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AN ANALYTICAL TREATMENT OF THE DELAYED FEEDBACK CONTROL ALGORITHM AT A SUBCRITICAL HOPF BIFURCATION

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We develop an analytical approach for the delayed feedback control algorithm applied to a dynamical system close to a subcritical Hopf bifurcation. A simple nonlinear electronic circuit is considered as a prototypical model of the subcritical Hopf bifurcation. The periodic orbit arising at this bifurcation is torsion free and cannot be controlled by the conventional delayed feedback algorithm. We show the necessity of employing an unstable degree of freedom in a feedback loop as well as a nonlinear coupling between the controlled system and controller. Close to the bifurcation point the system is weakly nonlinear and the problem is treated analytically using the method of averaging.

Keywords: chaos, dynamical systems, delayed feedback, subcritical Hopf bifurcation, analytical theory

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1. Introduction

Delayed feedback control (DFC) method [1] is used to stabilize unstable periodic orbits (UPOs) embedded in strange attractors of chaotic systems. The method uses a control signal composed as the difference between the current state of the controlled system and the state of the same system delayed by one period of the unstable periodic orbit. The method enables to treat the controlled system as a black box with an accessible signal in its output. We need to know *a priori* neither the location of the periodic orbit in the phase space nor the equations describing the controlled system. By guessing only the period of the unstable orbit the system under control automatically settles on the desired periodic motion, and stability of this motion is maintained with only tiny perturbations. Successful implementation of this algorithm has been attained in quite diverse experimental systems including electronic chaotic oscillators [2–5], mechanical pendulums [6, 7], lasers [8–10], gas discharge systems [11–13], a current-driven ion acoustic instability [14], a chaotic Taylor–Couette flow [15], chemical systems [16, 17], high-power ferromagnetic resonance [18], helicopter rotor blades [19], and a cardiac system [20].

A topological limitation of the delayed feedback control method has recently attracted much attention. It has been proven [21, 22] that the method fails for unstable periodic orbits with an odd number of real positive Floquet exponents (FEs). A similar limitation emerges in the simpler problem of adaptive stabilization of unknown steady states of dynamical systems [23]. To overcome this limitation an unstable delayed feedback control method was proposed [24].

The simplest situation giving rise to the topological limitation of the usual delayed feedback algorithm is the problem of stabilizing an UPO appearing in a dynamical system close to a subcritical Hopf bifurcation. We will consider this problem in the present paper. An unstable periodic orbit appearing at this bifurcation is torsion-free, hence we need an unstable controller. However, close to the bifurcation point the periodic orbit is only weakly unstable, and its stabilization is a relatively simple problem. Moreover, the problem can be treated analytically by means of standard asymptotic methods developed in the theory of weakly nonlinear oscillators.

2. Nonlinear circuit as a model of a subcritical Hopf bifurcation

The problem of controlling an unstable periodic orbit at a subcritical Hopf bifurcation can be considered in a general way, however, for the clarity of presentation we restrict ourselves to a specific example of dynamical system shown in Fig. 1.

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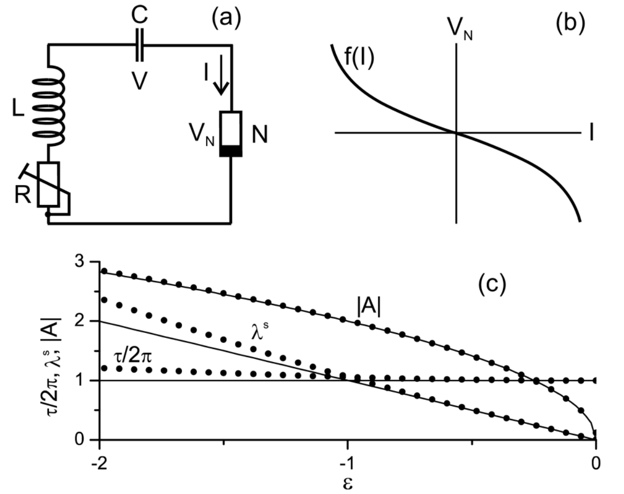

Fig. 1. (a) Circuit modelling a subcritical Hopf bifurcation. (b) Current versus voltage characteristic of the nonlinear element. (c) Amplitude $|A|$, period τ, and Floquet exponent λ^{s} of the unstable limit cycle as functions of the bifurcation parameter ε. Lines represent analytical results obtained from the averaged Eq. (5). Dots are the numerical results obtained from the exact Eqs. (2). The amplitude is defined as the maximum of the x variable on the limit cycle.

