# Cooperative Distributed Model Predictive Control☆

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**Abstract**

In this paper we propose a cooperative distributed linear model predictive control strategy applicable to any finite number of subsystems with the following features: hard input constraints are satisfied; the distributed control provides nominal stability for the same set of plants as centralized control; terminating the iteration of the distributed controllers prior to convergence retains closed-loop stability; in the limit of iterating to convergence, the control is plantwide Pareto optimal and equivalent to the centralized control solution; no coordination layer is employed.

We first prove exponential stability of suboptimal model predictive control and show the proposed cooperative control strategy is in this class. We also establish that under perturbation from a stable state estimator, the origin remains exponentially stable. For plants with sparsely coupled input constraints, we provide an extension in which the decision variable space of each suboptimization is augmented to achieve Pareto optimality. We conclude with a simple example showing the performance advantage of cooperative control compared to noncooperative and decentralized control strategies.

*Key words:* distributed control, cooperative control, large scale control, model predictive control

## 1. Introduction

Model predictive control (MPC) has been widely adopted in the petrochemical industry for controlling large, multi-variable processes [1]. MPC solves an online optimization to determine inputs, taking into account the current conditions of the plant, any disturbances affecting operation, and imposed safety and physical constraints. Over the last several decades, MPC technology has reached a mature stage. Closed-loop properties are well understood, and nominal stability is easily demonstrated for many applications [2].

Chemical plants usually consist of linked unit operations and can be subdivided into a number of subsystems. These subsystems are connected through a network of material, energy, and information streams. Because plants often take advantage of the economic savings available in material recycle and energy integration, the plantwide interactions of the network are difficult to elucidate. Plantwide control has traditionally been implemented in a decentralized fashion, in which each subsystem is controlled independently and network interactions are treated as local subsystem disturbances [**?** 3]. It is well known, however, that when the inter-subsystem interactions are strong, decentralized control is unreliable [4].

Centralized control, in which all subsystems are controlled via a single agent, can account for the plantwide interactions. Indeed, increased computational power, faster optimization software, and algorithms designed specifically for large-scale plantwide control have made centralized control more practical [5**?** ]. Objections to centralized control, however, are often not computational but organizational. All subsystems rely upon the central agent, making plantwide control difficult to coordinate and maintain. These obstacles deter implementation of centralized control for large-scale plants.

As a middle ground between the decentralized and centralized strategies, distributed control preserves the topology and flexibility of decentralized control yet possesses stability properties. Stability is achieved by two features: the network interactions between subsystems are explicitly modeled and open-loop information, usually input trajectories, is exchanged between subsystem controllers. Distributed control strategies differ in the utilization of the open-loop information. In *noncooperative* distributed control each subsystem controller anticipates the effect of network interactions only locally [6**?** ]. These strategies are described as noncooperative dynamic games [7], and the plantwide performance converges to the Nash equilibrium. If network interactions are strong, however, noncooperative control can destabilize the plant and performance may be, in these cases, poorer than decentralized control [**?** ].

Alternatively, *cooperative* distributed control improves performance by requiring each subsystem to consider the effect of local control actions on all subsystems in the network

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[? ]. Each controller optimizes a plantwide objective function, e.g., the centralized controller objective. Distributed optimization algorithms are used to ensure a decrease in the plantwide objective at each intermediate iterate. Under cooperative control, plantwide performance converges to the Pareto optimum, providing centralized-like performance. Because the optimization may be terminated before convergence, cooperative control is a form of suboptimal control for the plantwide control problem. Hence, stability is deduced from suboptimal control theory [8].

Other recent work in large-scale control has focused on coordinating an underlying MPC structure. ? ] develop a coordinating MPC that controls the plant variables with the greatest impact on plant performance, then allow the other decentralized controllers to react to the coordinator MPC. In a series of papers, ? present a controller for networked, nonlinear subsystems [? ? ]. A stabilizing decentralized control architecture and a control Lyapunov function are assumed to exist. The performance is improved via a coordinating controller that perturbs the network controller, taking into account the closed-loop response of the network. ? ] propose a distributed MPC that relies on a centralized dual optimization. This coordinator has the advantage that it can handle coupling dynamics and constraints optimally; however, it must wait for convergence of the plantwide problem before it can provide an implementable input trajectory. Cooperative distributed MPC differs from these methods in that a coordinator is not necessary and suboptimal input trajectories may be used to stabilize the plant [see ? ].

