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SOME FORMULAS FOR APOSTOL-EULER POLYNOMIALS ASSOCIATED WITH HURWITZ ZETA FUNCTION AT RATIONAL ARGUMENTS

Qiu-Ming Luo

We give some explicit relationships between the Apostol-Euler polynomials and generalized Hurwitz-Lerch Zeta function and obtain some series representations of the Apostol-Euler polynomials of higher order in terms of the generalized Hurwitz-Lerch Zeta function. Several interesting special cases are also shown.

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and Euler polynomials $E_n^{(\alpha)}(x)$ of order α (real or complex) are usually defined by means of the following generating functions (see, for details, [1], [5], [13], [15]):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \qquad (|z| < 2\pi) \tag{1.1}$$

and

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{z^n}{n!} \qquad (|z| < \pi). \tag{1.2}$$

Obviously, the classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \qquad (n \in \mathbb{N}_0), \tag{1.3}$$

respectively. The classical Bernoulli numbers B_n and Euler numbers E_n are defined by

$$B_n := B_n(0) \quad \text{and} \quad E_n := 2^n E_n\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}_0), \tag{1.4}$$

respectively.

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Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol [2, p. 165, Eq. (3.1)] and (more recently) by Srivastava [14, p. 83-84]. We begin by recalling Apostol's definitions as follows:

Definition 1.1 (Apostol [2]; see also Srivastava [14]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ in x are defined by means of the generating function:*

$$\frac{ze^{xz}}{\lambda e^z-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{z^n}{n!} \tag{1.5}$$

$$(|z|<2\pi \text{ when } \lambda=1;\ |z|<|\log\lambda| \text{ when } \lambda\neq 1)$$

with, of course,

$$B_n(x)=\mathcal{B}_n(x;1) \qquad \textit{and} \qquad \mathcal{B}_n(\lambda):=\mathcal{B}_n(0;\lambda), \tag{1.6}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers (In fact, it is a function in λ).

Recently, Luo and Srivastava extended further the Apostol-Bernoulli and Apostol-Euler polynomials and their generalizations as follows:

Definition 1.2 (*cf.* Luo and Srivastava [10, 12]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the generating function*

$$\left(\frac{z}{\lambda e^z-1}\right)^{\alpha} e^{xz}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\ \frac{z^n}{n!} \tag{1.7}$$

$$(|z|<2\pi \text{ when } \lambda=1;\ |z|<|\log\lambda| \text{ when } \lambda\neq 1)$$

with, of course,

$$\begin{aligned} B_n^{(\alpha)}(x)=\mathcal{B}_n^{(\alpha)}(x;1) \qquad &\textit{and} \qquad \mathcal{B}_n^{(\alpha)}(\lambda):=\mathcal{B}_n^{(\alpha)}(0;\lambda),\\ \mathcal{B}_n(x;\lambda):=\mathcal{B}_n^{(1)}(x;\lambda) \qquad &\textit{and} \qquad \mathcal{B}_n(\lambda):=\mathcal{B}_n(0;\lambda), \end{aligned} \tag{1.8}$$

where $\mathcal{B}_n(\lambda)$, $\mathcal{B}_n^{(\alpha)}(\lambda)$ and $\mathcal{B}_n(x;\lambda)$ denote the so-called Apostol-Bernoulli numbers, Apostol-Bernoulli numbers of order α and Apostol-Bernoulli polynomials respectively.

Definition 1.3 (*cf.* Luo [11]). *The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the generating function*

$$\left(\frac{2}{\lambda e^z+1}\right)^{\alpha} e^{xz}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \qquad (|z|<|\log(-\lambda)|), \tag{1.9}$$

with, of course,

$$\begin{aligned} E_n^{(\alpha)}(x)=\mathcal{E}_n^{(\alpha)}(x;1) \qquad &\textit{and} \qquad \mathcal{E}_n^{(\alpha)}(\lambda):=2^n\mathcal{E}_n^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right),\\ \mathcal{E}_n(x;\lambda):=\mathcal{E}_n^{(1)}(x;\lambda) \qquad &\textit{and} \qquad \mathcal{E}_n(\lambda):=2^n\mathcal{E}_n\left(\frac{\alpha}{2};\lambda\right), \end{aligned} \tag{1.10}$$

where $\mathcal{E}_n(\lambda)$, $\mathcal{E}_n^{(\alpha)}(\lambda)$ and $\mathcal{E}_n(x;\lambda)$ denote the so-called Apostol-Euler numbers, Apostol-Euler numbers of order α and Apostol-Euler polynomials respectively.

