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Steady-State Torsional Response With Viscous Damping

Introduction

Rotating machinery trains are rotationally free-free systems, and because of finite inertia and stiffness, they have many degrees of freedom. For most torsional vibration analysis the train may be adequately modeled as a close-coupled assembly of discrete inertias and linear, torsional springs. While there is only negligible damping in most of the rotating assembly, significant damping may be encountered in the torque-speed characteristics of some machines or included by design with damped couplings or untuned, free, viscous shear dampers.

The most common type of torsional vibration analysis is the steady-state response evaluation for constant average speed of operation and the associated periodic excitation torques. Steady-state response is determined by the particular solution for the equations of motion. This motion persists after damping action has effectively removed the homogeneous solution from the motion. Torsional steady-state response calculations have been standard machinery design practice for many years [1]. The analysis, including station-to-station, station-to-ground, and untuned, free, viscous shear damping actions, has been correctly formulated by a number of writers in the past [1-4]. While damping is often a nonlinear phenomenon, it has been standard practice to linearize the description, employing an equivalent viscous damping action. Even with this linearization, the computational difficulty associated with complex arithmetic has led to a rejection of the rigorous solution as impractical [3], or as feasible only in a tabular calculation format [1, 4, 5]. In view of the greatly increased computational capacity now readily available, it is appropriate to reexamine this key machinery design analysis and to recast it in terms familar to today's analysts and consistent with current computational methods.

The purpose of this paper is to develop the complete steady-state torsional response to periodic excitation in a form directly suited to machine computation. This will be a solution for the actual rotation of each station, not the relative rotation, as this makes the roles of steady twist and rigid body oscillations ("rolling modes") more clear. This is also the motion required for comparison with measurements made using single station, FFT measuring techniques [6]. The relative motion is readily available from these results simply by computing station-to-station differences. There follows below a description of the system leading to formulation of the equation of motion, the determination of the complete particular solution, and two example problems illustrating the methods and results obtained.

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System Description

A typical torsional system model is shown in Fig. 1. The primary sub-system consists of n_1 rotational inertias with elastic interconnections. Damping is indicated by station-to-station and station-to-ground viscous coupling. The varying torque-speed relation for a ship's propeller or damper winding action in a synchronous generator are typical sources of station-to-ground viscous coupling. Significant station-to-station damping is often found in couplings incorporating damper elements. Additional damping is indicated in the form of n_2 untuned, free, viscous shear dampers, sometimes known as "Houdaille dampers," although they have been manufactured by several companies. This type of damper will be called a free damper. The rotor of each free damper is viscously coupled to one of the primary inertias. Although this description has been in terms of the single shaft system shown in Fig. 1, it should be understood that the methods described apply to both single shaft and branched systems with any of the viscous damper types in any location. Examples of both single shaft and branched systems are given later.

Let each degree of freedom be represented by the rotation of a system inertia. Using the subscripts p and f to denote the primary system components and free damper components, respectively, the equation of motion for the system is

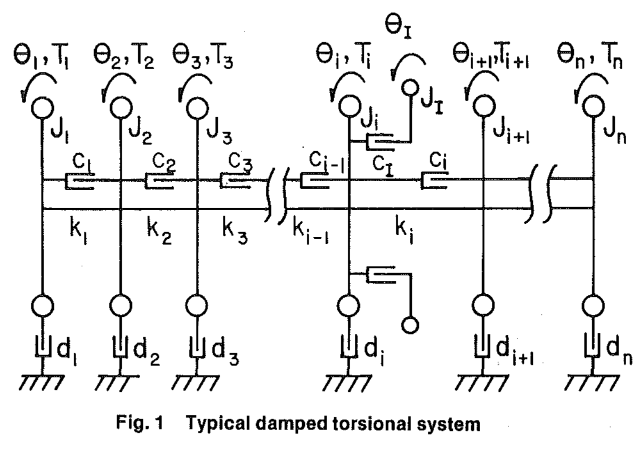

Fig. 1 Typical damped torsional system

$$\left[\begin{array}{c|c} J_p & 0 \\ \hline 0 & J_f \end{array}\right]\left\{\begin{array}{c} \ddot{\theta}_p \\ \hline \ddot{\theta}_f \end{array}\right\}+\left[\begin{array}{c|c} C_{pp} & C_{pf} \\ \hline C_{fp} & C_{ff} \end{array}\right]\left\{\begin{array}{c} \dot{\theta}_p \\ \hline \dot{\theta}_f \end{array}\right\}$$

$$+\left[\begin{array}{c|c} K_{pp} & 0 \\ \hline 0 & 0 \end{array}\right]\left\{\begin{array}{c} \theta_p \\ \hline \theta_f \end{array}\right\}=\left\{\begin{array}{c} \mathbf{T}_p(t) \\ \hline 0 \end{array}\right\} \quad (1)$$