In this paper, we state and prove the stability properties for cooperative distributed control under state and output feedback. In Section 2, we provide relevant theory for suboptimal control. Section 3 provides stability theory for cooperative control under state feedback. For ease of exposition, we introduce the theorems for the case of two controllers only. Section 4 extends these results to the output feedback case. The results are modified to the case of coupled input constraints in Section 5. We then show how stability extends to the case of any finite number of controllers. We conclude with an example comparing performance of cooperative control with other plantwide control strategies.

*Notation.* Given a vector $x \in \mathbb{R}^n$ the symbol $|x|$ indicates the Euclidean 2-norm; given a positive scalar $r$ the symbol $\mathbb{B}_r$ indicates a closed ball of radius $r$ centered at the origin, i.e. $\mathbb{B}_r = \{x \in \mathbb{R}^n, |x| \leq r\}$. Given two integers, $l \leq m$, we define the set $\mathbb{I}_{[l,m]} = \{l, l+1, \ldots, m-1, m\}$. The set of positive reals is denoted $\mathbb{R}_+$. Given an input sequence $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\} \in \mathbb{R}^{Nm}$ and the input constraints $u(k) \in \mathbb{U}$, $k \in \mathbb{I}_{[0,N-1]}$, in which $\mathbb{U}$ is compact, convex, and contains the origin in its interior. To compress notation we use $\mathbf{u} \in \mathbb{U}$ in addition to $\mathbf{u} \in \mathbb{U}^N$ to indicate that the constraints apply to each element of the sequence. The symbol $'$ indicates the transpose.

## 2. Suboptimal Model Predictive Control

In this section, we provide the definitions and theory of suboptimal MPC necessary for proving stability of cooperative MPC. Waiting for distributed MPC strategies to converge is equivalent to implementing centralized MPC with the optimization distributed over many processors. These strategies attain their full impact only when we allow termination prior to convergence, in which case they behave as a centralized suboptimal MPC.

We define the current state of the system as $x \in \mathbb{R}^n$, the trajectory of inputs $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\} \in \mathbb{R}^{Nm}$, and the state and input at time $k$ as $(x(k), u(k))$. For the latter, we often abbreviate the notation to $(x, u)$. Denote the input constraints $\mathbf{u} \in \mathbb{U}$, and denote $\mathscr{X}_N$ as the set of all $x$ for which there exists a feasible $\mathbf{u}$. Initialized with a feasible input trajectory $\tilde{\mathbf{u}}$, the controller performs $p$ iterations of a feasible path algorithm and computes $\mathbf{u}$ such that some performance metric is improved. At each sample time, the first input in the (suboptimal) trajectory is applied, $u = u(0)$. The state is updated by the state evolution equation $x^+ = f(x, u)$, in which $x^+$ is the state at the next iterate.

For any initial state $x(0)$, we initialize the suboptimal MPC with a feasible input sequence $\tilde{\mathbf{u}}(0) = \mathbf{h}(x(0))$ with $\mathbf{h}(\cdot)$ continuous. For subsequent decision times, we denote $\tilde{\mathbf{u}}^+$ as the *warm start*, a feasible input sequence for $x^+$ used to initialize the suboptimal MPC algorithm. Here, we set $\tilde{\mathbf{u}}^+ = \{u(1), \ldots, u(N-1), 0\}$. This sequence is obtained by discarding the first input, shifting the rest of the sequence forward one step and setting the last input to zero.

We observe that the input sequence at termination $\mathbf{u}^+$ is a function of the state initial condition $x^+$ and of the warm start $\tilde{\mathbf{u}}^+$. Noting that $x^+$ and $\tilde{\mathbf{u}}^+$ are both functions of $x$ and $\mathbf{u}$, the input sequence $\mathbf{u}^+$ can be expressed as a function of only $(x, \mathbf{u})$ by $\mathbf{u}^+ = g(x, \mathbf{u})$. We refer to the function $g$ as the iterate update.

Given a system $x^+ = f(x)$, with equilibrium point at the origin $0 = f(0)$, denote with $\phi(k, x(0))$ the solution $x(k)$ given the initial state $x(0)$. We consider the following definition.