The main object of the present paper is to give some explicit relationships between the Apostol-Euler polynomials and generalized Hurwitz-Lerch Zeta function and to investigate some series representations of the Apostol-Euler polynomials in terms of generalized Hurwitz-Lerch Zeta function.

2. Some explicit relationships between the Apostol-Euler polynomials and the generalized Hurwitz-Lerch Zeta function

A family of the Hurwitz-Lerch Zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ defined by (see *e.g.* [9, p. 727, Eq. (8)]

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s} \tag{2.1}$$

$$(\mu \in \mathbb{C};\ a,\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \rho,\sigma \in \mathbb{R}^+;\ \rho<\sigma \quad \text{when} \quad s,z\in\mathbb{C};$$
$$\rho=\sigma \quad \text{and} \quad s\in\mathbb{C} \quad \text{when} \quad |z|<1;\ \rho=\sigma \quad \text{and} \quad \Re(s-\mu+\nu)>1 \quad \text{when} \quad |z|=1),$$

contains, as its *special* cases, not only the Hurwitz-Lerch Zeta function

$$\Phi_{\nu,\nu}^{(\sigma,\sigma)}(z,s,a) = \Phi_{\mu,\nu}^{(0,0)}(z,s,a) = \Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{2.2}$$

and the Lipschitz-Lerch Zeta function (*cf.* [15, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi,a,s) := \Phi\left(e^{2\pi i\xi},s,a\right) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} \tag{2.3}$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \Re(s)>0 \quad \text{when} \quad \xi\in\mathbb{R}\setminus\mathbb{Z};\ \Re(s)>1 \quad \text{when} \quad \xi\in\mathbb{Z}\right),$$

but also the following generalized Hurwitz-Lerch Zeta functions introduced and studied earlier by Goyal and Laddha [7, p. 100, Eq. (1.5)]

$$\Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_\mu(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}, \tag{2.4}$$

which, for convenience, are called the *Goyal-Laddha-Hurwitz-Lerch Zeta function.* Here the symbol $(a)_k$ denotes the Pochhammer symbol or the shifted factorial defined, $a\in\mathbb{C}$, by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & (k=0) \\ a(a+1)\cdots(a+k-1) & (k\in\mathbb{N}), \end{cases} \tag{2.5}$$

where $\Gamma(x)$ is the usual Gamma function.

Recently, Garg *et al.* [6] obtained the following interesting formula:

$$\mathcal{B}_n^{(l)}(a;\lambda) = (-n)_l \Phi_l(\lambda, l-n, a) \qquad (n,l\in\mathbb{N};\ n\geqq l;\ |\lambda|<1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-). \tag{2.6}$$

Clearly, we have

$$\mathcal{B}_n(a;\lambda) = -n\Phi(\lambda,1-n,a) \qquad (n\in\mathbb{N};\ |\lambda|\leqq 1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-). \tag{2.7}$$

Below we give the following explicit relationships between the a family of Euler polynomials and a family of Zeta function.

THEOREM 2.1. *For* $n\in\mathbb{N};\ -1<\lambda\leqq 1;\ \alpha\in\mathbb{C};\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-$*, the following relationship*

$$\mathcal{E}_n^{(\alpha)}(a;\lambda) = 2^\alpha \Phi_\alpha(-\lambda,-n,a) \tag{2.8}$$

between the Apostol-Euler polynomials of higher order and the Goyal-Laddha-Hurwitz-Lerch Zeta function.

