On the negative-order norm accuracy of a local-structure-preserving LDG method

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Abstract

The accuracy in negative-order norms is examined for a local-structure-preserving local discontinuous Galerkin method for the Laplace equation [Li and Shu, Methods and Applications of Analysis, v13 (2006), pp.215-233]. With its distinctive feature in using harmonic polynomials as local approximating functions, this method has lower computational complexity than the standard local discontinuous Galerkin method while keeping the same order of accuracy in both the energy and the L^2 norms. In this note, numerical experiments are presented to demonstrate some accuracy loss of the method in negative-order norms.

Keywords: discontinuous Galerkin method, Laplace equation, local-structure-preserving, harmonic polynomial, negative-order norm

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1 Introduction

A local-structure-preserving (LSP) local discontinuous Galerkin (LDG) method was introduced in [12] for the Laplace equation. The method is based on the standard LDG method for the second order elliptic problems [3], and its distinctive feature is to use harmonic polynomials (polynomials which satisfy $\triangle u = 0$, the Laplace equation) to approximate the solution inside each mesh element. Using this local-structure-preserving discrete space significantly reduces the size of the final algebraic system and therefore the overall computational complexity. Meanwhile, the method keeps the same order of accuracy in both the energy and the L^2 norms as the standard LDG method (see [3, 12] and section 2). This work is among the series of developments in [6, 10, 11] to design discontinuous Galerkin (DG) methods with better cost efficiency for certain differential equations. Such efficiency is achieved by incorporating the *a priori* knowledge of the exact solutions into the choice of local approximating functions in DG formulations, and it is mainly due to the flexibility of these methods in using various local discrete spaces. Some other examples of DG methods utilizing this flexibility include [17, 8].

The objective of this note is to investigate the accuracy of the aforementioned LSP LDG method in negative-order norms. With harmonic polynomials as local approximations, the standard duality argument can not be applied to obtain the error estimates in negative-order norms for the LSP LDG method. On the other hand, negative-order norm error estimates often contain the information on the oscillatory nature of the error, which can be used to enhance the accuracy of the numerical solutions. In fact, a local post-processing technique was applied to finite element solutions of elliptic problems in [2] and to DG solutions of hyperbolic problems in [7, 15], and it filtered out the oscillation in the error and enhanced the accuracy in the L^2 norm up to the order of the error estimates in negative-order norms. The success of this technique relies only on a negative-order norm error estimate of the numerical solution and a local translation invariance of the mesh. Based on this, we apply the post-processing technique of [2, 7, 15] to the numerical solutions of our LSP LDG method.

By examining the accuracy of the post-processed solutions, we indirectly study the accuracy of the method in negative-order norms. Numerical experiments indicate that the LSP LDG method for the Laplace equation, though having lower computational complexity while keeping the same order of accuracy as the standard LDG method [3] in commonly used norms, has some accuracy loss in negative-order norms. Mathematical understanding of this result is yet to be established.

The rest of this note is organized as follows. In section 2, both the standard and the LSP LDG methods are reviewed for solving the Laplace equation. Computational complexity and error estimates in the energy and the L^2 norms are also briefly discussed. In section 3, numerical experiments are presented to indicate some accuracy loss of the LSP LDG method in negative-order norms. Concluding remarks are given in section 4.

2 Numerical methods

In this section, the standard [3] and the LSP [12] LDG methods will be reviewed for the Laplace equation

$$-\triangle u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_D} = g_D, \quad \frac{\partial u}{\partial \mathbf{n}}|_{\Gamma_N} = \mathbf{g}_N \cdot \mathbf{n}, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with $\mathbf{n}$ being the outward unit normal along the domain boundary $\bar{\Gamma} = \bar{\Gamma}_N \cup \bar{\Gamma}_D$, Γ_N and Γ_D are disjoint, and $|\Gamma_D|_{\mathbb{R}^{d-1}} > 0$. Though these methods can be formulated for general space dimension, they are presented here only for $d = 2$.