**Definition 1** (Exponential stability on a set $\mathbb{X}$)**.** The origin is exponentially stable on the set $\mathbb{X}$ if for all $x(0) \in \mathbb{X}$, the solution $\phi(k, x(0)) \in \mathbb{X}$ and there exists $\alpha > 0$ and $0 < \gamma < 1$ such that

$$\left|\phi(k, x(0))\right| \leq \alpha\, |x(0)|\, \gamma^k$$

for all $k \geq 0$.

The following lemma is an extension of [8, Theorem 1] for exponential stability.

**Lemma 2** (Exponential stability of suboptimal MPC)**.** *Consider a system*

$$\begin{pmatrix} x^+ \\ \mathbf{u}^+ \end{pmatrix} = \begin{pmatrix} F(x,\mathbf{u}) \\ g(x,\mathbf{u}) \end{pmatrix} = \begin{pmatrix} f(x,u) \\ g(x,\mathbf{u}) \end{pmatrix} \qquad (x(0), \mathbf{u}(0)) \text{ given} \tag{2.1}$$

*with a steady-state solution* $(0,0) = (f(0,0), g(0,0))$. *Assume that the function* $V(\cdot): \mathbb{R}^n \times \mathbb{R}^{Nm} \to \mathbb{R}_+$ *and input trajectory* $\mathbf{u}$

*satisfy*

$$a|(x,\mathbf{u})|^2 \le V(x,\mathbf{u}) \le b|(x,\mathbf{u})|^2 \tag{2.2a}$$
$$V(x^+,\mathbf{u}^+) - V(x,\mathbf{u}) \le -c|(x,u(0))|^2 \tag{2.2b}$$
$$|\mathbf{u}| \le d|x| \quad x \in \mathbb{B}_r \tag{2.2c}$$

*in which $a,b,c,d,r>0$. If $\mathscr{X}_N$ is forward invariant for the system $x^+ = f(x,u)$, the origin is exponentially stable for all $x(0) \in \mathscr{X}_N$.*

Notice in the second inequality (2.2b), only the first input appears in the norm $|(x,u(0))|^2$. In the sequel, this norm is used as a bound for the stage cost. Note also that the last inequality applies only for $x$ in a ball of radius $r$, which may be chosen arbitrarily small.

*Proof of Lemma 2.* First we establish the origin of extended system (2.1) is exponentially stable for all $(x(0),\mathbf{u}(0)) \in \mathscr{X}_N \times \mathbb{U}$. For $x \in \mathbb{B}_r$, we have $|\mathbf{u}| \le d|x|$. Consider the optimization

$$s = \max_{\mathbf{u}\in\mathbb{U}} |\mathbf{u}|$$

The solution exists by the Weierstrass theorem since $\mathbb{U}$ is compact and by definition we have that $s>0$. Then we have $|\mathbf{u}| \le (s/r)|x|$ for $x \notin \mathbb{B}_r$. Therefore, for all $x \in \mathscr{X}_N$, we have $|\mathbf{u}| \le \bar{d}|x|$ in which $\bar{d} = \max(d, s/r)$, and

$$|(x,\mathbf{u})| \le |x| + |\mathbf{u}| \le (1+\bar{d})|x| \le (1+\bar{d})|(x,u(0))|$$

which gives $|(x,u(0))| \ge \bar{c}|(x,\mathbf{u})|$ with $\bar{c} = 1/(1+\bar{d}) > 0$. Therefore the extended state $(x,\mathbf{u})$ satisfies

$$V(x^+,\mathbf{u}^+) - V(x,\mathbf{u}) \le -\tilde{c}|(x,\mathbf{u})|^2 \quad (x,\mathbf{u}) \in \mathscr{X}_N \times \mathbb{U} \tag{2.3}$$

in which $\tilde{c} = c(\bar{c})^2$. Together with (2.2), (2.3) establishes that $V(\cdot)$ is a Lyapunov function of the extended state $(x,\mathbf{u})$ for all $x \in \mathscr{X}_N$ and $\mathbf{u} \in \mathbb{U}$. Hence for all $(x(0),\mathbf{u}(0)) \in \mathscr{X}_N \times \mathbb{U}$ and $k \ge 0$, we have

$$|(x(k),\mathbf{u}(k))| \le \alpha|(x(0),\mathbf{u}(0))|\gamma^k$$

in which $\alpha > 0$ and $0 < \gamma < 1$. Notice that $\mathscr{X}_N \times \mathbb{U}$ is forward invariant for the extended system (2.1).

