help of the temporal Talbot effect, already observed experimentally in the context of Bose–Einstein condensate of sodium atoms diffracted by the pulsed grating [10].

We describe the Bose–Einstein condensate, as the authors of [8], with the time-dependent dissipative Gross–Pitaevskii equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{tr}} + g|\psi|^2 - i\gamma|\psi|^4\right)\psi, \tag{1}$$

where $\psi(\mathbf{r}, t)$ is the macroscopic wavefunction of Bose–Einstein condensate of ^{7}Li atoms, $V_{\mathrm{tr}}(z, \rho) = m\left(\omega_z^2 z^2 + \omega_\perp^2 \rho^2\right)/2 + V_{\mathrm{box}}$ is the axially symmetric trapping potential, $g = 4\pi\hbar^2 a_s/m$ (a_s being the scattering length that determines the strength of the interaction). The number of atoms equals $N = 10^4$, the geometry of the harmonic confinement is given by $\omega_z = 2\pi \times 4$ Hz and $\omega_\perp = 2\pi \times 800$ Hz and the box-like potential V_{box} (the 'end caps') of length L is positioned symmetrically with respect to the centre of the harmonic trap. The initial value of the scattering length of ^{7}Li condensate prepared in the end caps is positive and equals $100a_0$ (a_0—Bohr radius). Then the scattering length is changed within approximately 10 ms to its final value $a_s = -3a_0$ and the condensate is additionally kept in the box-like potential for a certain time. Finally, the end caps are turned off.

The imaginary term in equation (1) describes the losses due to three-body recombination processes [11, 8]. Since such losses were not investigated experimentally, we follow the references just mentioned and put $\gamma = 2.05 \times 10^{-26}$ cm^6 s^{-1}. In our simulation the dissipative term is turned on, according to the observation in [4], when the interaction strength becomes negative and then the losses are kept constant. Decreasing the dissipative term leads to the condensate collapse. Hence the dissipation plays a crucial role in stabilizing the system. Certainly, it has an influence on dynamics of the collapse studied recently in [9]. However, the authors of [9] ignore the losses term completely.

Our calculations show that after switching off the box-like potential, the bosonic cloud starts oscillating in harmonic trap. The cloud actually breaks into several peaks which propagate in the potential for many oscillatory cycles. However, depending on the time the box-like potential is removed, we observe qualitatively different response of the system (see figure 1). First, we discover the existence of *time windows*, i.e. the time intervals for removing the end caps that support the generation of train of real solitons (multipeak structures with conserved number of peaks that undergo only elastic collisions). It is illustrated in figure 1(a), where the condensate density is imaged at various times showing, however, always five distinguishable peaks. Here, the end caps are separated by $L = 4.0$ osc. units (75.6 μm) and are off 61 ms after the interaction strength is changed. There is a waiting time equal to the nonlinear time scale $T_{\mathrm{non}} = \hbar/(gN/V)$ (several milliseconds in our case) after which the well separated peaks in the density profile are formed for the first time. Calculations also show the existence of further time windows, especially for shorter length of the box-like potential. The duration of the time window is about 3 ms and increases when the size of the box gets larger. Only positioning the box-like potential symmetrically with respect to the centre of the harmonic trap allows for having the same number of peaks during the evolution. In contrast,

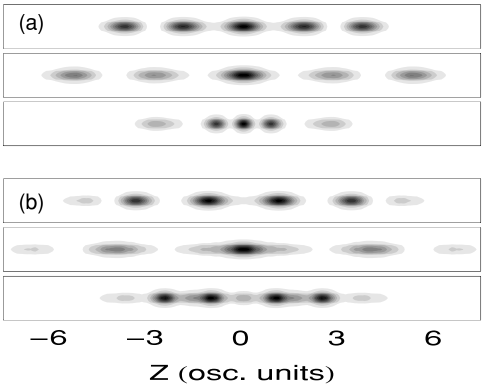

Figure 1. Illustration of the importance of a time when the end caps are switched off. Frame (a) shows the condensate density at 30 ms, 66 ms and 108 ms after the end caps are off, i.e. 61 ms after the interaction strength is changed. This is the case of the time window when the structure consists of the same number of peaks. As opposed, frame (b) is an illustration of what we call the missing soliton structure. Here, the number of peaks changes during the evolution. Successive snapshots correspond to 30 ms, 66 ms and 139 ms after the end caps are turned off outside the time window (52 ms after the change of the interaction strength).

when the end caps are off outside the time window, the number of peaks changes (frame (b) in figure 1) during the evolution. Certain peaks disappear and reappear later. Such structures cannot be, in fact, considered as solitons.

