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The Existence of Positive Solutions for an Integral Boundary Value Problems

Hongling Geng and Fuyi Xu

School of Science
Shandong University of Technology, Zibo, Shandong
255049, People's Republic of China

Abstract

In this paper, by using fixed point theorem of cone expansion-compression type and suitable conditions, we present the existence of single positive solution for the integral boundary value problems. We derive an explicit interval of λ such that for any λ in this interval, the existence of at least one positive solution to the boundary value problems is guaranteed.

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1 Introduction

In this paper, we consider the integral boundary value problem

$$\begin{cases} (p(t)u'(t))' - q(t)u(t) + \lambda f(t,u) = 0, \qquad 0 \le t \le 1, \\ au(0) - bp(0)u'(0) = \int_r^R \alpha(t)u(t)dt, \\ cu(1) + dp(1)u'(1) = \int_r^R \beta(t)u(t)dt \end{cases} \tag{1.1}$$

where $a, b, c, d \in [0, +\infty), 0 < r < R < 1$ are given constants, $p, q \in C([0, 1], (0, +\infty))$, $\alpha, \beta \in C([0, 1], [0, +\infty))$.

Integral boundary value problems arise in a variety of different areas of applied mathematics and physics. Boundary value problems(BVP) (1.1) becomes a generic multi-point boundary value problem, some special cases of which have been extensively studied in many papers in recent years. Moreover, boundary

[1]E-mail addresses: ghl1102@163.com(H.Geng), xfy_02@163.com(F.Xu).

value problem (1.1)includes the two-point,three-point and multi-point boundary value problems as special cases. So we can see our work naturally extend and unify some known results both for two-point boundary value problem and for multi-point boundary value problem in the literature.

The mail tool of this paper is the following well known Krasnoselskii's fixed point theorem.

Theorem 1.1(See[10]). Suppose E is a Banach space, $K \subset E$ is a cone, let Ω_1, Ω_2 be two bounded open sets of E such that $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let operator $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be completely continuous. Suppose that one of the following two conditions holds

(i) $\|Tx\| \leq \|x\|, \ \ \forall\, x \in K \cap \partial\Omega_1, \ \ \|Tx\| \geq \|x\|, \ \ \forall\, x \in K \cap \partial\Omega_2$; or

(ii) $\|Tx\| \geq \|x\|, \ \ \forall\, x \in K \cap \partial\Omega_1, \ \ \|Tx\| \leq \|x\|, \ \ \forall\, x \in K \cap \partial\Omega_2$.

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries and Lemmas

In the rest of the paper, we make the following assumptions:

(H_1) $f \in C([0, +\infty), [0, +\infty))$;

(H_2) $h \in C([0, 1], [0, +\infty))$ and there exists $x_0 \in (0,)$ such that $h(x_0) > 0$;

(H_3) $a \in C[0, 1]$,$b \in C([0, 1], (-\infty, 0))$.

In this section , we present some lemmas that are important to our main results .

Lemma 2.1(See[5]). Assume that (H_3) holds. Let $\phi_1(t), \phi_2(t)$ are the positive solutions of

$$\begin{cases} \phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t) = 0, & 0 \leq t \leq 1, \\ \phi_1(0) = 0, \quad \phi_1(1) = 1, \end{cases}$$

$$\begin{cases} \phi_2''(t) + a(t)\phi_2'(t) + b(t)\phi_2(t) = 0, & 0 \leq t \leq 1, \\ \phi_2(0) = 1, \quad \phi_2(1) = 0. \end{cases}$$

Then

(i) $\phi_1(t)$ is strictly increasing on [0,1], and$\phi_1(t) > 0$ on $(0, 1]$;

(ii) $\phi_2(t)$ is strictly decreasing on [0,1], and$\phi_1(t) > 0$ on $[0, 1)$.

Set $\rho = p(0)\begin{vmatrix} \phi_2(0) & \phi_1(0) \\ \phi_2'(0) & \phi_1'(0) \end{vmatrix}$, $\Delta = \begin{vmatrix} -\int_r^R \alpha(t)\phi_1(t)dt & \rho - \int_r^R \alpha(t)\phi_2(t)dt \\ \rho - \int_r^R \beta(t)\phi_1(t)dt & -\int_r^R \beta(t)\phi_2(t)dt \end{vmatrix}$.