We start with a mesh $\mathcal{T}_h = \{K\}$ for the domain Ω, with the triangular or rectangular element being denoted as K, the edge as e, the diameter of K as h_K, and the meshsize of $\mathcal{T}_h$ as $h = \max_{K \in \mathcal{T}_h} h_K$. We further denote the union of all interior edges as $\mathcal{E}_i$, the union of boundary edges in Γ_D (resp. Γ_N) as $\mathcal{E}_D$ (resp. $\mathcal{E}_N$), and $\mathcal{E} = \mathcal{E}_i \cup \mathcal{E}_D \cup \mathcal{E}_N$. With an auxiliary variable $\mathbf{q}$, (2.1) can be rewritten as

$$\mathbf{q} = \nabla u, \quad -\nabla \cdot \mathbf{q} = 0 \ \text{ in } \Omega, \quad u|_{\Gamma_D} = g_D, \quad \mathbf{q} \cdot \mathbf{n}|_{\Gamma_N} = \mathbf{g}_N \cdot \mathbf{n}. \tag{2.2}$$

Based on [3], a general LDG method for (2.2) can be formulated as: finding $(u_h, \mathbf{q}_h) \in (V_h, \mathbf{M}_h)$, such that

$$\int_K \mathbf{q}_h \cdot \mathbf{r} dx = -\int_K u_h \nabla \cdot \mathbf{r} dx + \int_{\partial K} \hat{u}_h \mathbf{r} \cdot \mathbf{n}_K ds, \qquad \int_K \mathbf{q}_h \cdot \nabla v dx = \int_{\partial K} v \hat{\mathbf{q}}_h \cdot \mathbf{n}_K ds \quad (2.3)$$

for any $(v, \mathbf{r}) \in (V_h, \mathbf{M}_h)$ and $K \in \mathcal{T}_h$. Here $\mathbf{n}_K$ is the outward unit normal of K, $(V_h, \mathbf{M}_h)$ is a discrete space pair to approximate $(u, \mathbf{q})$, and $(\hat{u}_h, \hat{\mathbf{q}}_h)$ are the so-called numerical fluxes, which are single-valued and approximate $(u, \mathbf{q})$ along $\mathcal{E}$. To finalize the scheme, one needs to specify $(\hat{u}_h, \hat{\mathbf{q}}_h)$ and $(V_h, \mathbf{M}_h)$.

The LSP LDG method in [12] employs the same numerical fluxes as the standard LDG method in [3]. That is, for an interior edge $e \in \mathcal{E}_i$,

$$\hat{\mathbf{q}}_h = \{\{\mathbf{q}_h\}\} - C_{11}[\![u_h]\!] - \mathbf{C}_{12}[\![\mathbf{q}_h]\!], \qquad \hat{u}_h = \{\{u_h\}\} + \mathbf{C}_{12} \cdot [\![u_h]\!]. \quad (2.4)$$

Here the standard notations are used for the average $\{\{\cdot\}\}$ and the jump $[\![\cdot]\!]$: given $e = K^+ \cap K^- \in \mathcal{E}_i$, and with $\mathbf{n}^\pm = \mathbf{n}_{K^\pm}$ and $(v^\pm, \mathbf{r}^\pm) = (v, \mathbf{r})|_{K^\pm}$, we define on e

$$\{\{v\}\} = (v^+ + v^-)/2, \;\; \{\{\mathbf{r}\}\} = (\mathbf{r}^+ + \mathbf{r}^-)/2, \;\; [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^-, \;\; [\![\mathbf{r}]\!] = \mathbf{r}^+ \cdot \mathbf{n}^+ + \mathbf{r}^- \cdot \mathbf{n}^-.$$

And for a boundary edge $e \in \partial K$, with $(v^+, \mathbf{r}^+) = (v, \mathbf{r})|_K$

$$\hat{\mathbf{q}}_h = \begin{cases} \mathbf{q}_h^+ - C_{11}(u_h^+ - g_D)\mathbf{n} & \text{for } e \in \mathcal{E}_D, \\ \mathbf{g}_N & \text{for } e \in \mathcal{E}_N, \end{cases} \qquad \hat{u}_h = \begin{cases} g_D & \text{for } e \in \mathcal{E}_D, \\ u_h^+ & \text{for } e \in \mathcal{E}_N. \end{cases} \quad (2.5)$$

The parameters C_{11} and $\mathbf{C}_{12}$ in (2.4)-(2.5) can be chosen edge by edge, and their values may affect the accuracy and stability of LDG methods as well as the matrix structures in the final algebraic system [3].