Finally, we remove the input sequence and establish that the origin is exponentially stable for the closed-loop system $x^+ = f(x,u)$. We have for all $x(0) \in \mathscr{X}_N$ and $k \ge 0$

$$\begin{aligned}|x(k)| &\le |(x(k),\mathbf{u}(k))| \le \alpha|(x(0),\mathbf{u}(0))|\gamma^k \\ &\le \alpha(|x(0)| + |\mathbf{u}(0)|)\gamma^k \le \alpha(1+\bar{d})|x(0)|\gamma^k \\ &\le \bar{\alpha}|x(0)|\gamma^k\end{aligned}$$

in which $\bar{\alpha} = \alpha(1+\bar{d}) > 0$, and we have established exponential stability of the origin by observing that $\mathscr{X}_N$ is forward invariant for the closed-loop system $x^+ = f(x,u)$. □

*Remark* 1. For Lemma 2, we use the fact that $\mathbb{U}$ is compact. For unbounded $\mathbb{U}$, however, exponential stability may instead be established by compactness of $\mathscr{X}_N$.

## 3. Cooperative Model Predictive Control

We now show cooperative MPC is a form of suboptimal MPC and prove stability. To simplify the exposition and proofs, in Sections 3-5 we assume the plant consists of only two subsystems. We establish in Section 6, however, that the results extend to any finite number of subsystems.

### 3.1. Definitions

#### 3.1.1. Models

We assume for each subsystem $i$, there exist a collection of linear models that denote the effect of inputs of subsystem $j$ on the states of subsystem $i$ for all $(i,j) \in \mathbb{I}_{[1,2]} \times \mathbb{I}_{[1,2]}$

$$x_{ij}^+ = A_{ij}x_{ij} + B_{ij}u_j$$

in which $x_{ij} \in \mathbb{R}^{n_{ij}}$, $u_j \in \mathbb{R}^{m_j}$, $A_{ij} \in \mathbb{R}^{(n_{ij}\times n_{ij})}$, and $B_{ij} \in \mathbb{R}^{(n_{ij}\times m_j)}$. For a discussion of identification of this model choice, see [9]. Considering subsystem 1, we collect the states to form

$$\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}^+ = \begin{bmatrix} A_{11} & \\ & A_{12} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ B_{12} \end{bmatrix} u_2$$

which denotes the model for subsystem 1. To ease notation, we define the equivalent model

$$x_1^+ = A_1x_1 + \bar{B}_{11}u_1 + \bar{B}_{12}u_2$$

for which

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \quad A_1 = \begin{bmatrix} A_{11} & \\ & A_{12} \end{bmatrix} \quad \bar{B}_{11} = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \quad \bar{B}_{12} = \begin{bmatrix} 0 \\ B_{12} \end{bmatrix}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $A_1 \in \mathbb{R}^{(n_1\times n_1)}$, and $\bar{B}_{1j} \in \mathbb{R}^{(n_1\times m_j)}$ with $n_1 = n_{11} + n_{12}$. Forming a similar model for subsystem 2, the plantwide model is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} u_2$$

We further simplify the plantwide model

$$x^+ = Ax + B_1u_1 + B_2u_2$$

for which

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \quad B_1 = \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} \quad B_2 = \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix}$$

#### 3.1.2. Objective Functions

Consider subsystem 1, for which we define the quadratic stage cost and terminal penalty, respectively

$$\ell_1(x_1,u_1) = \frac{1}{2}(x_1'Q_1x_1 + u_1'R_1u_1) \tag{3.1a}$$
$$V_{1f}(x_1) = \frac{1}{2}x_1'P_{1f}x_1 \tag{3.1b}$$

in which $Q_1 = \mathrm{diag}(Q_{11}, Q_{12}) \in \mathbb{R}^{(n_1 \times n_1)}$, $R_1 \in \mathbb{R}^{(m_1 \times m_1)}$, and $P_{1f} \in \mathbb{R}^{(n_1 \times n_1)}$. We define the objective function for subsystem 1

$$V_1\big(x_1(0), \mathbf{u}_1, \mathbf{u}_2\big) = \sum_{k=0}^{N-1} \ell_1\big(x_1(k), u_1(k)\big) + V_{1f}\big(x_1(N)\big)$$