Lemma 2.2. Assume that (H_3) and (H_4) hold, $\Delta \neq 0$. Let $y \in C[0, 1]$, then the problems

$$\begin{cases} (p(t)u'(t))' - q(t)u(t) + y(t) = 0, & 0 \le t \le 1, \\ au(0) - bp(0)u'(0) = \int_r^R \alpha(t)u(t)dt, \\ cu(1) + dp(1)u'(1) = \int_r^R \beta(t)u(t)dt. \end{cases} \tag{2.1}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)y(s)ds + A(y)\phi_1(t) + B(y)\phi_2(t), \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} \phi_1(t)\phi_2(s), & t \le s, \\ \phi_1(s)\phi_2(t), & s \le t, \end{cases} \tag{2.3}$$

$$A(y) = \frac{1}{\Delta}\begin{vmatrix} \int_r^R \alpha(t)\int_0^1 G(t,s)y(s)dsdt & \rho - \int_r^R \alpha(t)\phi_2(t)dt \\ \int_r^R \beta(t)\int_0^1 G(t,s)y(s)dsdt & -\int_r^R \beta(t)\phi_2(t)dt \end{vmatrix}, \tag{2.4}$$

$$B(y) = \frac{1}{\Delta}\begin{vmatrix} -\int_r^R \alpha(t)\phi_1(t) & \int_r^R \alpha(t)\int_0^1 G(t,s)y(s)dsdt \\ \rho - \int_r^R \beta(t)\phi_1(t)dt & \int_r^R \beta(t)\int_0^1 G(t,s)y(s)dsdt \end{vmatrix}. \tag{2.5}$$

Moreover, $u(t) \ge 0$ on [0,1] provided $y(t) \ge 0$.

Set

$$q(t) = \min\left(\frac{\phi_1(t)}{\|\phi_1\|}, \frac{\phi_2(t)}{\|\phi_2\|}\right),$$

where $\|.\|$ denote the supernum norm. From (2.3) we have

$$G(s,s) \ge G(t,s) \ge q(t)G(s,s), \quad \text{for } \ t \in [0,1].$$

Choose $\delta \in (0, \frac{1}{2})$ such that there exists $x_0 \in (\delta, 1-\delta)$, take

$$\gamma = \min\{q(t) | t \in [\delta, 1-\delta]\}.$$

It follows from Lemma 2.1 that $0 < \gamma < 1$

Lemma 2.3. Let (H_2), (H_3) and (H_4) hold. Assume $\Delta < 0, \rho - \int_r^R \alpha(t)\phi_2(t)dt > 0, \rho - \int_r^R \beta(t)\phi_1(t)dt > 0$. Then $y \in C[0,1]$ with $y \ge 0$, the unique solution of the problem (2.1) satisfies $u(t) \ge 0$ and

$$u(t) \ge \gamma\|u\|, \ t \in [\delta, 1-\delta].$$

Proof. By

$$0 \le G(t,s) \le G(s,s), \quad \text{for } \ t \in [0,1],$$

which implies

$$u(t) \leq \int_0^1 G(s,s)p(s)y(s)ds + A\phi_1(t) + B\phi_2(t).$$

By

$$G(t,s) \geq q(t)G(s,s), \quad \text{for} \quad t \in [0,1],$$

we have that for $t \in [\delta, 1-\delta]$,

$$G(t,s) \geq \gamma G(s,s).$$

Thus for $t \in [\delta, 1-\delta]$,

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)p(s)y(s)ds + A\phi_1(t) + B\phi_2(t) \\ &= \int_0^1 \frac{G(t,s)}{G(s,s)} G(s,s)p(s)y(s)ds + A\phi_1(t) + B\phi_2(t) \\ &\geq \gamma \int_0^1 G(s,s)p(s)y(s)ds + A\phi_1(t) + B\phi_2(t) \\ &\geq \gamma (\int_0^1 G(s,s)p(s)y(s)ds + A\phi_1(t) + B\phi_2(t)) \\ &\geq \gamma \|u\|. \end{aligned}$$

3 The Main Results

Let

$$f_0 = \lim_{x \longrightarrow 0^+} \min_{t \in [0,1]} \frac{f(t,x)}{x}, \qquad f_\infty = \lim_{x \longrightarrow \infty} \min_{t \in [0,1]} \frac{f(t,x)}{x},$$

$$f^0 = \lim_{x \longrightarrow 0^+} \max_{t \in [0,1]} \frac{f(t,x)}{x}, \qquad f^\infty = \lim_{x \longrightarrow \infty} \max_{t \in [0,1]} \frac{f(t,x)}{x},$$