We now turn to the choice of the discrete spaces $(V_h, \mathbf{M}_h)$. For the standard LDG method in [3], $(V_h, \mathbf{M}_h) = (V_h^k, \mathbf{M}_h^k) = (V_h^{k,STD}, \mathbf{M}_h^{k,STD})$ is taken, with

$$\begin{aligned} V_h^{k,STD} &= \{u \in L^2(\Omega) : u|_K \in P^k(K), \forall K \in \mathcal{T}_h\}, \\ \mathbf{M}_h^{k,STD} &= \{\mathbf{q} \in [L^2(\Omega)]^d : \mathbf{q}|_K \in [P^k(K)]^d, \forall K \in \mathcal{T}_h\}, \end{aligned}$$

where $P^k(K)$ is the set of polynomials of the total degree at most k on K. For the LSP LDG method in [12], we use $(V_h, \mathbf{M}_h) = (V_h^k, \mathbf{M}_h^k) = (V_h^{k,LSP}, \mathbf{M}_h^{k,LSP})$, with

$$\begin{aligned} V_h^{k,LSP} &= \{u \in L^2(\Omega) : u|_K \in P^k(K), \triangle u|_K = 0, \forall K \in \mathcal{T}_h\}, \\ \mathbf{M}_h^{k,LSP} &= \{\mathbf{q} \in [L^2(\Omega)]^d : \mathbf{q}|_K \in [P^k(K)]^d, \nabla \cdot \mathbf{q}|_K = 0, \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Another choice, $(V_h^k, \mathbf{M}_h^k) = (V_h^{k,LSP}, \widetilde{\mathbf{M}}_h^{k,LSP})$, is also considered in [12] with $\widetilde{\mathbf{M}}_h^{k,LSP} = \mathbf{M}_h^{k,STD}$. In both cases, the approximating functions in V_h for the LSP LDG method are piecewise harmonic polynomials, and such functions satisfy the Laplace equation exactly in each element K.

With (2.4)-(2.5), $\mathbf{q}_h$ in (2.3) can be solved locally in terms of u_h, so the size of the final algebraic system of the LDG method depends only on the dimension of V_h. By incorporating the *a priori* knowledge of the exact solution to the discrete space V_h, the LSP LDG method results in a smaller linear system especially when polynomials of higher degrees are used, and therefore has lower computational complexity. More specifically, the dimension of the local-structure-preserving space $V_h^{k,LSP}$ on each element $K \in \mathcal{T}_h$ is $2k+1$ which depends on k linearly, whereas the dimension of the standard polynomial space $V_h^{k,STD}$ on K is $(k+2)(k+1)/2$, which depends on k quadratically. Indeed, this local-structure-preserving approximating space $V_h^{k,LSP}$ can be used in any of the DG methods discussed in [1] to provide high order numerical methods for the Laplace equation with low computational complexity. The actual cost efficiency of such methods certainly needs additional investigation.

The reduction of the computational complexity discussed above does not compromise the overall accuracy of the LSP LDG method when it is measured in the L^2 norm and the energy norm. In fact, with the meshes $\{\mathcal{T}_h\}_h$ being regular [4] and for the sufficiently smooth exact solution u, one can establish the error estimate $||\nabla u - \mathbf{q}_h||_{L^2(\Omega)} = O(h^k)$ for both standard and LSP LDG methods (see [3, 12]) with $C_{11} = O(1/h)$ or $O(1)$. If the full elliptic regularity is further assumed for the adjoint problem of (2.1)

$$-\triangle \psi = f \quad \text{in } \Omega, \qquad \psi|_{\Gamma_D} = 0, \qquad \frac{\partial \psi}{\partial \mathbf{n}}|_{\Gamma_N} = 0, \tag{2.6}$$

namely, $||\psi||_{2,\Omega} \leq C_r||f||_{L^2(\Omega)}$, $\forall\ f \in L^2(\Omega)$ with a constant C_r solely depending on Ω and $||\cdot||_{t,\Omega}$ with $t > 0$ being the standard Sobolev norm, then a duality argument can lead to the L^2 error estimate $||u - u_h||_{L^2(\Omega)} = O(h^{k+1})$ with $C_{11} = O(1/h)$. This estimate was given in [3] for the standard LDG method. And it can also be established for the LSP LDG method by following the general analysis in [1] based on the primal formulation of the scheme. With the focus of this note in mind, we will not present the proof but mention that the duality argument uses the fact of $V_h^{1,LSP} = V_h^{1,STD}$, which ensures that the discrete space $V_h^{1,LSP}$ can be used to approximate the solutions of both the Laplace equation and its adjoint problem (2.6). One can refer to [9] for the details of the proof.