Notice $V_1$ is implicitly a function of both $\mathbf{u}_1$ and $\mathbf{u}_2$ because $x_1$ is a function of both $u_1$ and $u_2$. For subsystem 2, we similarly define an objective function $V_2$. We define the plantwide objective function

$$V\big(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2\big) = \rho_1 V_1\big(x_1(0), \mathbf{u}_1, \mathbf{u}_2\big) + \rho_2 V_2\big(x_2(0), \mathbf{u}_1, \mathbf{u}_2\big)$$

in which $\rho_1, \rho_2 > 0$ are relative weights. For notational simplicity, we write $V(x, \mathbf{u})$ for the plant objective.

#### 3.1.3. *Constraints*

We require the inputs satisfy

$$u_1(k) \in \mathbb{U}_1 \qquad u_2(k) \in \mathbb{U}_2 \qquad k \in \mathbb{I}_{[0,N-1]}$$

in which $\mathbb{U}_1$ and $\mathbb{U}_2$ are compact and convex such that 0 is in the interior of $\mathbb{U}_i$ $\forall i \in \mathbb{I}_{[1,2]}$.

*Remark* 2. The constraints are termed *uncoupled* because the feasible region of $\mathbf{u}_1$ is not affected by $\mathbf{u}_2$, and vice-versa.

#### 3.1.4. *Assumptions*

The following assumptions are used to establish stability.

**Assumption 3.** For all $(i, j) \in \mathbb{I}_{[1,2]} \times \mathbb{I}_{[1,2]}$

(a) The systems $(A_{ij}, B_{ij})$ are stabilizable.
(b) The input penalties $R_i > 0$.
(c) The state penalties $Q_i \geq 0$.
(d) The systems $(A_{ij}, Q_{ij})$ are detectable.
(e) $N \geq \max_{j \in \mathbb{I}_{[1,2]}} (\sum_{i \in \mathbb{I}_{[1,2]}} n^u_{ij})$, in which $n^u_{ij}$ is the number of unstable modes of $A_{ij}$, i.e., number of $\lambda \in \mathrm{eig}(A_{ij})$ such that $|\lambda| \geq 1$.

The assumption 3(e) is required so that the horizon $N$ is sufficiently large to zero the unstable modes.

#### 3.1.5. *Unstable Modes*

For an unstable plant, we constrain the unstable modes to zero at the end of the horizon to maintain closed-loop stability. To construct this constraint, consider the real Schur decomposition of $A_{ij}$ for each $(i, j) \in \mathbb{I}_{[1,2]} \times \mathbb{I}_{[1,2]}$

$$A_{ij} = \begin{bmatrix} S^s_{ij} & S^u_{ij} \end{bmatrix} \begin{bmatrix} A^s_{ij} & - \\ & A^u_{ij} \end{bmatrix} \begin{bmatrix} {S^s_{ij}}' \\ {S^u_{ij}}' \end{bmatrix} \tag{3.2}$$

in which $A^s_{ij}$ is stable and $A^u_{ij}$ has all unstable eigenvalues. Let $\Sigma_{ij}$ denote the solution of the Lyapunov equation

$${A^s_{ij}}' \Sigma_{ij} A^s_{ij} - \Sigma_{ij} = -{S^s_{ij}}' Q_{ij} S^s_{ij} \tag{3.3}$$

#### 3.1.6. *Terminal Penalty*

Given the definition of the Schur decomposition (3.2), we define the matrices

$$S^s_i = \mathrm{diag}(S^s_{i1}, S^s_{i2}) \qquad A^s_i = \mathrm{diag}(A^s_{i1}, A^s_{i2}) \quad \forall i \in \mathbb{I}_{[1,2]} \tag{3.4a}$$
$$S^u_i = \mathrm{diag}(S^u_{i1}, S^u_{i2}) \qquad A^u_i = \mathrm{diag}(A^u_{i1}, A^u_{i2}) \quad \forall i \in \mathbb{I}_{[1,2]} \tag{3.4b}$$