It is important to understand the role of the location of the box-like potential. We have checked numerically that the time windows persist even if the end caps are shifted from the centre of the harmonic potential by several per cent, depending on the size of the box. However, we do not see the time windows when the size of the box is too large nor when the box is positioned at the slope of the harmonic potential (as in the experiment of [4]). It is clear that the discussed phenomenon is somehow modified by the presence of the harmonic trap. As will be explained later, the disappearance of time windows is caused by the loss of Talbot-type recurrence due to nonuniform shift of energy levels of the box potential forced by the harmonic trap. Certainly, the most favourable conditions for the existence of time windows are those when there is no axial harmonic confinement. Such calculations have already been performed, see figure 1 in [8]. However, the authors of [8] have not investigated the influence of the delay time between the removal of the end caps and switching of the scattering length to the negative value. In figure 2, we plot the axial density profiles (frames (b) and (c)) in the case when the box-like potential is turned off within the time window. The number of peaks equals 8 not 4 nor 5 as might be suggested by the density profile at the time when the end caps are removed (frame (b)). Surprisingly, this number coincides with the number of distinguishable peaks in momentum distribution at the time when the condensate is released (frame (a)).

Figure 2 also shows that the observed peaks are indeed solitons. We compare here the longitudinal shapes of the peaks with the analytical solitonic solutions obtained by Zakharov [1] and given by $(1/\sqrt{|g_{\rm eff}|}a/\cosh[a(z-z_0)])^2$, where parameters a and z_0 determine the height and the width and the position of the centre of the soliton, respectively. The Zakharov solution of the Gross–Pitaevskii equation is given only in one-dimensional space. However, assuming that the three-dimensional solution can be factorized, i.e. written as a product of functions depending on single variables, one can reduce (after integration over radial

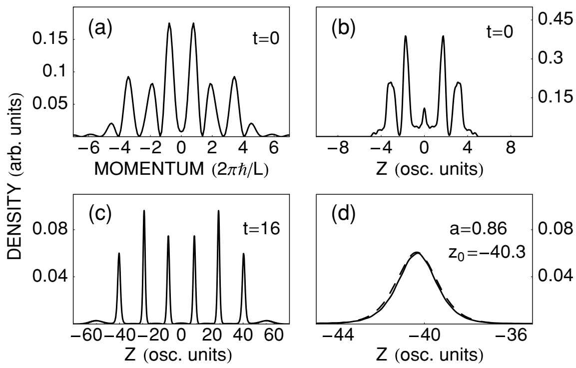

Figure 2. Momentum (frame (a)) and spatial but radially integrated (frames (b) and (c)) density profiles of a condensate of 10^4 ^{7}Li atoms during the evolution under no axial confinement. The size of the box equals 189 μm and the end caps are turned off within the time window (314 ms after the scattering length is changed). Time is measured beginning from the instant when the end caps are removed (time unit is equal to 39.8 ms). Frame (d) is a comparison between numerical calculations and the theory [1]. It shows a density profile of the second extreme peak going to the left at $t = 16$ accompanied by the fit (dashed line) according to the analytical Zakharov solution (see the text for the details).

directions) the three-dimensional Gross–Pitaevskii equation to the one-dimensional equation with the effective value of the interaction strength $g_{\rm eff}$. Indeed, there exists a single value of $g_{\rm eff}$ allowing for excellent fits with the help of a and z_0 parameters only (see figure 2(d)). By using the virial theorem $2E_{\rm kin} + E_{\rm int} = 0$ [3] one finds for the Zakharov solution $\hbar^2/6m\tilde{z}^2 - |g_{\rm eff}|N/6\tilde{z} = 0$, where $\tilde{z}$ $(=1/a)$ is the soliton width and N is the number of atoms within the peak. Therefore, $N = 2\hbar^2/m\tilde{z}g_{\rm eff}$ and equals 526 in the case of figure 2(d) which remains in good agreement with direct integration of the density distribution that gives 532 atoms. Certainly, peaks visible in figure 2(c) are solitons.