3 Accuracy in negative-order norms

For many applications, numerical methods are regarded as being accurate if they are accurate in commonly used norms such as the energy and L^2 norms. In certain applications (with an example indicated below), one may also be interested in the accuracy of the methods in negative-order norms $||\cdot||_{-s,\Omega}$, namely, $||v||_{-s,\Omega} = \sup_{\phi \in C_0^\infty(\Omega)} \frac{\int_\Omega v(x)\phi(x)dx}{||\phi||_{s,\Omega}}$, with any natural number s. The duality argument in [3] can be used to show the error estimates in negative-order norms for the standard LDG method. However, such argument can not be applied directly to the LSP LDG method in section 2 to get similar estimates, due to that $V_h^{k,LSP}$ with $k > 1$ is not a suitable discrete space for the adjoint problem (2.6) of the Laplace equation.

On the other hand, error estimates in negative-order norms often contain the information on the oscillatory nature of the error, and this has been used to enhance the accuracy of some numerical methods by a local post-processing technique, which was originally developed by Bramble and Schatz [2] in the context of continuous finite element methods for elliptic problems and later by Cockburn et al. [7], Ryan et al. [15], and Ryan and Cockburn [13] in the context of DG methods for hyperbolic equations, and was also applied to LDG methods for convection-diffusion equations [7] and for differential equations with even higher spatial

derivatives [16]. With a negative-order norm error estimate of the numerical solution and a local translation invariance of the mesh, this local post-processing technique can filter out the (possible) oscillation in the error and recover the accuracy in the L^2 norm, up to the order of the error estimates in the negative-order norm. Note that a negative-order norm of a function is no bigger than its L^2 norm, that is, $||v||_{-s,\Omega} \leq ||v||_{0,\Omega}$ for $s \geq 1$. When the error estimate in the negative-order norm for a numerical method is of higher order than its L^2 error estimate, the post-processed solution will be of higher order accuracy than the original numerical solution in the L^2 norm. Based on this, we next apply the post-processing technique of [2, 7, 15] to the numerical solutions of our LSP LDG method. Instead of aiming at enhancing the accuracy of the computed solution, we use this technique as an indirect tool to study the accuracy of the method in negative-order norms. To facilitate our understanding, the results for the standard LDG method are also presented.

We consider an example with the smooth exact solution $u(x, y) = e^{-x}\cos(y)$ in $\Omega = [0, 1]^2$ and the Dirichlet boundary condition. The numerical fluxes are taken as $\hat{\mathbf{q}} = \mathbf{q}^+ - C_{11}[\![u]\!]$, $\hat{u} = u^-$ for the interior edge $e = K^+ \cap K^- \in \mathcal{E}_i$, where (u^-, q^+) is either $(u|_{K^+}, q|_{K^-})$ or $(u|_{K^-}, q|_{K^+})$. Such choice results in a smaller local dependence stencil and hence a sparser matrix in the final system. Both the standard and the LSP LDG methods are simulated on uniform rectangular meshes in quadruple precision. We then apply the local post-processing technique of [7, 15] by convoluting the numerical solution in $(V_h^k, \mathbf{M}_h^k)$ with the two-dimensional kernel $K^{2(k+1),k+1}$ defined in [15]. The tensor product of a six-point Gaussian quadrature formula in one dimension is used to compute the errors, and this simplifies the post-processing step into small matrix vector multiplications of the pre-stored matrices and the coefficients which represent the numerical solution in the neighboring mesh elements [15]. Since the kernel $K^{2(k+1),k+1}$ involves a symmetric stencil, to avoid the boundary effect, the errors before and after the post-processing step are computed in a sub-domain $\Omega_c = [0.25, 0.75]^2$. Alternatively, one can combine the one-sided post-processing technique developed in [14] for errors in the whole domain Ω.

For $k = 1, 2, 3$, convergence orders in the L^2 norm are summarized in Table 3.1 for numerical solutions before and after the post-processing procedure. The non-negative $\mu_i(k)$ with $i, k = 1, 2, 3$ are defined as

$$\mu_1(k) = \begin{cases} 0, & \text{for } k = 1, \\ 1, & \text{for } k = 2, \\ 0, & \text{for } k = 3, \end{cases} \quad \mu_2(k) = \begin{cases} 1, & \text{for } k = 1, \\ 1, & \text{for } k = 2, \\ 0.4 \sim 0.5, & \text{for } k = 3, \end{cases} \quad \mu_3(k) = \begin{cases} 1, & \text{for } k = 1, \\ 0, & \text{for } k = 2, \\ 0, & \text{for } k = 3. \end{cases} \tag{3.7}$$