**Lemma 4.** *The matrices (3.4) satisfy the Schur decompositions*

$$A_i = \begin{bmatrix} S^s_i & S^u_i \end{bmatrix} \begin{bmatrix} A^s_i & - \\ & A^u_i \end{bmatrix} \begin{bmatrix} {S^s_i}' \\ {S^u_i}' \end{bmatrix} \qquad \forall i \in \mathbb{I}_{[1,2]}$$

We further define the matrices

$$\Sigma_i = \mathrm{diag}(\Sigma_{i1}, \Sigma_{i2}) \quad \forall i \in \mathbb{I}_{[1,2]} \tag{3.5}$$

**Lemma 5.** *The matrices (3.5) satisfy the Lyapunov equations*

$${A^s_1}' \Sigma_1 A^s_1 - \Sigma_1 = -{S^s_1}' Q_1 S^s_1 \qquad {A^s_2}' \Sigma_2 A^s_2 - \Sigma_2 = -{S^s_2}' Q_2 S^s_2$$

We then choose the terminal penalty for each subsystem to be the cost to go under zero control, such that

$$P_{1f} = S^s_1 \Sigma_1 {S^s_1}' \qquad\qquad P_{2f} = S^s_2 \Sigma_2 {S^s_2}' \tag{3.6}$$

#### 3.1.7. *Cooperative Model Predictive Control Algorithm*

Let $\boldsymbol{v}^0$ be the initial condition for the cooperative MPC algorithm (see Section 3.2 for the discussion of initialization). At each iterate $p \geq 0$, the following optimization problem is solved for subsystem $i$, $i \in \mathbb{I}_{[1,2]}$

$$\min_{\boldsymbol{v}_i} V(x_1(0), x_2(0), \boldsymbol{v}_1, \boldsymbol{v}_2) \tag{3.7a}$$

subject to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} v_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} v_2 \tag{3.7b}$$
$$\boldsymbol{v}_i \in \mathbb{U}_i \tag{3.7c}$$
$${S^u_{ji}}' x_{ji}(N) = 0 \quad j \in \mathbb{I}_{[1,2]} \tag{3.7d}$$
$$|\boldsymbol{v}_i| \leq d_i \sum_{j \in \mathbb{I}_{[1,2]}} \big| x_{ji}(0) \big| \quad \text{if } x_{ji}(0) \in \mathbb{B}_r \quad \forall j \in \mathbb{I}_{[1,2]} \tag{3.7e}$$
$$\boldsymbol{v}_j = \boldsymbol{v}^p_j \quad j \in \mathbb{I}_{[1,2]} \setminus i \tag{3.7f}$$

in which we include the hard input constraints, the stabilizing constraint on the unstable modes, and the Lyapunov stability constraint. We denote the solutions to these problems as

$$\boldsymbol{v}^*_1(x_1(0), x_2(0), \boldsymbol{v}^p_2), \qquad \boldsymbol{v}^*_2(x_1(0), x_2(0), \boldsymbol{v}^p_1)$$

Given the prior, feasible iterate $(\boldsymbol{v}^p_1, \boldsymbol{v}^p_2)$, the next iterate is defined to be

$$(\boldsymbol{v}^{p+1}_1, \boldsymbol{v}^{p+1}_2) = w_1\Big(\boldsymbol{v}^*_1(\boldsymbol{v}^p_2), \boldsymbol{v}^p_2\Big) + w_2\Big(\boldsymbol{v}^p_1, \boldsymbol{v}^*_2(\boldsymbol{v}^p_1)\Big) \tag{3.8}$$
$$w_1 + w_2 = 1, \quad w_1, w_2 > 0$$

for which we omit the state dependence of $\boldsymbol{v}^*_1$ and $\boldsymbol{v}^*_2$ to reduce notation. This distributed optimization is of the Gauss-Jacobi type [see **?** , pp.219–223]. At the last iterate $\bar{p}$, we set $\mathbf{u} \leftarrow (\boldsymbol{v}^{\bar{p}}_1, \boldsymbol{v}^{\bar{p}}_2)$ and inject $u(0)$ into the plant.