The submatrices $[J_p]$ and $[J_f]$ are each diagonal and composed of the component mass moments of inertia. The elastic torques developed in the primary sub-system are described by $[K_{pp}]$ while the other partitions of the stiffness matrix are null, reflecting the absence of elastic coupling to the free damper rotors. The partitions of the damping coefficient matrix are:

$[C_{pp}]$, a matrix of viscous coefficients reflecting station-to-station and station-to-ground torques on the primary inertias;

$[C_{pf}]$, $[C_{fp}]$, cross coupling matrices containing free damper coefficients reflecting the interaction between primary inertias and free damper rotors; and

$[C_{ff}]$, a diagonal matrix of free damper coefficients.

For systems reducible to an equivalent single shaft, the martices $[K_{pp}]$ and $[C_{pp}]$ are of triple band diagonal form; for branched systems, the form of these matrices depends on the nature of the branching and the station numbering scheme employed. The cross coupling matrices, $[C_{pf}]$ and $[C_{fp}]$, are usually sparse.

The external torques, acting on the primary system inertias, compose the upper partition of the excitation vector. The lower partition is null since no torques external to the system act on the free damper rotors. The steady-state condition implies 1) constant average angular velocity, ω, and 2) periodic excitation torques at all stations, expressible in terms of a common fundamental frequency Ω. In most cases there is a direct relation between ω and Ω, but this is not necessary; the two frequencies are carried separately here. A typical external torque, acting on the ith primary inertia, may be expanded in a Fourier series

$$\mathbf{T}_i(t)=\frac{1}{2}a_{i0}+a_{i1}\cos\Omega t+a_{i2}\cos 2\Omega t+\ldots$$
$$+b_{i1}\sin\Omega t+b_{i2}\sin 2\Omega t+\ldots \quad (2)$$

For internal combustion engines the series coefficients may be estimated using Porter's data [7] while for many other machines the manufacturers can provide the required values. In some cases it will be necessary to compute the coefficients by a harmonic analysis of measured data. For present purposes, it is convenient to express the series in complex form [8]:

$$\mathbf{T}_i(t)=\sum_{m=-\infty}^{\infty}\alpha_{im}e^{jm\Omega t} \quad (3)$$

where the complex series coefficients are related to the real coefficients by

$$\alpha_{im}=\frac{1}{2}(a_{im}-jb_{im}) \quad (4)$$

The entire excitation vector may be written as

$$\left\{\begin{array}{c} \mathbf{T}_p(t) \\ \hline 0 \end{array}\right\}=\ldots+\left\{\begin{array}{c} \alpha_{-1} \\ \hline 0 \end{array}\right\}e^{-j\Omega t}+\left\{\begin{array}{c} \alpha_0 \\ \hline 0 \end{array}\right\}+\left\{\begin{array}{c} \alpha_1 \\ \hline 0 \end{array}\right\}e^{j\Omega t}$$
$$+\left\{\begin{array}{c} \alpha_2 \\ \hline 0 \end{array}\right\}e^{j2\Omega t}+\ldots\ldots+\left\{\begin{array}{c} \alpha_m \\ \hline 0 \end{array}\right\}e^{jm\Omega t}+\ldots \quad (5)$$

Particular Solution

For the steady-state response, the particular solution is required for the equations of motion, equation (2), with the excitation given by equation (5). The required solution may be assumed in the form

$$\left\{\begin{array}{c} \theta_p \\ \hline \theta_f \end{array}\right\}=\omega t\{1\}+\ldots+\{\mathbf{R}_{-1}\}e^{-j\Omega t}+\{\mathbf{R}_0\}$$
$$+\{\mathbf{R}_1\}e^{j\Omega t}+\ldots+\{\mathbf{R}_m\}e^{jm\Omega t}+\ldots \quad (6)$$

where $\{1\}$ denotes a column vector of unit elements, used here to reflect uniform rotation. In order to determine the response vectors, the assumed form is differentiated and substituted into the differential equations,

$$\left[\begin{array}{c|c} J_p & 0 \\ \hline 0 & J_f \end{array}\right]\left\langle\ldots-m^2\Omega^2\{\mathbf{R}_m\}e^{jm\Omega t}-\ldots\right\rangle$$

$$+\left[\begin{array}{c|c} C_{pp} & C_{pf} \\ \hline C_{fp} & C_{ff} \end{array}\right]\left\langle\omega\{1\}+\ldots+jm\Omega\{\mathbf{R}_m\}e^{jm\Omega t}+\ldots\right\rangle$$

$$+\left[\begin{array}{c|c} K_{pp} & 0 \\ \hline 0 & 0 \end{array}\right]\left\langle\omega t\{1\}+\ldots+\{\mathbf{R}_0\}+\ldots+\{\mathbf{R}_m\}e^{jm\Omega t}+\ldots\right\rangle=$$