The more detailed errors and convergence orders are reported in Tables 3.2-3.5. Here the LSP LDG method I uses $(V_h^{k,LSP}, \mathbf{M}_h^{k,LSP})$ as the discrete space, and the LSP LDG method II uses $(V_h^{k,LSP}, \widetilde{\mathbf{M}}_h^{k,LSP})$. In addition, $(u, \mathbf{q})$, $(u_h, \mathbf{q}_h)$, and $(\mathcal{P}u_h, \mathcal{P}\mathbf{q}_h)$ represent the exact solution, the numerical solution, and the post-processed numerical solution, respectively. We take $C_{11} = 1/h$ and 10. Based on the numerical results and [7], for $k = 1, 2, 3$, we conclude that

(1) for standard LDG approximations in $(V_h^{k,STD}, \mathbf{M}_h^{k,STD})$, the post-processing technique enhances the accuracy of $(u_h, \mathbf{q}_h)$ from $(k+1, k)$ to $(2k, 2k)$ in the L^2 norm. This is consistent to the error estimate for u_h in negative k-th order norm, namely, $||u - u_h||_{-k,\Omega} = O(h^{2k})$. Such estimate was not stated explicitly yet it is a direct consequence of Lemma 2.4, Lemma 3.3, and Lemma 3.6 in [3]. The accuracy enhancement after the post-processing step also suggests $||\mathbf{q} - \mathbf{q}_h||_{-k,\Omega} = O(h^{2k})$.

(2) for LSP LDG approximations in either $(V_h^{k,LSP}, \mathbf{M}_h^{k,LSP})$ or $(V_h^{k,LSP}, \widetilde{\mathbf{M}}_h^{k,LSP})$, the post-processing technique in general does not improve the accuracy order for u_h in the L^2 norm, though the actual errors of the post-processed $\mathcal{P}u_h$ are smaller. Such post-processing step does improve the accuracy for $\mathbf{q}_h$ by at least one order. These results indicate that the error for $u_h \in V_h^{k,LSP}$ in the negative k-th order norm is generally of the same order as its L^2 error, namely, $||u - u_h||_{-k,\Omega} = O(h^{k+1+\nu})$ with $\nu \geq 0$, and the error for $\mathbf{q}_h \in \mathbf{M}_h^{k,LSP}$ or $\widetilde{\mathbf{M}}_h^{k,LSP}$ in the negative k-th order norm is of at least one order higher than its L^2 error, namely, $||\mathbf{q} - \mathbf{q}_h||_{-k,\Omega} = O(h^{k+\nu})$, $\nu \geq 1$.

Table 3.1: The summary of the convergence orders of the LDG approximation $(u_h, \mathbf{q}_h)$ and the post-processed approximations $(\mathcal{P}u_h, \mathcal{P}\mathbf{q}_h)$. $\mu_i(k)$ with $i, k = 1, 2, 3$ are non-negative and they are defined by (3.7).

C_{11}	Method	$\|\|u - u_h\|\|_{0,\Omega}$	$\|\|u - \mathcal{P}u_h\|\|_{0,\Omega}$	$\|\|\mathbf{q} - \mathbf{q}_h\|\|_{0,\Omega}$	$\|\|\mathbf{q} - \mathcal{P}\mathbf{q}_h\|\|_{0,\Omega}$
$1/h$	LSP LDG method I	k+1	k+1	k	k+1+$\mu_1(k)$
	LSP LDG method II	k+1	k+1	k	k+1
	Standard LDG method	k+1	2k	k	2k
10	LSP LDG method I	k+1	k+1	k+$\mu_2(k)$	k+2
	LSP LDG method II	k+1	k+1+$\mu_3(k)$	k+$\mu_3(k)$	k+1+$\mu_3(k)$
	Standard LDG method	k+1	2k+$\mu_3(k)$	k+$\mu_3(k)$	2k+$\mu_3(k)$

Compared with the standard LDG method, the LSP LDG method has some accuracy loss when measured in negative-order norms. In practice, one can always apply the local post-processing technique to the LDG approximations to reduce the errors and therefore to enhance the resolution of the numerical solutions.