$$\underline{f}^0 = \lim_{x \longrightarrow 0^+} \inf \max_{t \in [0,1]} \frac{f(t,x)}{x}, \qquad \underline{f}^\infty = \lim_{x \longrightarrow \infty} \inf \max_{t \in [0,1]} \frac{f(t,x)}{x},$$

$$\overline{f}_0 = \lim_{x \longrightarrow 0^+} \sup \min_{t \in [0,1]} \frac{f(t,x)}{x}, \qquad \overline{f}_\infty = \lim_{x \longrightarrow \infty} \sup \min_{t \in [0,1]} \frac{f(t,x)}{x},$$

$$A_1 = \int_0^1 G(s,s)ds + A_0\phi_1(1) + B_0\phi_2(0), A_2 = \max_{t \in [0,1]} \left(\int_\delta^{1-\delta} G(t,s)ds \right),$$

where

$$A_0 = \frac{1}{\Delta}\begin{vmatrix} \int_r^R \alpha(t)\int_0^1 G(s,s)dsdt & \rho - \int_r^R \alpha(t)\phi_2(t)dt \\ \int_r^R \beta(t)\int_0^1 G(s,s)dsdt & -\int_r^R \beta(t)\phi_2(t)dt \end{vmatrix},$$

$$B_0 = \frac{1}{\Delta}\begin{vmatrix} -\int_r^R \alpha(t)\phi_1(t) & \int_r^R \alpha(t)\int_0^1 G(s,s)dsdt \\ \rho - \int_r^R \beta(t)\phi_1(t)dt & \int_r^R \beta(t)\int_0^1 G(s,s)dsdt \end{vmatrix}.$$

Let $E = C[0,1]$, then E is Banach space, with respect to the norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$.

Now BVP(1.1) has a solution $u = u(t)$ if and only if u is a solution of the operator equation

$$(Tu)(t) = \lambda\left(\int_0^1 G(t,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(t) + B(f(s,u(s)))\phi_2(t)\right),$$

where p and G are defined in Lemma2.1.

We define a cone in E by

$$P = \{u \in E | u \geq 0, \min u(t)_{t\in[\delta,1-\delta]} \geq \gamma\|u\|\}.$$

It is easy to check $T(P) \subset P$ and T is completely continuous.

In this section, we discuss the existence conditions of at least one positive solution for BVP(1.1). We obtain the following existence results.

Theorem 3.1. Suppose conditions (H_1)-(H_4) hold. In addition,

(1) If $A_1 f_0 < \gamma A_2 f_\infty$, then for each $\lambda \in (\frac{1}{\gamma A_2 f_\infty}, \frac{1}{A_1 f_0})$, the BVP(1.1) has at least one positive solution.

(2) If $f_0 = 0$ and $f_\infty = \infty$, then for any $\lambda \in (0,\infty)$, the BVP(1.1) has at least one positive solution.

(3) If $f_\infty = \infty$, $0 < f_0 < \infty$, then for each $\lambda \in (0, \frac{1}{A_1 f_0})$, the BVP(1.1) has at least one positive solution.

(4) If $f_0 = 0$, $0 < f_\infty < \infty$, then for each $\lambda \in (\frac{1}{\gamma A_2 f_\infty}, \infty)$, the BVP(1.1) has at least one positive solution.

Proof. We only prove the case (1). We construct the set Ω_{R_1}, $\Omega_{R_2^*}$ in order to apply Theorem1.1. Let $\lambda \in (\frac{1}{\gamma A_2 f_\infty}, \frac{1}{A_1 f_0})$, and choose $\varepsilon > 0$ such that

$$\frac{1}{\gamma A_2(f_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{A_1(f^0 + \varepsilon)}.$$

By the definition of f^0, we choose $R_1 > 0$ such that $f(t,u) \leq (f^0+\varepsilon)u$, for $(t,u) \in [0,1]\times[0,R_1]$. Thus for $u \in P$ and $\|u\| = R_1$, we have

$$\begin{aligned}(Tu)(t) &= \lambda\left(\textstyle\int_0^1 G(t,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(t) + B(f(s,u(s)))\phi_2(t)\right)\\ &\leq \lambda\left(\textstyle\int_0^1 G(s,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(1) + B(f(s,u(s)))\phi_2(0)\right)\\ &\leq \lambda A_1(f_0+\varepsilon)\|u\| \leq \|u\|.\end{aligned}$$

Let $\Omega_{R_1} = \{u \in E | \|u\| < R_1\}$, it follows that

$$\|Tu\| \leq \|u\|, \text{for} \quad u \in P \cap \partial\Omega_{R_1}. \tag{3.1}$$