Nomenclature

a = cosine coefficient in excitation series
b = sine coefficient in excitation series
c = station-to-station or free damper coefficient
d = station-to-ground damping coefficient
e = Naperian base
j = complex unit, $\sqrt{-1}$
k = shaft stiffness
u = response cosine coefficient
v = response sine coefficient
C = damping matrix
J = inertia matrix
K = stiffness matrix
$\mathbf{R}$ = response vector
$\mathbf{T}$ = torque vector
α = complex Fourier series coefficient
θ = station rotation angle
ω = average angular velocity, a constant
Ω = fundamental excitation frequency

Subscripts

f = referring to the free dampers
i = station index
m = typical oscillatory mode index
0 = refers to a constant term
p = refers to a primary train component

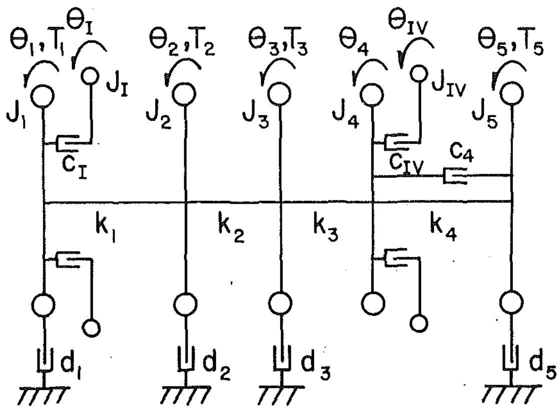

Fig. 2 Single shaft system with two free dampers

Table 1 System parameters for Example 1

Station i	Moment of inertia J_i (kg.m^2)	Damper coefficients d_i (N.m.s)	c_i (N.m.s.)	Stiffness k_i (N.m.s./rad)
1	1.70	0.10	–	5600
2	2.50	0.30	–	8200
3	2.30	0.35	–	12000
4	1.40	0.00	9.90	4000
5	3.70	0.40	–	–
I	0.15	–	0.20	–
IV	0.17	–	0.22	–

$$\ldots + \left\{ \begin{array}{c} \alpha_0 \\ \hline 0 \end{array} \right\} + \ldots + \left\{ \begin{array}{c} \alpha_m \\ \hline 0 \end{array} \right\} e^{jm\Omega t} + \ldots \tag{7}$$

The terms of this equation may be grouped according to the nature of their time dependence, with each such group expressing an independent relation. The term linear in t is identically the null vector and provides no information.

Steady Twist Term. The group of constant terms describes the steady twist associated with the net transfer of power through the train. The positions of the free damper rotors are immaterial to the solution and may be taken as zero. The upper partition of the steady solution vector is $\{\mathbf{R}_{p0}\}$, to be determined by

$$[K_{pp}]\{\mathbf{R}_{p0}\} = \{\alpha_0\} - \omega[d]\{1\} \tag{8}$$

Since $[K_{pp}]$ is of rank $n_1 - 1$, it is singular, and the elements of $\{\mathbf{R}_{p0}\}$ can not be directly determined. If one station is chosen as a reference and assigned zero steady rotation, the remaining elements of $\{\mathbf{R}_{p0}\}$, representing steady rotations relative to the reference station, may be determined from $n_1 - 1$ of the relations expressed in equation (8).

Oscillatory Terms. The group of terms associated with the mth harmonic define the response in that harmonic according to

$$\left\langle \left[\begin{array}{c|c} K_{pp} & 0 \\ \hline 0 & 0 \end{array} \right] - m^2\Omega^2 \left[\begin{array}{c|c} J_p & 0 \\ \hline 0 & J_f \end{array} \right] + jm\Omega \left[\begin{array}{c|c} C_{pp} & C_{pf} \\ \hline C_{fp} & C_{ff} \end{array} \right] \right\rangle \{\mathbf{R}_m\} = \left\{ \begin{array}{c} \alpha_m \\ - \\ 0 \end{array} \right\} \tag{9}$$