When $C_{11} = O(1)$, another relevant work is the superconvergence result in [5], which is established for the LDG method in [3] when the numerical fluxes are suitably chosen, and the finite elements with *tensor structures* are used on Cartesian meshes. It was proved that $||u - u_h||_{L^2(\Omega)} = O(h^{k+1})$ and $||\mathbf{q} - \mathbf{q}_h||_{L^2(\Omega)} = O(h^{k+\frac{1}{2}})$. These results are sharp, and the estimate in $\mathbf{q}_h$ is $\frac{1}{2}$ order higher than that of the general LSP LDG methods. The numerical experiments in this section with $C_{11} = O(1)$ are also on Cartesian meshes with the same type of numerical fluxes as in [5], and the only difference is in discrete spaces. Numerically, compared with the LDG method in [5], the LSP LDG method I is $\frac{1}{2}$ order more accurate in $\mathbf{q}_h$ when $k = 1$ and 2, and the LSP LDG method II is $\frac{1}{2}$ order less accurate in $\mathbf{q}_h$ when $k \neq 1$, while the convergence orders in u_h are the same among all methods in this note and in [5].

4 Concluding remarks

To certain extent, this note reports some negative finding for the LSP LDG method in [12] to solve the Laplace equation, and the mathematical understanding of this result is yet to be

Table 3.2: Errors and convergence orders of the LDG approximation u_h and the post-processed approximation $\mathcal{P}u_h$. $C_{11} = 1/h$. h is the meshsize with $h_0 = 0.05$.

		$\|\|u - u_h\|\|_{0,\Omega}$		$\|\|u - \mathcal{P}u_h\|\|_{0,\Omega}$	
	h	error	order	error	order
LSP LDG method I					
P^1	h_0	1.18e-04	-	1.18e-05	-
	$h_0/2$	2.96e-05	2.00	3.27e-06	1.85
	$h_0/4$	7.40e-06	2.00	8.56e-07	1.94
	$h_0/8$	1.85e-06	2.00	2.18e-07	1.97
P^2	h_0	1.32e-06	-	1.55e-07	-
	$h_0/2$	1.65e-07	3.00	1.99e-08	2.96
	$h_0/4$	2.06e-08	3.00	2.52e-09	2.98
	$h_0/8$	2.58e-09	3.00	3.17e-10	2.99
P^3	h_0	1.26e-08	-	8.11e-09	-
	$h_0/2$	7.90e-10	4.00	5.07e-10	4.00
	$h_0/4$	4.94e-11	4.00	3.17e-11	4.00
LSP LDG method II					
P^1	h_0	1.18e-04	-	1.27e-05	-
	$h_0/2$	2.96e-05	2.00	3.39e-06	1.91
	$h_0/4$	7.40e-06	2.00	8.71e-07	1.96
	$h_0/8$	1.85e-06	2.00	2.20e-07	1.98
P^2	h_0	9.20e-07	-	7.56e-08	-
	$h_0/2$	1.15e-07	3.00	9.62e-09	2.97
	$h_0/4$	1.44e-08	3.00	1.22e-09	2.98
	$h_0/8$	1.80e-09	3.00	1.53e-10	2.99
P^3	h_0	9.51e-09	-	5.24e-09	-
	$h_0/2$	5.93e-10	4.00	3.27e-10	4.00
	$h_0/4$	3.70e-11	4.00	2.04e-11	4.00
Standard LDG method					
P^1	h_0	1.18e-04	-	1.27e-05	-
	$h_0/2$	2.96e-05	2.00	3.39e-06	1.91
	$h_0/4$	7.40e-06	2.00	8.71e-07	1.96
	$h_0/8$	1.85e-06	2.00	2.20e-07	1.98
P^2	h_0	9.20e-07	-	1.75e-08	-
	$h_0/2$	1.15e-07	3.00	1.13e-09	3.96
	$h_0/4$	1.44e-08	3.00	7.14e-11	3.98
P^3	h_0	6.80e-09	-	5.44e-12	-
	$h_0/2$	4.24e-10	4.01	8.55e-14	5.99
	$h_0/4$	2.64e-11	4.00	1.34e-15	6.00

Table 3.3: Errors and convergence orders of the LDG approximation u_h and the post-processed approximation $\mathcal{P}u_h$. $C_{11} = 10$. h is the meshsize with $h_0 = 0.05$.