The system represents a nonlinear circuit described by

$$L\dot{I} = -IR - V - f(I)\,, \quad C\dot{V} = I\,. \tag{1}$$

Here I is the current and V is the voltage on the capacitor C. The function $f(I)$ describes the voltage versus current characteristic $V_N = f(I)$ of a nonlinear element N placed in a series with the LC circuit. We assume that this element has a negative differential resistivity and for small I can be approximated by the function $f(I) = -aI - bI^3 + O(I^5)$ with positive parameters a and b. Using the dimensionless variables $x = I/I_0$ and $y = V/V_0$, where $I_0 = \sqrt{\rho/3b}$, $V_0 = I_0\rho$, $\rho = \sqrt{L/C}$, and normalizing the time to the characteristic period $T = \sqrt{LC}$ of the LC circuit, Eqs. (1) are simplified to

$$\dot{x} = -y + \varepsilon x + \frac{x^3}{3}\,, \quad \dot{y} = x\,. \tag{2}$$

The only dimensionless parameter $\varepsilon = (a - R)/\rho$ can be easily controlled by varying the resistor R. The system (2) can be presented in the form $\ddot{x} + x - (\varepsilon + x^2)\dot{x} = 0$ similar to the well known van der Pol equation, with the only difference that the term $x^2\dot{x}$ comes here with a negative sign. For small ε, we apply the method of averaging to obtain an approximate solution of this equation.

Defining the complex amplitude $A(t)$ by putting

$$y = \frac{A\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}t}}{2}\,, \quad x = \frac{\mathrm{i}A\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A^*\mathrm{e}^{-\mathrm{i}t}}{2} \tag{3}$$

and inserting these into Eqs. (2) we get

$$\dot{A} = \frac{A}{8}(4\varepsilon + |A|^2) - \frac{A^*}{8}(4\varepsilon + |A|^2)\mathrm{e}^{-\mathrm{i}2t} - \frac{A}{24}\mathrm{e}^{\mathrm{i}2t} + \frac{A^*}{24}\mathrm{e}^{-\mathrm{i}4t}\,. \tag{4}$$

Close to the bifurcation point $\varepsilon = 0$, slow variations of the amplitude $A(t)$ can be determined by averaging Eq. (4) over the period of the fast oscillations $\tau = 2\pi$. This averaging is equivalent to neglecting the terms containing fast oscillations ($\mathrm{e}^{\pm\mathrm{i}t}$, $\mathrm{e}^{\pm\mathrm{i}2t}$, etc.). Thus the averaged equation for the amplitude reads:

$$\dot{A} = \frac{A}{8}(4\varepsilon + |A|^2)\,. \tag{5}$$

For $\varepsilon < 0$, this equation has two steady state solutions $A = 0$ and $|A| = 2\sqrt{-\varepsilon}$. The first represents a stable fixed point of the system at the origin $(x, y) = (0, 0)$, and the second corresponds to an unstable limit cycle with the period $\tau = 2\pi$, amplitude $2\sqrt{-\varepsilon}$, and a real positive Floquet exponent $\lambda^{\mathrm{s}} = -\varepsilon$. For $\varepsilon > 0$, the limit cycle disappears, and the fixed point at the origin becomes unstable. Thus at $\varepsilon = 0$ we have a subcritical Hopf bifurcation. As is seen from Fig. 1 (c), the analytical results obtained from averaged Eq. (5) are in a good quantitative agreement with numerical results determined from the exact Eqs. (2) when the system is in the vicinity of the bifurcation point.

3. Nonlinear delayed feedback controller

Now we assume that the current x is an observable accessible in experiment. To stabilize the unstable limit cycle appearing for $\varepsilon < 0$ we consider the following delayed feedback control algorithm:

$$\dot{x} = -y + \varepsilon x + \frac{x^3}{3} + wx\,, \tag{6a}$$

$$\dot{y} = x\,, \tag{6b}$$

$$\dot{w} = \lambda^{\mathrm{c}} w - k(x - x_\tau)x\,. \tag{6c}$$

The term wx in Eq. (6a) is the control perturbation introduced in the circuit as an additional voltage source. Equation (6c) describes an unstable delayed feedback controller with $\lambda^{\mathrm{c}} > 0$. Here w is a dynamical variable of the controller and k defines the feedback strength. We use the notation $x_\tau \equiv x(t - \tau)$. Note that the perturbation does not change the solution of the free system corresponding to the UPO of period τ, since for $x = x_\tau$

Eq. (6c) is satisfied by $w = 0$ and the perturbation wx in Eq. (6a) vanishes.