Proof. By (1.9) and the generalized binomial theorem, yields

(2.9)
$$\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(a;\lambda)\frac{z^n}{n!}=\left(\frac{2}{\lambda e^z+1}\right)^{\alpha}e^{az}=2^{\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-\lambda)^k e^{(k+a)z}$$
$$=\sum_{n=0}^{\infty}\left[2^{\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-\lambda)^k(k+a)^n\right]\frac{z^n}{n!}=\sum_{n=0}^{\infty}\left[2^{\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\frac{(-\lambda)^k}{(k+a)^{-n}}\right]\frac{z^n}{n!}.$$

Hence, the formula (2.8) follows. □

Corollary 2.2. *For $n\in\mathbb{N};\ -1<\lambda\leqq 1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-$, the following relationship*

$$\mathcal{E}_n(a;\lambda)=2\Phi(-\lambda,-n,a) \tag{2.10}$$

holds true between the Apostol-Euler polynomials and the Hurwitz-Lerch Zeta function.

It is well-known that the following relationship between the Bernoulli polynomials and the Hurwitz Zeta function (see Apostol [3, p. 264, Theorem 12.13])

$$B_n(a)=-n\zeta(1-n,a)\qquad(n\in\mathbb{N}), \tag{2.11}$$

where $\zeta(s,a)$ denotes the Hurwitz Zeta function defined by

$$\zeta(s,a):=\Phi(1,s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\qquad(\Re(s)>1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-).$$

An alternating series version of the Hurwitz Zeta function is given as follows:

Definition 2.3. *The L-function is defined by*

$$L(s,a):=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^s}\qquad\left(\Re(s)>1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-\right). \tag{2.12}$$

In the same method, it is not difficult, we give a quasi formula of (2.11) as follows:

Theorem 2.4. *For $n\in\mathbb{N};\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-$, the following relationship*

$$E_n(a)=2L(-n,a) \tag{2.13}$$

holds true between the Euler polynomials and the L-function.

It is well-known that the following relationship between the Bernoulli numbers and the Riemann Zeta function (see [3, p. 266, Theorem 12.16])

$$B_n=-n\zeta(1-n)\qquad(n\in\mathbb{N}), \tag{2.14}$$

where $\zeta(s)$ denotes the Riemann Zeta function defined by

$$\zeta(s):=\zeta(s,1)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

An alternating series version of the Riemann Zeta function is given as follows:

Definition 2.5. *For $\Re(s)>0$, the l-function defined by*

$$l(s):=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}. \tag{2.15}$$

Similarly, we give an analogue of the formula (2.14) as follows:

Theorem 2.6. *For $n \in \mathbb{N}$, the following relationship*

$$E_n = 2l(-n) \tag{2.16}$$

holds true between the Euler numbers and the l-function.

3. Explicit series representations for the Apostol-Euler polynomials of order α

It is not difficult, we make use of the elementary series identity:

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q} \sum_{k=0}^{\infty} f(qk+j), \qquad (q \in \mathbb{N}), \tag{3.1}$$

to the Hurwitz-Lerch Zeta function (2.2), yields that

$$\Phi(z,s,a) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{a+j-1}{q}\right) z^{j-1}. \tag{3.2}$$

Obviously, a special case of (3.2) when

$$z = \exp\left(\frac{2p\pi i}{q}\right) \qquad (p \in \mathbb{Z},\ q \in \mathbb{N})$$

is the following summation formula for the Lipschitz-Lerch Zeta function $\phi(\xi, a, s)$ defined by (2.3):

$$\begin{aligned} \Phi\left(\exp\left(\frac{2p\pi i}{q}\right), s, a\right) &= \phi\left(\frac{p}{q}, a, s\right) \\ &= q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right) \exp\left(\frac{2(j-1)p\pi i}{q}\right), \end{aligned} \tag{3.3}$$

in terms of the Hurwitz Zeta function $\zeta(s,a)$.