		$\|\|u - u_h\|\|_{0,\Omega}$		$\|\|u - \mathcal{P}u_h\|\|_{0,\Omega}$	
	h	error	order	error	order
LSP LDG method I					
P^1	h_0	1.24e-04	-	4.86e-06	-
	$h_0/2$	3.19e-05	1.96	5.66e-07	3.10
	$h_0/4$	8.11e-06	1.98	6.81e-08	3.06
	$h_0/8$	2.05e-06	1.99	1.53e-08	2.16
P^2	h_0	1.68e-06	-	1.73e-07	-
	$h_0/2$	2.52e-07	2.74	2.38e-08	2.86
	$h_0/4$	3.53e-08	2.84	3.14e-09	2.92
	$h_0/8$	4.71e-09	2.91	4.03e-10	2.96
P^3	h_0	1.50e-08	-	9.72e-09	-
	$h_0/2$	1.08e-09	3.80	6.99e-10	3.80
	$h_0/4$	7.47e-11	3.86	4.79e-11	3.87
LSP LDG method II					
P^1	h_0	1.24e-04	-	6.19e-06	-
	$h_0/2$	3.19e-05	1.96	8.30e-07	2.90
	$h_0/4$	8.11e-06	1.98	1.07e-07	2.95
	$h_0/8$	2.05e-06	1.99	1.36e-08	2.98
P^2	h_0	9.30e-07	-	8.14e-08	-
	$h_0/2$	1.17e-07	2.99	1.06e-08	2.94
	$h_0/4$	1.47e-08	2.99	1.36e-09	2.97
	$h_0/8$	1.84e-09	3.00	1.71e-10	2.99
P^3	h_0	9.67e-09	-	5.30e-09	-
	$h_0/2$	6.10e-10	3.99	3.33e-10	3.99
	$h_0/4$	3.83e-11	3.99	2.09e-11	3.99
Standard LDG method					
P^1	h_0	1.24e-04	-	6.19e-06	-
	$h_0/2$	3.19e-05	1.96	8.30e-07	2.90
	$h_0/4$	8.11e-06	1.98	1.07e-07	2.95
	$h_0/8$	2.05e-06	1.99	1.36e-08	2.98
P^2	h_0	9.29e-07	-	1.42e-08	-
	$h_0/2$	1.17e-07	2.99	8.12e-10	4.13
	$h_0/4$	1.47e-08	2.99	4.84e-11	4.07
P^3	h_0	7.02e-09	-	5.09e-12	-
	$h_0/2$	4.46e-10	3.98	7.68e-14	6.05
	$h_0/4$	2.81e-11	3.99	1.18e-15	6.03

Table 3.4: Errors and convergence orders of the LDG approximation $\mathbf{q}_h$ and the post-processed approximation $\mathcal{P}\mathbf{q}_h$. $C_{11} = 1/h$. h is the meshsize with $h_0 = 0.05$.

		$\|\|\mathbf{q} - \mathbf{q}_h\|\|_{0,\Omega}$		$\|\|\mathbf{q} - \mathcal{P}\mathbf{q}_h\|\|_{0,\Omega}$	
	h	error	order	error	order
LSP LDG method I					
P^1	h_0	2.66e-03	-	5.51e-05	-
	$h_0/2$	1.35e-03	0.98	1.44e-05	1.94
	$h_0/4$	6.82e-04	0.99	3.67e-06	1.97
	$h_0/8$	3.42e-04	0.99	9.25e-07	1.99
P^2	h_0	7.44e-05	-	5.79e-09	-
	$h_0/2$	1.85e-05	2.00	2.89e-10	4.33
	$h_0/4$	4.63e-06	2.00	1.73e-11	4.06
	$h_0/8$	1.16e-06	2.00	1.08e-12	4.00
P^3	h_0	6.67e-07	-	2.70e-09	-
	$h_0/2$	8.34e-08	3.00	1.69e-10	3.99
	$h_0/4$	1.04e-08	3.00	1.06e-11	4.00
LSP LDG method II					
P^1	h_0	2.66e-03	-	5.52e-05	-
	$h_0/2$	1.35e-03	0.98	1.44e-05	1.94
	$h_0/4$	6.82e-04	0.99	3.67e-06	1.97
	$h_0/8$	3.42e-04	0.99	9.26e-07	1.99
P^2	h_0	1.25e-04	-	2.54e-07	-
	$h_0/2$	3.11e-05	2.01	3.18e-08	3.00
	$h_0/4$	7.76e-06	2.00	3.97e-09	3.00
	$h_0/8$	1.94e-06	2.00	4.95e-10	3.00
P^3	h_0	9.94e-07	-	6.00e-09	
	$h_0/2$	1.24e-07	3.00	3.75e-10	4.00
	$h_0/4$	1.55e-08	3.00	2.35e-11	4.00
Standard LDG method					
P^1	h_0	2.66e-03	-	5.52e-05	-
	$h_0/2$	1.35e-03	0.98	1.44e-05	1.94
	$h_0/4$	6.82e-04	0.99	3.67e-06	1.97
	$h_0/8$	3.42e-04	0.99	9.26e-07	1.99
P^2	h_0	1.24e-04	-	1.45e-08	-
	$h_0/2$	3.10e-05	2.01	9.57e-10	3.92
	$h_0/4$	7.73e-06	2.00	6.24e-11	3.94
P^3	h_0	7.92e-07	-	1.12e-11	-
	$h_0/2$	9.91e-08	3.00	1.76e-13	5.99
	$h_0/4$	1.24e-08	3.00	2.76e-15	5.99