Unlike the control algorithm considered in Ref. [24] here we introduce nonlinear coupling terms, namely, the products wx and $(x - x_\tau)x$ in Eqs. (6a) and (6c), respectively. The nonlinearity is a necessary ingredient of the DFC algorithm when considering the stabilization of UPOs close to the bifurcation point. It is easy to verify that any linear coupling terms (e. g., w in Eq. (6a) and $x - x_\tau$ in Eq. (6c)) vanish due to the averaging procedure and thus result in uncoupled averaged equations for the slow dynamics of the controller and the controlled system. To provide an interrelation between these two subsystems in the averaged equations we need a nonlinear coupling in the original equations.

For small values of the parameters ε and λ^c, the averaged equations for the closed loop system are obtained by inserting Eqs. (3) into system (6) and neglecting the fast oscillating terms:

$$\dot{A} = \frac{A}{8}(4\varepsilon + |A|^2) + \frac{A}{2}w\,, \tag{7a}$$

$$\dot{w} = \lambda^c w - \frac{k}{4}(2|A|^2 - AA_\tau^* - A_\tau A^*)\,. \tag{7b}$$

Using the ansatz $A(t) = r(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$, from the imaginary part of Eq. (7a) it is easy to derive an equation for the phase, $r\dot{\varphi} = 0$. It follows that the phase is independent of time, $\varphi = constant$. For the slowly varying real amplitude $r(t)$ and controller variable $w(t)$ we obtain:

$$\dot{r} = \frac{r}{8}(4\varepsilon + r^2) + \frac{r}{2}w\,, \tag{8a}$$

$$\dot{w} = \lambda^c w - \frac{k}{2}r(r - r_\tau)\,. \tag{8b}$$

This system can be even more simplified. Taking into account that $r(t)$ is a slow variable the delay term r_τ can be approximated by the first derivative, $r_\tau = r(t-\tau) \approx r(t) - \tau\dot{r}$. This approximation is valid for $\tau|\dot{r}|/r \ll 1$. Then the time-delay system (8) transforms to a system of ordinary differential equations:

$$\dot{r} = \frac{r}{8}(4\varepsilon + r^2) + \frac{r}{2}w\,, \tag{9a}$$

$$\dot{w} = \lambda^c w - \frac{k}{2}\tau r\dot{r}\,. \tag{9b}$$

The eigenvalues λ of the fixed point $(r_0, w_0) = (2\sqrt{-\varepsilon}, 0)$ of this system satisfy the characteristic equation

$$\lambda^2 - (\lambda^c - \varepsilon + \varepsilon k\tau)\lambda - \varepsilon\lambda^c = 0\,. \tag{10}$$

They correspond to two leading nonzero FEs of the controlled UPO (The zero FE is defined by the equation for the phase $\dot{\varphi} = 0$ derived above). Note that the UPO satisfying the time-delay system (6) has an infinite number of FEs, and most of them are lost in this approximation. A more precise characteristic equation for the FEs can be derived from the averaged Eqs. (8) without using the approximation for the time-delay term r_τ. Linearization of Eqs. (8) around the fixed point $(r_0, w_0) = (2\sqrt{-\varepsilon}, 0)$ leads to the transcendental equation

$$\lambda^2 - (\lambda^c - \varepsilon)\lambda - \varepsilon\lambda^c - \varepsilon k(1 - \mathrm{e}^{-\lambda\tau}) = 0\,. \tag{11}$$