The following properties follow immediately.

**Lemma 6** (Feasibility)**.** *Given a feasible initial guess, the iterates satisfy*

$$(\boldsymbol{v}_1^p, \boldsymbol{v}_2^p) \in \mathbb{U}_1 \times \mathbb{U}_2$$

*for all* $p \geq 1$.

**Lemma 7** (Convergence)**.** *The cost* $V(x(0), \boldsymbol{v}^p)$ *is nonincreasing for each iterate* $p$ *and converges as* $p \to \infty$.

**Lemma 8** (Optimality)**.** *As* $p \to \infty$ *the cost* $V(x(0), \boldsymbol{v}^p)$ *converges to the optimal value* $V^0(x(0))$, *and the iterates* $(\boldsymbol{v}_1^p, \boldsymbol{v}_2^p)$ *converge to* $(\mathbf{u}_1^0, \mathbf{u}_2^0)$ *in which* $\mathbf{u}^0 = (\mathbf{u}_1^0, \mathbf{u}_2^0)$ *is the Pareto (centralized) optimal solution.*

See proofs in Appendix.

*Remark* 3. This paper presents the distributed optimization algorithm with subproblem (3.7) and iterate update (3.8) so that the Lemmas 6–8 are satisfied. This choice, however, is nonunique and other optimization methods may exist satisfying these properties.

### 3.2. *Stability of Cooperative Model Predictive Control*

We define the steerable set $\mathcal{X}_N$ as the set of all $x$ such that there exists a $\mathbf{u} \in \mathbb{U}$ satisfying (3.7d).

**Assumption 9.** Given $r > 0$, for all $i \in \mathbb{I}_{[1,2]}$, $d_i$ is chosen large enough such that there exists a $\mathbf{u}_i \in \mathbb{U}$ satisfying $|\mathbf{u}_i| \leq d_i \sum_{j \in \mathbb{I}_{[1,2]}} |x_{ij}|$ and (3.7d) for all $x_{ij} \in \mathbb{B}_r \ \forall j \in \mathbb{I}_{[1,2]}$.

*Remark* 4. Given Assumption 9, $\mathcal{X}_N$ is forward invariant.

We now show the stability of the closed-loop system by treating cooperative MPC as a form of suboptimal MPC. We define the warm start for each subsystem as

$$\tilde{\mathbf{u}}_1^+ = \{u_1(1), u_1(2), \ldots, u_1(N-1), 0\}$$
$$\tilde{\mathbf{u}}_2^+ = \{u_2(1), u_2(2), \ldots, u_2(N-1), 0\}$$

The warm start $\tilde{\mathbf{u}}_i^+$ is used as the initial condition for the cooperative MPC problem in each subsystem $i$. We define the functions $g_1^p$ and $g_2^p$ as the outcome of applying the cooperative control iteration (3.8) $p$ times

$$\mathbf{u}_1^+ = g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \qquad \mathbf{u}_2^+ = g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2)$$

The system evolution is then given by

$$\begin{pmatrix} x_1^+ \\ x_2^+ \\ \mathbf{u}_1^+ \\ \mathbf{u}_2^+ \end{pmatrix} = \begin{pmatrix} A_1 x_1 + \bar{B}_{11} u_1 + \bar{B}_{12} u_2 \\ A_2 x_2 + \bar{B}_{21} u_1 + \bar{B}_{22} u_2 \\ g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \\ g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \end{pmatrix}$$

for which we simplify into

$$\begin{pmatrix} x^+ \\ \mathbf{u}^+ \end{pmatrix} = \begin{pmatrix} Ax + B_1 u_1 + B_2 u_2 \\ g^p(x, \mathbf{u}) \end{pmatrix}$$

**Theorem 10** (Exponential stability)**.** *Given Assumptions 3 and 9, the origin* $(x = 0)$ *of the closed-loop system* $x^+ = Ax + B_1 u_1 + B_2 u_2$ *is exponentially stable on the set* $\mathcal{X}_N$.