Table 3.5: Errors and convergence orders of the LDG approximation $\mathbf{q}_h$ and the post-processed approximation $\mathcal{P}\mathbf{q}_h$. $C_{11} = 10$. h is the meshsize with $h_0 = 0.05$.

		$\|\|\mathbf{q} - \mathbf{q}_h\|\|_{0,\Omega}$		$\|\|\mathbf{q} - \mathcal{P}\mathbf{q}_h\|\|_{0,\Omega}$	
	h	error	order	error	order
LSP LDG method I					
P^1	h_0	1.52e-03	-	2.98e-05	-
	$h_0/2$	4.11e-04	1.88	4.07e-06	2.87
	$h_0/4$	1.07e-04	1.94	5.32e-07	2.94
	$h_0/8$	2.74e-05	1.97	6.79e-08	2.97
P^2	h_0	5.10e-05	-	5.82e-09	-
	$h_0/2$	7.81e-06	2.71	3.20e-10	4.18
	$h_0/4$	1.10e-06	2.83	1.99e-11	4.01
	$h_0/8$	1.47e-07	2.90	1.26e-12	3.98
P^3	h_0	4.96e-07	-	1.89e-09	-
	$h_0/2$	4.46e-08	3.48	7.45e-11	4.67
	$h_0/4$	4.21e-09	3.41	2.67e-12	4.80
LSP LDG method II					
P^1	h_0	1.51e-03	-	3.00e-05	-
	$h_0/2$	4.11e-04	1.88	4.10e-06	2.87
	$h_0/4$	1.07e-04	1.94	5.36e-07	2.93
	$h_0/8$	2.74e-05	1.97	6.86e-08	2.97
P^2	h_0	1.24e-04	-	2.71e-07	-
	$h_0/2$	3.09e-05	2.01	3.48e-08	2.96
	$h_0/4$	7.71e-06	2.00	4.41e-09	2.98
	$h_0/8$	1.93e-06	2.00	5.54e-10	2.99
P^3	h_0	9.90e-07	-	6.07e-09	-
	$h_0/2$	1.23e-07	3.00	3.82e-10	3.99
	$h_0/4$	1.54e-08	3.00	2.40e-11	3.99
Standard LDG method					
P^1	h_0	1.51e-03	-	3.00e-05	-
	$h_0/2$	4.11e-04	1.88	4.10e-06	2.87
	$h_0/4$	1.07e-04	1.94	5.36e-07	2.93
	$h_0/8$	2.74e-05	1.97	6.86e-08	2.97
P^2	h_0	1.24e-04	-	1.26e-08	-
	$h_0/2$	3.08e-05	2.01	7.38e-10	4.09
	$h_0/4$	7.68e-06	2.00	4.66e-11	3.99
P^3	h_0	7.84e-07	-	1.05e-11	-
	$h_0/2$	9.78e-08	3.00	1.58e-13	6.05
	$h_0/4$	1.22e-08	3.00	2.43e-15	6.03

established. In many applications when it is sufficient to have schemes which are accurate in commonly used norms such as the energy and the L^2 norms, this LSP LDG method is still competitive due to its low computational complexity.

In the end, we want to mention that such post-processing technique was also applied to the locally divergence-free DG approximations for time-dependent Maxwell equations in [6]. This is another example of local-structure-preserving DG methods. Numerical results in [6] suggest that using local-structure-preserving discrete spaces in DG frameworks does not necessarily lead to accuracy loss in negative-order norms. In fact, the accuracy order of the numerical solutions in [6] was enhanced from $k+1$ to $2k+1$ in the L^2 norm by the post-processing technique, same as what occurs to the standard DG approximations. This indicates that such local-structure-preserving DG approximations have the same $(2k+1)$-st order of accuracy in negative k-th order norm as the standard DG approximations [7] when solving the Maxwell equations. Mathematical justification for this is still an open problem.

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