Theorem 3.1. *For $n, q \in \mathbb{N};\ p \in \mathbb{Z};\ \xi \in \mathbb{R},\ \alpha \in \mathbb{C}$, the following formula of the Apostol-Euler polynomials of order α*

$$\begin{aligned} \mathcal{E}_n^{(\alpha)}\left(\frac{p}{q}; e^{2\pi i \xi}\right) =& \frac{i\ 2^{\alpha} \cdot n!}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \binom{\alpha-1}{k} \left(k - \frac{p}{q} + 1\right)_{\alpha-k-1} \sum_{j=0}^{k} \binom{k-1}{j-1} (n+1)_j B_{k-j}^{(k)} \\ & \cdot (2\pi q)^{-n-j-1} \left\{ \sum_{r=1}^{q} \zeta\left(n+j+1, \frac{2\xi+2r-1}{2q}\right) \right. \\ & \cdot \exp\left[\left(\frac{n+j}{2} - \frac{(2\xi+2r-1)p}{2}\right)\pi i\right] - \sum_{r=1}^{q} \zeta\left(n+j+1, \frac{2r-2\xi+1}{2q}\right) \\ & \left. \cdot \exp\left[\left(-\frac{n+j}{2} + \frac{(2r-2\xi+1)p}{2}\right)\pi i\right] \right\}, \end{aligned} \tag{3.4}$$

holds true in terms of the Hurwitz Zeta function.

PROOF. We now rewrite the result of Lin *et al.* as follows (see [8, p. 823, Theorem]):

$$\begin{aligned}\Phi_\mu(z,s,a) =& iz^{-a}\Gamma(1-s)\sum_{k=0}^{\infty}\frac{(k-a+1)_{\mu-k-1}}{k!\Gamma(\mu-k)}\sum_{j=0}^{k}\binom{k-1}{j-1}(1-s)_j B_{k-j}^{(k)}(2\pi)^{s-j-1}\\ &\cdot\left[\exp\left(-\frac{1}{2}(s-j)\pi i\right)\Phi\left(e^{-2\pi a i},1-s+j,\frac{\log z}{2\pi i}\right)\right.\\ &\left.-\exp\left[\left(2a+\frac{1}{2}(s-j)\right)\pi i\right]\Phi\left(e^{2\pi a i},1-s+j,1-\frac{\log z}{2\pi i}\right)\right]\quad(\mu\in\mathbb{C}).\end{aligned}\tag{3.5}$$

Setting

$$z=-e^{2\pi i\xi},\qquad a=\frac{p}{q}\qquad\text{and}\qquad s\mapsto -s,\qquad \mu\mapsto\alpha$$

and by applying the series identity (3.3), we find that

$$\begin{aligned}\Phi_\alpha\left(-e^{2\pi i\xi},-s,\frac{p}{q}\right) =& \frac{i\Gamma(s+1)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\binom{\alpha-1}{k}\left(k-\frac{p}{q}+1\right)_{\alpha-k-1}\sum_{j=0}^{k}\binom{k-1}{j-1}(s+1)_j B_{k-j}^{(k)}\\ &\cdot(2\pi q)^{-s-j-1}\left\{\sum_{r=1}^{q}\zeta\left(s+j+1,\frac{2\xi+2r-1}{2q}\right)\right.\\ &\cdot\exp\left[\left(\frac{s+j}{2}-\frac{(2\xi+2r-1)p}{q}\right)\pi i\right]-\sum_{r=1}^{q}\zeta\left(s+j+1,\frac{2r-2\xi+1}{2q}\right)\\ &\left.\cdot\exp\left[\left(-\frac{s+j}{2}+\frac{(2r-2\xi+1)p}{q}\right)\pi i\right]\right\}\quad(\alpha\in\mathbb{C}).\end{aligned}\tag{3.6}$$