Considering the definition of f_∞, there exists $R_2 > R_1$ such that $f(t,u) \geq (f_\infty - \varepsilon)u$ for $(t,u) \in [0,1] \times [R_2, \infty)$. Let $R_2^* = \max\{2R_1, \frac{R_2}{\gamma}\}$ and $\Omega_{R_2^*} = \{u \in E | \|u\| < R_2^*\}$, then $u \in P$ and $\|u\| = R_2^*$ implies

$$\min_{t \in [\delta, 1-\delta]} u(t) \geq \gamma \|u\| \geq R_2.$$

So we have

$$\begin{aligned} ||(Tu)|| &= \max\nolimits_{t\in[0,1]} \lambda \left(\int_0^1 G(t,s) f(s,u(s))ds + A(f(s,u(s)))\phi_1(t) + B(f(s,u(s)))\phi_2(t) \right) \\ &\geq \max\nolimits_{t\in[0,1]} \lambda \left(\int_0^1 G(t,s) f(s,u(s))ds \right) \\ &\geq \max\nolimits_{t\in[0,1]} \lambda \left(\int_\delta^{1-\delta} G(t,s) f(s,u(s))ds \right) \\ &\geq \lambda (f_\infty - \varepsilon)\gamma A_2 \|u\| \geq \|u\|. \end{aligned}$$

Hence

$$\|Tu\| \geq \|u\|, \text{for} \quad u \in P \cap \partial\Omega_{R_2^*}. \tag{3.2}$$

From (3.1) and (3.2) and Theorem1.1[10], thus T has a fixed point in $u \in P \cap (\overline{\Omega}_{R_2^*} \setminus \Omega_{R_1})$, which is a positive solution of BVP(1.1).

Theorem 3.2. Suppose condition (H_1)-(H_4) hold. In addition,

(1) If $A_1 f_\infty < \gamma A_2 f_0$, then for each $\lambda \in (\frac{1}{\gamma A_2 f_0}, \frac{1}{A_1 f_\infty})$, the BVP(1.1) has at least one positive solution.

(2) If $f_0 = \infty$ and $f_\infty = 0$, then for any $\lambda \in (0, \infty)$, the BVP(1.1) has at least one positive solution.

(3) If $f_\infty = \infty$, $0 < f_0 < \infty$, then for each $\lambda \in (0, \frac{1}{A_1 f_0})$, the BVP(1.1) has at least one positive solution.

(4) If $f_0 = 0$, $0 < f_\infty < \infty$, then for each $\lambda \in (\frac{1}{\gamma A_2 f_\infty}, \infty)$, the BVP(1.1) has at least one positive solution..

Proof. We only prove the case(1). Let $\lambda \in (\frac{1}{\gamma A_2 f_0}, \frac{1}{A_1 f^\infty})$, and choose $\varepsilon > 0$ such that

$$\frac{1}{\gamma A_2 (f_0 - \varepsilon)} \leq \lambda \leq \frac{1}{A_1 (f^\infty + \varepsilon)}.$$

By the definition of f_0, there exists $\rho_0 \in (0, \infty)$ such that $f(t,u) \geq (f_0 - \varepsilon)u$ for $(t,u) \in [0,1] \times [0, \rho_0]$. Let $\Omega_{\rho_0} = \{u \in E | \quad \|u\| < \rho_0\}$, then $u \in P$ and $\|u\| = \rho_0$, we have

$$\begin{aligned} ||(Tu)|| &= \max\nolimits_{t\in[0,1]} \lambda \left(\int_0^1 G(t,s) f(s,u(s))ds + A(f(s,u(s)))\phi_1(t) + B(f(s,u(s))\phi_2(t) \right) \\ &\geq \max\nolimits_{t\in[0,1]} \lambda \left(\int_\delta^{1-\delta} G(t,s) f(s,u(s))ds \right) \\ &\geq \lambda (f_0 - \varepsilon)\gamma A_2 \|u\| \geq \|u\|. \end{aligned}$$

So

$$\|Tu\| \geq \|u\|, \text{for } \ u \in P \cap \partial\Omega_{\rho_0}. \tag{3.3}$$

By the definition of f^∞, there exists ρ_1 such that $f(t,u) \leq (f_\infty + \varepsilon)u$ for $(t,u) \in [0,1] \times [\rho_1, \infty)$.