In the following, we shall indicate that the origin of time windows is due to the temporal Talbot effect well known in the linear case. This effect can also survive a weak nonlinearity as shown in the experiment of [10]. Let us consider first the linear dynamics of the symmetric wave packet in a finite depth rectangular one-dimensional box. The walls of the box are very high and therefore we will expand the initial wave packet on the basis of the symmetric eigenstates of infinitely deep well potential that are $\sqrt{2/L}\cos(k_n z)$ with $k_n = \frac{\pi}{L}(2n+1)$, where $n = 0, 1, \ldots$ and L is the length of the well. Consequently, the wavefunctions are set to zero in the classically forbidden region. The Fourier transform of the wave packet reads

$$\tilde{\psi}(k,t) = \sqrt{\frac{L}{2}} \sum_n \alpha_n \,\mathrm{e}^{-\mathrm{i}E_0(2n+1)^2 t/\hbar} \left(\frac{\sin\frac{L}{2}(k-k_n)}{\frac{L}{2}(k-k_n)} + \frac{\sin\frac{L}{2}(k+k_n)}{\frac{L}{2}(k+k_n)} \right), \qquad (2)$$

where $E_0 = \hbar^2\pi^2/(2mL^2)$ and $\{\alpha_n\}$ are the coefficients of the expansion of $\psi(z,0)$. According to the above formula the momentum distribution is recovered after a period of $T_{\rm rev} = \pi/4$ (in units of $\hbar/E_0$) and this is the result of the Talbot-type recurrence originating from the particular form of the phase factor ('$n(n+1)$' dependence on the quantum number n). However, this formula exhibits more structure. At time $T_{\rm win}^{\rm lin} = \pi/8$ the resulting momentum distribution forms a set of fully separated groups centred at momenta $k = k_n$. This can be verified (e.g. numerically) taking into account several facts: (1) the form of the phase factor in equation (2); (2) the localization of the function $\sin(k)/k$ around $k = 0$; (3) assumed weak dependence of coefficients α_n on n. An example is given in figure 3, frame (b), where we took initially

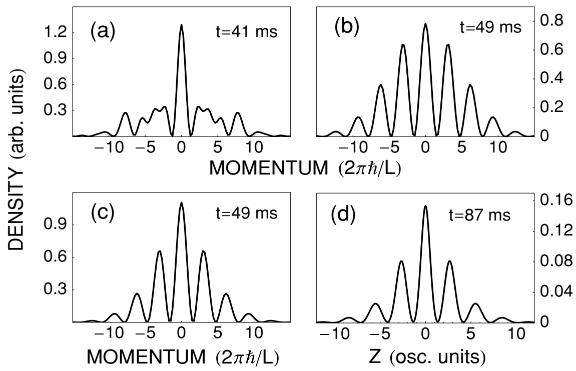

Figure 3. Illustration of a temporal Talbot effect. Frames (a) and (b) show the one-dimensional momentum density in the linear case just before the appearance of the time window and within it, respectively. Frame (c) proves that the time window survives when a small nonlinearity is present in the dynamics. Here, the time window appears almost at the same time as in the linear case. After the nonlinear time T_{non} ($\approx$40 ms) groups in momentum space are transformed into groups in position space (frame (d)).

the Gaussian profile of the wave packet. Frame (a) shows the momentum distribution a few microseconds before the first time window appears.

If the box-like potential is switched off within certain interval of time centred around $T_{\text{win}}^{\text{lin}} = \pi/8$ (50.5 ms in the case of figure 3), all groups of momenta transform into separated wave packets which simply spread during further evolution. This spreading can be stopped by including the nonlinear term in the dynamics. First, frame (c) in figure 3 proves that even under a presence of the nonlinearity the time window still exists. The nonlinear time equals $T_{\text{non}} =$ 40 ms in this case and after this time the well separated peaks appear in the density distribution (frame (d)). These peaks fit the Zakharov solitons [1], i.e. their shape is a secans-hyperbolicus one, with the same quality as those in figure 2(d). Hence, the nonlinearity is essential to built the solitons from the separated wave packets. Here, the separated wave packets originate from the temporal Talbot effect combined with the appropriate initial conditions. In a real experiment, initial conditions are formed as a result of modulational instability driven by the condensate collapse.