Table 2 Response vectors for Example 1

Station i	$\{u_0\}$ (10^{-3}rad)	$\{u_1\}$ (10^{-3}rad)	$\{v_1\}$ (10^{-3}rad)	$\{u_2\}$ (10^{-3}rad)	$\{v_2\}$ (10^{-3}rad)
1	0.000	2.47	1.33	−1.48	4.33
2	1.19	1.93	1.03	−0.16	0.50
3	3.72	−1.31	0.60	0.89	−4.39
4	−11.29	−5.42	0.22	1.10	−3.61
5	−33.99	−16.25	−1.69	−0.18	2.23
I	–	−0.06	0.13	−0.11	−0.03
IV	–	−0.02	−0.26	0.09	0.02

For finite damping the coefficient matrix is never singular; it may be inverted entirely in terms of real operations. For each positive harmonic index, the complex response vector $\{\mathbf{R}_m\}$ is determined by solving equation (9). The response vector for a negative harmonic index is the complex conjugate of that for the corresponding positive harmonic index, as can be shown using equation (9) and the properties of complex numbers. In terms of real quantities, the mth harmonic displacements are

$$\left\{ \begin{array}{c} \theta_p \\ \hline \theta_f \end{array} \right\}_m = 2 \text{ Real } \{\mathbf{R}_m\} \cos(m\Omega t) - 2 \text{ Imag } \{\mathbf{R}_m\} \sin(m\Omega t) \tag{10}$$

The particular solution, equation (6), could be assumed in a form displaying rigid body oscillations and twisting oscillations separately. The first involves oscillatory displacement without deformation, while the second requires deformation, and both motions do in fact occur. When formulated separately, at each harmonic, the two responses add to produce $\{\mathbf{R}_m\}$ as presented here, and they cannot be determined separately.

As presented here, a factor 0.5 is introduced in defining α_i (equation (4)) which is effectively cancelled by the factor 2.0 appearing in equation (10). The factor 0.5 was introduced to be consistent with the accepted relation between real and complex Fourier series coefficients [8]. This is also consistent with the steady twist calculation where no cancellation occurs. For numerical implementation, there may be a small advantage to deleting both factors in the oscillatory response calculation.

Examples

Single Shaft System. A single shaft system incorporating two free dampers is shown in Fig. 2. Numerical values for the system parameters are given in Table 1. Roman numeral subscripts are associated with the free dampers in both Fig. 2 and Table 1. The shaft rotation rate is 27.0 rad/s, which is also the fundamental excitation frequency. The steady-state response of this system is to be determined for the external excitation torques given by

$$\{\mathbf{T}_p(t)\} = \left\{ \begin{array}{c} -3.95 \\ 55.0 \\ 60.0 \\ 0.0 \\ -80.0 \end{array} \right\} + \left\{ \begin{array}{c} 0.0 \\ 20.0 \\ 25.0 \\ 0.0 \\ 0.0 \end{array} \right\} \cos(27t) + \left\{ \begin{array}{c} 0.0 \\ 15.0 \\ -20.0 \\ 0.0 \\ 0.0 \end{array} \right\} \sin(54t) \tag{11}$$

Table 3 System parameters for example 2

Station i	Moment of inertia J_i ($kg.m^2$)	Damper coefficients d_i (N.m.s)	c_i (N.m.s.)	Stiffness k_i (N.m.s./rad)
1	0.068	–	–	170000
2	0.068	–	–	290000
3	1.360	–	10.00	79000
4	2.260	0.230	–	90000
5	0.136	–	–	–
6	0.056	–	–	11000
7	0.045	–	–	11000
8	0.169	0.080	–	–
I	0.220	–	7.00	–

Gear ratio: $\mathbf{R}=N_3/N_6=0.5$

Similar to equation (6), but modified along the lines of equation (10), the response is expected to be of the form

$$\{\theta(t)\}=27t\{1\}+\{u_0\} + \{u_1\}\cos(27t)+\{v_1\}\sin(27t) + \{u_2\}\cos(54t)+\{v_2\}\sin(54t) \quad (12)$$

The response vectors, $\{u_i\}$ and $\{v_i\}$, have been computed as described above and are tabulated in Table 2.

Branched System. The branched system shown in Fig. 3 represents a two cylinder engine with flywheel driving a generator with a rotating exciter. A free damper is mounted at the free end of the crankshaft. The branch models a cam shaft and blower. The gear ratio from the main shaft to the branch is $\mathbf{R}=0.5$, as appropriate for a four-stroke engine. Engine speed is 1800 rpm for which ω is 188.5 rad/s. Expressed in N•m units, the nonzero external torques on the system are

$$\begin{aligned} T_1 &= 106.06+123.0\cos(188.5t)+235.0\sin(188.5t) \\ T_2 &= 106.06-123.0\cos(188.5t)-235.0\sin(188.5t) \quad (13) \\ T_4 &= -160.0 \\ T_8 &= -10.0 \end{aligned}$$