For $|\lambda|\tau \ll 1$, it coincides with Eq. (10) due to the approximation $\mathrm{e}^{-\lambda\tau} \approx 1 - \lambda\tau$. In Figs. 2 (a) and (c) we compare the FEs defined by Eqs. (10) and (11) with the 'exact' values of the FEs obtained numerically from the non-averaged variational equations

$$\delta\dot{x} = -\delta y + [\varepsilon + x_0^2(t)]\delta x + x_0(t)\delta w\,, \tag{12a}$$

$$\delta\dot{y} = \delta x\,, \tag{12b}$$

$$\delta\dot{w} = \lambda^c\delta w - kx_0(t)(\delta x - \delta x_\tau) \tag{12c}$$

derived from the original system (6). Here $\delta x = x - x_0(t)$, $\delta y = y - y_0(t)$ are small deviations from the periodic orbit $[x_0(t), y_0(t)] = [x_0(t+\tau), y_0(t+\tau)]$ that satisfies the free system (2), and $\delta w = w$.

For $|\varepsilon|\tau \ll 1$, all three above results are in good quantitative agreement (Fig. 2(a)). Thus the leading FEs of the controlled UPO can be reliably obtained from the simple quadratic Eq. (10). The stability conditions of this equation for $\varepsilon < 0$ are

$$\lambda^c > 0\,, \quad k > k_0 = \frac{\lambda^c - \varepsilon}{-\varepsilon\tau}\,. \tag{13}$$

The first condition confirms the general statement that the torsion-free UPOs can be stabilized only with an unstable controller. The second condition can be rewritten in the form $k\tau > 1 + \lambda^c/\lambda^s$, where λ^c is the eigenvalue of the free controller and $\lambda^s = -\varepsilon$ is the FE of the unstable limit cycle of the free system. The mechanism of stabilization is evident from Fig. 2 (b). For $k = 0$, two real positive solutions of Eq. (10) $\lambda = \lambda^s$ and $\lambda = \lambda^c$ describe unstable eigenvalues of the free system and the free controller, respectively. With increasing k, the eigenvalues approach each other on the real axis, then collide and pass to the complex plane. For $k = k_0$, they cross the imaginary axis and move symmetrically into the left half-plane, i. e. both the system and the controller become stable. An optimal value

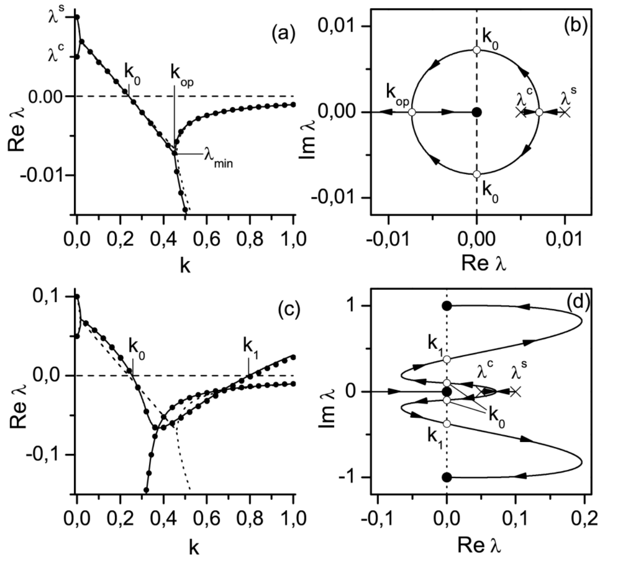

Fig. 2. (a) Real parts of leading Floquet exponents of the controlled UPO as functions of the control gain for $\varepsilon = -0.01$, $\lambda^c = 0.005$. Dotted and solid lines show the solutions of the characteristic equations (10) and (11), respectively. Dots correspond to the values of Floquet exponents obtained from the exact variational equations (12). (b) Root loci of Eq. (11) as k varies from 0 to ∞ for the same parameter value as in (a). Crosses and black dots denote the location of the roots for $k = 0$ and $k = \infty$, respectively. (c) and (d) are the same diagrams as in (a) and (b) but for $\varepsilon = -0.1$ and $\lambda^c = 0.05$.