The presence of the box-like potential is essential since it allows the condensate wavefunction for multiple reflexions from the walls resulting in the interference pattern found above. However, 'small addition' of the harmonic potential does not spoil the picture. This can be proved by the perturbation calculation. It turns out that for not too large size of the box as compared to the characteristic length scale of the harmonic potential ($L < 5$ osc. units), all the levels but the lowest one are shifted approximately uniformly. In such a way the new time scale that is proportional to the reciprocal of the difference of the shifts of the two lowest levels (and approximately equals the trap period) can be introduced. It follows from formula (2) that only those time windows survive that occur at times shorter than the trap period.

We can also prove that the temporal Talbot effect survives under the presence of the nonlinearity. First, note that for small enough attraction the time windows appear almost exactly at the same time as in the linear case (compare figures 3(b) and (c)). This can be explained by observing that small nonlinearity uniformly shifts all single-particle energy levels. Therefore, from formula (2), one should expect the formation of groups in momentum space at the same times and of the same duration as in the linear case.

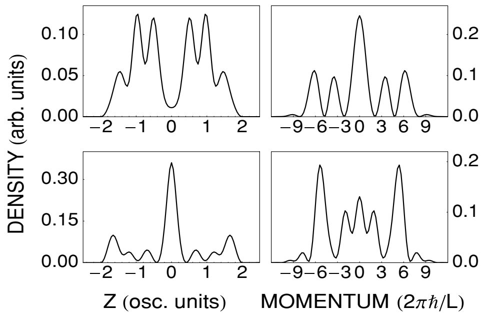

Figure 4. Spatial (left column) and momentum (right column) densities at the time when the end caps are off. The upper frames illustrate the case of the time window, i.e. when the momentum density shows well-developed groups of momenta. The lower frames correspond to the case when there are no separated peaks in a momentum space and consequently the probability flow leads to missing soliton structures.

We have checked numerically that the temporal Talbot effect persists also when the nonlinearity is larger and comparable to that present in the experiment of [4]. As it is the case, we plot in figure 4 the axial momentum densities (at the time when the end caps are off) corresponding to the frames (a) and (b) of figure 1. Figure 4 proves that the time windows survive under the conditions when both the box-like and harmonic confinements and the nonlinearity are present. When the end caps are off just within the time window (the upper frames), the momentum density shows five distinguishable peaks whereas the density in the position space consists of two broad peaks with partially developed three subpeaks. Upper frames in figure 1 confirm that afterwards five (neither two nor six) solitons are developed. It means that all momenta groups have been transformed to peaks in position space and the time needed for that is of the order of nonlinear scale T_{non}. These peaks oscillate with the trap period, collide when they meet at the centre of the trap and then reappear. Their motion is particle-like. If one considers a point-like particle moving according to the Newton equation, initially placed at the trap centre and having the initial velocity determined by the maximum value of the momenta group, the particle will follow the density peak. Hence, it is reasonable to use the name solitons for the density peaks.

When the end caps are switched off outside the time window (lower frames in figure 4) the number of density peaks changes during the evolution (see the lower part of figure 1). This is because the momentum density does not consist of well separated groups and the probability can flow from one group to the other forming the missing soliton structures. This is, in fact, the case of numerical calculations reported in [8]. Figure 5 of [8] shows that the number of peaks changes in time while the condensate is axially confined. On the other hand, when the axial confinement is off, the condensate always ends in a state with a fixed number of solitons although this happens after very long time.

In conclusion, we have investigated the role of the box-like potential in the process of generating bright solitons in the attractive Bose–Einstein condensates. Although no theory of solitons in a confined potential exists, there are some obvious properties the structures have to possess to be considered as solitons. In particular, the solitons do not disappear in consequence of collisions. We showed that to satisfy this condition the condensate needs to be kept in the box-like potential for appropriate time, until it is decomposed into separated groups in momentum space. In fact, what we propose is to utilize the Talbot effect that is well known from the linear physics. It turns out, however, that this effect survives under the condition of not too strong nonlinearity. It is not surprising since the temporal Talbot effect was already experimentally observed for a condensate of sodium atoms, although in a different situation when the condensate was diffracted by a pair of pulsed gratings and the periodicity with respect to the time delay between pulses in a series of condensate images was discovered [10].

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