Taking $s=n$ in (3.6) and noting that (2.8) of Theorem 2.1, of course, with

$$\lambda=e^{2\pi i\xi}\qquad\text{and}\qquad a=\frac{p}{q},$$

we obtain the desire (3.4). This proof is complete. □

THEOREM 3.2. *For $n,q,l\in\mathbb{N}$; $p\in\mathbb{Z}$; $\xi\in\mathbb{C}$, the following formula of the Apostol-Euler polynomials of order l*

$$\begin{aligned}\mathcal{E}_n^{(l)}\left(\frac{p}{q};e^{2\pi i\xi}\right) =& \frac{i2^l\cdot n!}{(l-1)!}\sum_{k=0}^{l-1}\binom{l-1}{k}\left(k-\frac{p}{q}+1\right)_{l-k-1}\sum_{j=0}^{k}\binom{k-1}{j-1}(n+1)_j B_{k-j}^{(k)}\\ &\cdot(2\pi q)^{-n-j-1}\left\{\sum_{r=1}^{q}\zeta\left(n+j+1,\frac{2\xi+2r-1}{2q}\right)\right.\\ &\exp\left[\left(\frac{n+j}{2}-\frac{(2\xi+2r-1)p}{2}\right)\pi i\right]-\sum_{r=1}^{q}\zeta\left(n+j+1,\frac{2r-2\xi+1}{2q}\right)\\ &\left.\exp\left[\left(-\frac{n+j}{2}+\frac{(2r-2\xi+1)p}{2}\right)\pi i\right]\right\},\end{aligned}\tag{3.7}$$

holds true in terms of the Hurwitz Zeta function.

PROOF. Let $\alpha=l$ $(l\in\mathbb{N})$ in (3.6) we may obtain the assertion (3.7). □

THEOREM 3.3. *For $n, q, l \in \mathbb{N}$; $p \in \mathbb{Z}$; $\xi \in \mathbb{C}$, the following formula of the Apostol-Euler polynomials of order l*

(3.8)

$$\mathcal{E}_n^{(l)}\left(\frac{p}{q}; e^{2\pi i\xi}\right) = -\frac{i(-2)^l \cdot n!}{(l-1)!}\sum_{k=0}^{l-1}\binom{l-1}{k}B_{l-k-1}^{(l)}q^{-n-k-1}\sum_{j=0}^{k}\binom{k}{j}\binom{-n-1}{j}j!p^{k-j}(2\pi)^{-n-j-1}$$
$$\cdot\left\{\sum_{r=1}^{q}\zeta\left(n+j+1, \frac{2\xi+2r-1}{2q}\right)\exp\left[\left(\frac{n+j}{2}-\frac{(2\xi+2r-1)p}{q}\right)\pi i\right]\right.$$
$$\left.-\sum_{r=1}^{q}\zeta\left(n+j+1, \frac{2r-2\xi+1}{2q}\right)\exp\left[\left(-\frac{n+j}{2}+\frac{(2r-2\xi+1)p}{q}\right)\pi i\right]\right\},$$

holds true in terms of the Hurwitz Zeta function.

PROOF. Setting $\mu = m$ $(m \in \mathbb{N})$ in (3.5), we obtain the following transformation formula:

$$\Phi_m(z,s,a) = \frac{iz^{-a}\Gamma(1-s)}{(m-1)!}\sum_{k=0}^{m-1}\binom{m-1}{k}B_{m-k-1}^{(m)}$$
$$\cdot\sum_{j=0}^{k}(-1)^{m-k+j-1}\binom{k}{j}\binom{s-1}{j}j!(-a)^{k-j}(2\pi)^{s-j-1} \tag{3.9}$$
$$\cdot\left[\exp\left(\frac{1}{2}(s-j)\pi i\right)\Phi\left(e^{-2\pi ai}, 1-s+j, \frac{\log z}{2\pi i}\right)\right.$$
$$\left.-\exp\left[\left(2a+\frac{1}{2}(s-j)\right)\pi i\right]\Phi\left(e^{2\pi ai}, 1-s+j, 1-\frac{\log z}{2\pi i}\right)\right] \quad (m \in \mathbb{N}).$$