We consider the following two cases:

Case(i) If f is unbounded, there exists $\rho_2^* > \max\{2\rho_0, \gamma^{-1}\rho_1\}$ such that $f(t,u) \leq f(t,\rho_2^*)$ for $(t,u) \in [0,1] \times [0,\rho_2^*]$. Then, for $u \in P$ and $\|u\| = \rho_2^*$, we have

$$\begin{aligned}(Tu)(t) &= \lambda\left(\int_0^1 G(t,s)f(s,u(s))ds + A(fs,(u(s)))\phi_1(t) + B(f(s,u(s)))\phi_2(t)\right)\\ &\leq \lambda\left(\int_0^1 G(s,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(1) + B(f(s,u(s)))\phi_2(0)\right)\\ &\leq \lambda(f_\infty + \varepsilon)A_1\|u\| \leq \|u\|.\end{aligned}$$

Case(ii) If f is bounded, say $f(t,u) \leq N_1$ for all $(t,u) \in [0,1] \times [0,+\infty)$, taking $\rho_2^* \geq \{2\rho_0, \lambda N_1 A_1\}$ for $u \in P$ and $\|u\| = \rho_2^*$, we have

$$\begin{aligned}(Tu)(t) &= \lambda\left(\int_0^1 G(t,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(t) + B(f(s,u(s)))\phi_2(t)\right)\\ &\leq \lambda\left(\int_0^1 G(s,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(1) + B(f(s,u(s)))\phi_1(0)\right)\\ &\leq \lambda N_1 A_1 \leq \rho_2^* \leq \|u\|.\end{aligned}$$

Hence, in either case, we may put $\Omega_{\rho_2^*} = \{u \in E | \|u\| < \rho_2^*\}$ such that

$$\|Tu\| \leq \|u\|, \text{for } \ u \in P \cap \partial\Omega^*_{\rho_2}. \tag{3.4}$$

From(3.3),(3.4) and Theorem1.1[10], thus T has a fixed point in $u \in P \cap (\overline{\Omega}_{\rho_2^*} \setminus \Omega_{\rho_0})$, which is a positive solution of BVP(1.1).

Theorem 3.3. Suppose condition (H_1)-(H_4)hold. In addition, assume there exist two positive constants $R_1 \neq R_2$ such that

(1) $f(t,u) \leq \frac{R_1}{\lambda A_1}, \forall (t,u) \in [0,1] \times [0,R_1]$;

(2) $f(t,u) \geq \frac{R_2}{\lambda A_2}, \forall (t,u) \in [0,1] \times [\gamma R_2, R_2]$. Then the BVP(1.1) has at least one positive solution u such that $\|u\|$ between R_1 and R_2.

Proof. We only consider the case $R_1 < R_2$, the case $R_1 > R_2$ follows in a similar way. Let $\Omega_{R_1} = \{u \in E | \|u\| < R_1\}, \Omega_{R_2} = \{u \in E | \|u\| < R_2\}$. It follows from (1) for any $u \in P \cap \Omega_{R_1}$,

$$\begin{aligned}(Tu)(t) &= \lambda\left(\int_0^1 G(t,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(t) + B(f(s,u(s)))\phi_2(t)\right)\\ &\leq \lambda\left(\int_0^1 G(s,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(1) + B(f(s,u(s)))\phi_2(0)\right)\\ &\leq \lambda A_1 \frac{R_1}{\lambda A_1} = \|u\|.\end{aligned}$$

Therefore,

$$||Tu|| \leq ||u||, \quad \text{for} \quad u \in P \cap \Omega_{R_1}. \tag{3.5}$$

On the other hand, for any $u \in P \cap \Omega_{R_2}$, we have $\gamma R_2 \leq |u(t)| \leq R_2$, for $t \in [\delta, 1-\delta]$. It follows that for any $u \in P \cap \Omega_{R_2}$,

$$\begin{aligned} ||Tu|| &= \max_{t\in[0,1]} \lambda \left(\int_0^1 G(t,s)f(s,u(s))ds + A(f(s,u(s)))\phi_1(t) + B(f(s,u(s)))\phi_2(t)\right) \\ &\geq \max_{t\in[0,1]} \lambda \left(\int_0^1 G(t,s)f(s,u(s))ds\right) \\ &\geq \max_{t\in[0,1]} \lambda \left(\int_\delta^{1-\delta} G(t,s)f(s,u(s))ds\right) \\ &\geq R_2 = \|u\|. \end{aligned}$$

So

$$\|Tu\| \geq \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_{R_2}. \tag{3.6}$$

From(3.5),(3.6) and Theorem1.1[10], thus T has a fixed point in $u \in P \cap (\overline{\Omega}_{R_2} \setminus \Omega_{R_1})$, which is a positive solution of BVP(1.1) between R_1 and R_2..

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