Assuming no compliance in the gearing, the equations of motion are of the form

$$[J]\{\ddot{\phi}\}+[C]\{\dot{\phi}\}+[K]\{\phi\}=\{\mathbf{T}\} \quad (14)$$

where the variables are

$$\{\phi\}=\mathrm{col}\,(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_7/\mathbf{R},\theta_8/\mathbf{R},\theta_T) \quad (15)$$

The excitation vector and coefficient matrices are

$$\{\mathbf{T}\}=\mathrm{col}\,(\mathbf{T}_1,\mathbf{T}_2,0,\mathbf{T}_4,0,0,\mathbf{R}\mathbf{T}_8,0) \quad (16)$$

$$[J]=\mathrm{diag}\,(J_1,J_2,J_3+\mathbf{R}^2J_6,J_4,J_5,\mathbf{R}^2J_7,\mathbf{R}^2J_8,J_I) \quad (17)$$

$$[C]=\begin{bmatrix} 7.00 & & & & & & & \\ 0.00 & 0.00 & & & \text{Symmetric} & & & \\ 0.00 & 0.00 & 10.00 & & & & & \\ 0.00 & 0.00 & -10.00 & 10.23 & & & & \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & & & \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & & \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.02 & \\ -7.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 7.00 \end{bmatrix} \quad (18)$$

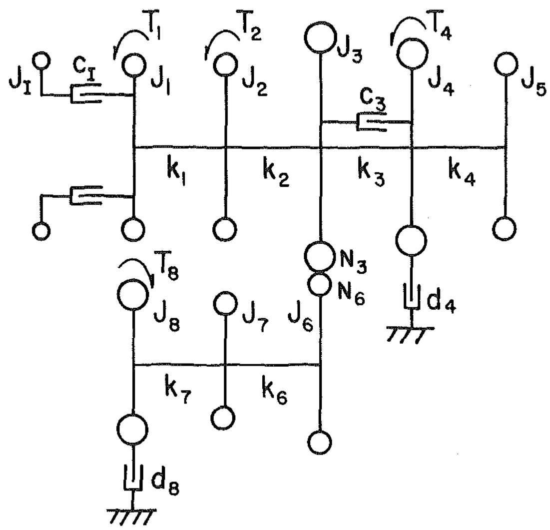

Fig. 3 Branched torsional system with damping

Table 4 Response vectors for example 2

Station i	$\{u_0\}$ (10^{-3}rad)	$\{u_1\}$ (10^{-5}rad)	$\{v_1\}$ (10^{-5}rad)
1	0.000	−72.263	143.100
2	0.624	0.129	2.100
3	−1.355	− 0.151	0.464
4	−3.929	− 0.009	−5.763
5	−3.929	− 0.010	−6.090
6	−0.670	− 0.076	0.232
7	−2.272	0.204	−0.673
8	−3.867	0.455	−1.478
I	–	−21.483	15.824

$$(10^{-3}[K]=\begin{bmatrix} 170 & & & & & & & \\ -170 & 460 & & & \text{Symmetric} & & & \\ 0 & -290 & 371.8 & & & & & \\ 0 & 0 & -79 & 169 & & & & \\ 0 & 0 & 0 & -90 & 90 & & & \\ 0 & 0 & -2.8 & 0 & 0 & 5.5 & & \\ 0 & 0 & 0 & 0 & 0 & -2.8 & 2.8 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \quad (19)$$

The coefficient of the free damper is evident in the corners of [C]. The branched structure of the problem is evident in the form of the stiffness matrix.

The steady state response, expressed in terms of physical rotations and neglecting the rigid body displacements, will be of the form

$$\{\theta\} = \{u_0\} + \{u_1\} \cos(188.5t) + \{v_1\} \sin(188.5t) \quad (20)$$

The response vectors $\{u_0\}$, $\{u_1\}$, and $\{v_1\}$ are tabulated in Table 4.

Conclusion

For free-free machine trains incorporating all of the common viscous damping actions, the rigorous description of steady-state torsional vibration has been formulated and solved in matrix form. The resulting solutions are well-suited to computer implementation. While the resulting coefficient matrices are too large for satisfactory manual computation, they are easily within the capabilities of today's computers. In obtaining these solutions, the entire motion has been addressed in a common framework, including both steady and oscillatory twists; the role of rigid body oscillation has been clarified. The analysis is further unified in that there is no need to consider resonant and nonresonant conditions separately as has been done in some previous analyses. Similarly, single shaft and branched systems are handled in the same manner; there is no distinction between them after the coefficient matrices have been defined.

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