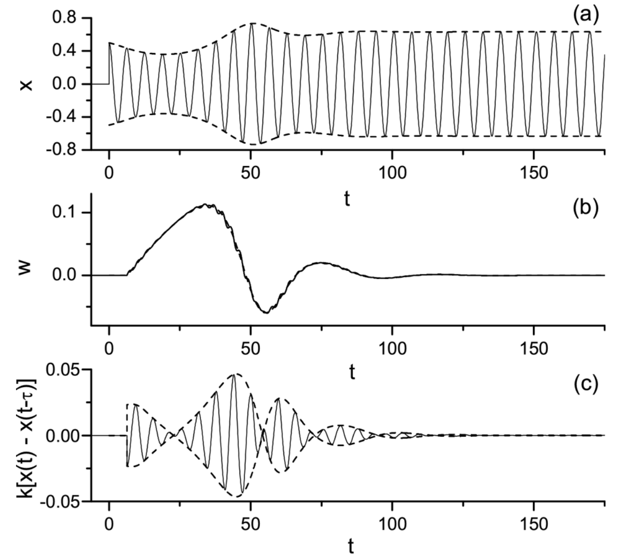

Fig. 3. Dynamics of (a) current x, (b) controller variable w, and (c) delayed feedback perturbation $k(x - x_\tau)$. Solid lines are the solutions of the nonlinear system (6) with initial conditions $x(t) = 0$ for $-\tau \le t < 0$, $x(0) = 0.5$, $y(0) = 0$, $w(0) = 0$. Dashed lines represent the solution of averaged Eqs. (8) with initial conditions $r(t) = 0$ for $-\tau \le t < 0$, $r(0) = 0.5$, $w(0) = 0$. The values of parameters are $\varepsilon = -0.1$, $\lambda^c = 0.05$, $\tau = 6.2871$, $k = 0$ for $t < \tau$, and $k = 0.35$ for $t > \tau$.

of the control gain is $k_{\rm op} = k_0 + 2\sqrt{\lambda^c/\lambda^s}/\tau$ since it provides the fastest convergence to the stabilized UPO with the characteristic rate $\lambda_{\min} = -\sqrt{\lambda^s\lambda^c}$.

For large values of $|\varepsilon|$, the root loci diagram is more complicated, see Fig. 2 (d). For $|\varepsilon|\tau \sim 1$ the approximation of the delay term r_τ by the derivative is not valid, however for $|\varepsilon| \ll 1$ we can use the averaged Eq. (8) as well as the transcendental characteristic Eq. (11). Figure 2 (c) shows that Eq. (11) indeed yields good quantitative results, while Eq. (10) is no longer valid. Now the eigenvalues due to the delay term come into play. As a result, there appears a second stability threshold k_1 such that the stabilization of the UPO becomes possible only in a finite interval of the control gain, $k_0 < k < k_1$.

Direct integration of the nonlinear Eqs. (6) confirms the results of linear analysis. Figure 3 shows the successful stabilization of the UPO close to the bifurcation point. After a transient process the controlled system approaches the previously unstable orbit, and the feedback perturbation vanishes. The envelopes of the transient are well described by averaged Eqs. (8). This confirms the validity of the averaging procedure applied to the time-delay system (6).

4. Conclusions

We have developed an analytical approach for the delayed feedback control of an unstable periodic orbit without torsion which could not be stabilized by conventional delay technique. The approach is demonstrated for the specific problem of a nonlinear electronic circuit as a prototypical model of a subcritical Hopf bifurcation. The analytical results are of general importance since they are applicable to any dynamical system close to the bifurcation point.

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ANALIZINIS VALDYMO ALGORITMO VĖLUOJANČIU GRĮŽTAMUOJU RYŠIU TYRIMAS ARTI SUBKRITINĖS HOPFO BIFURKACIJOS

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Santrauka

Nagrinėjame vėluojančio grįžtamojo ryšio metodo taikymą nestabiliai periodinei orbitai, atsirandančiai dinaminėje sistemoje arti subkritinės Hopfo bifurkacijos. Bifurkacijos taško aplinkoje sistema yra silpnai netiesinė ir uždavinys sprendžiamas analiziškai, naudojant suvidurkinimo metodą. Aptariame būtinybę įjungti papildomąjį nestabilųjį laisvės laipsnį į grįžtamojo ryšio grandinę bei panaudoti netiesinį ryšį tarp valdomos sistemos ir valdiklio. Analiziniam tyrimo būdui pademonstruoti buvo išnagrinėta paprasta netiesinė elektroninė grandinė, kuri modeliuoja subkritinę Hopfo bifurkaciją. Išplėtota analizinė teorija tinka bet kuriai dinaminei sistemai, kai ji yra arti subkritinės Hopfo bifurkacijos.