Letting

$$z = -e^{2\pi i\xi}, \qquad a = \frac{p}{q} \qquad \text{and} \qquad s \mapsto -s, \qquad m \mapsto l$$

and by applying the series identity (3.3), we obtain the following consequence

$$\Phi_l\left(-e^{2\pi i\xi}, -s, \frac{p}{q}\right)$$
$$= \frac{i(-1)^{l-1}\Gamma(s+1)}{(l-1)!}\sum_{k=0}^{l-1}\binom{l-1}{k}B_{l-k-1}^{(l)}q^{-s-k-1}\sum_{j=0}^{k}\binom{k}{j}\binom{-s-1}{j}j!p^{k-j}(2\pi)^{-s-j-1} \tag{3.10}$$
$$\cdot\left\{\sum_{r=1}^{q}\zeta\left(s+j+1, \frac{2\xi+2r-1}{2q}\right)\exp\left[\left(\frac{s+j}{2}-\frac{(2\xi+2r-1)p}{q}\right)\pi i\right]\right.$$
$$\left.-\sum_{r=1}^{q}\zeta\left(s+j+1, \frac{2r-2\xi+1}{2q}\right)\exp\left[\left(-\frac{s+j}{2}+\frac{(2r-2\xi+1)p}{q}\right)\pi i\right]\right\} \quad (l \in \mathbb{N}),$$

Further taking $s = n$ in (3.10) and noting that (2.8) of Theorem 2.1, of course, with

$$\lambda = e^{2\pi i\xi} \qquad \text{and} \qquad a = \frac{p}{q} \qquad (p \in \mathbb{Z};\ q \in \mathbb{N};\ \xi \in \mathbb{R}).$$

Therefore, the formula (3.8) follows. This proof is complete. □

4. Further observations and consequences

Recently, Srivastava found the following elegant formula for Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ (see [14, p. 84, Eq. (4.6)]):

$$\begin{aligned}\mathcal{B}_n\left(\frac{p}{q};e^{2\pi i\xi}\right) = -\frac{n!}{(2q\pi)^n}&\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{n}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right.\\ &\left.+\sum_{j=1}^{q}\zeta\left(n,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\} \quad (4.1)\\ &(n\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}).\end{aligned}$$

When $\xi\in\mathbb{Z}$ in (4.1), we can deduce a known result given earlier by Cvijović and Klinowski [4, p. 1529, Theorem A]:

$$B_n\left(\frac{p}{q}\right) = -\frac{2\cdot n!}{(2q\pi)^n}\sum_{j=1}^{q}\zeta\left(n,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right) \quad (4.2)$$

$$(n\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N}).$$

It follows that we set $\alpha=1$ in (3.4), or $l=1$ in (3.7) and (3.8). Then we obtain the following interesting formula for the Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$.

Theorem 4.1. *For $n,q\in\mathbb{N}$; $p\in\mathbb{Z}$; $\xi\in\mathbb{R}$, the following formula of the Apostol-Euler polynomials at rational arguments*

(4.3)
$$\begin{aligned}\mathcal{E}_n\left(\frac{p}{q};e^{2\pi i\xi}\right) =&\frac{2\cdot n!}{(2q\pi)^{n+1}}\left\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{2\xi+2j-1}{2q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{(2\xi+2j-1)p}{q}\right)\pi i\right]\right.\\ &\left.+\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned}$$

holds true in terms of the Hurwitz Zeta function.

A special case of formula (4.3) when $\xi\in\mathbb{Z}$, is just a known result given earlier by Cvijović and Klinowski:

Corollary 4.2 ([4, p. 1529, Theorem B]). *For $n,q\in\mathbb{N}$; $p\in\mathbb{Z}$, the following formula of the classical Euler polynomials*

$$E_n\left(\frac{p}{q}\right) = \frac{4\cdot n!}{(2q\pi)^{n+1}}\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)p\pi}{q}-\frac{n\pi}{2}\right), \quad (4.4)$$

holds true in terms of the Hurwitz Zeta function.

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