*Proof.* First we show that $V(\cdot)$ satisfies (2.2a). By the definition of $\ell_i(\cdot)$, we can write $V(x, \mathbf{u})$ in the form $|\mathcal{A}x|^2_{\mathcal{Q}} + |\mathcal{B}\mathbf{u}|^2_{\mathcal{R}}$ by eliminating the states in $V(\cdot)$. Defining $\mathcal{H} = \operatorname{diag}(\mathcal{A}'\mathcal{Q}\mathcal{A}, \mathcal{B}'\mathcal{R}\mathcal{B}) > 0$, and choosing $a = \min_i(\lambda_i(\mathcal{H}))$ and $b = \max_i(\lambda_i(\mathcal{H}))$ satisfies (2.2a). Next we show $V(\cdot)$ satisfies (2.2b). Using the warm start at the next sample time, we have the following cost

$$\begin{aligned} V(x^+, \tilde{\mathbf{u}}^+) &= V(x, \mathbf{u}) - \rho_1 \ell_1(x_1, u_1) - \rho_2 \ell_2(x_2, u_2) \\ &+ \frac{1}{2}\rho_1 x_1(N)' \Big(A_1' P_{1f} A_1 - P_{1f} + Q_1\Big) x_1(N) \\ &+ \frac{1}{2}\rho_2 x_2(N)' \Big(A_2' P_{2f} A_2 - P_{2f} + Q_2\Big) x_2(N) \end{aligned} \tag{3.9}$$

Using the Schur decomposition defined in Lemma 4, the constraints (3.7d) and (3.6), the last two terms of (3.2) can be written as

$$\begin{aligned} &\frac{1}{2}\rho_1 x_1(N)' S_1^s \Big(A_1^{s\prime} \Sigma_1 A_1^s - \Sigma_1 + S_1^{s\prime} Q_1 S_1^s\Big) S_1^{s\prime} x_1(N) \\ +&\frac{1}{2}\rho_2 x_2(N)' S_2^s \Big(A_2^{s\prime} \Sigma_2 A_2^s - \Sigma_2 + S_2^{s\prime} Q_2 S_2^s\Big) S_2^{s\prime} x_2(N) = 0 \end{aligned}$$

These terms are zero because of Lemma 5. Using this result and applying the iteration of the controllers gives

$$V(x^+, \mathbf{u}^+) \leq V(x, \mathbf{u}) - \rho_1 \ell_1(x_1, u_1) - \rho_2 \ell_2(x_2, u_2)$$

Because $\ell_i$ is quadratic in both arguments, there exists a $c > 0$ such that

$$V(x^+, \mathbf{u}^+) - V(x, \mathbf{u}) \leq -c\,|(x, u)|^2$$

The Lyapunov stability constraint (3.7e) for $x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{B}_r$ implies for $(x_1, x_2) \in \mathbb{B}_r$ that $|(\mathbf{u}_1, \mathbf{u}_2)| \leq 2\hat{d}\,|(x_1, x_2)|$ in which $\hat{d} = \max(d_1, d_2)$, satisfying (2.2c). Therefore the closed-loop system satisfies Lemma 2. Hence the closed-loop system is exponentially stable. □

## 4. Output Feedback

We now consider the stability of the closed-loop system with estimator error.

### 4.1. *Models*

For all $(i, j) \in \mathbb{I}_{[1,2]} \times \mathbb{I}_{[1,2]}$

$$x_{ij}^+ = A_{ij} x_{ij} + B_{ij} u_j \tag{4.1a}$$

$$y_i = \sum_{j \in \mathbb{I}_{[1,2]}} C_{ij} x_{ij} \tag{4.1b}$$

in which $y_i \in \mathbb{R}^{p_i}$ is the output of subsystem $i$ and $C_{ij} \in \mathbb{R}^{(p_i \times n_{ij})}$. Consider subsystem 1. As above, we collect the states to form $y_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ and use the simplified notation $y_1 = C_1 x_1$ to form the output model for subsystem 1.

**Assumption 11.** For all $(i, j) \in \mathbb{I}_{[1,2]} \times \mathbb{I}_{[1,2]}$, $(A_{ij}, C_{ij})$ is detectable.

### 4.2. Estimator

We construct a decentralized estimator. Consider subsystem 1, for which the local measurement $y_1$ and both inputs $u_1$ and $u_2$ are available but for which $x_1$ must be estimated

$$\hat{x}_1^+ = A_1\hat{x}_1 + \bar{B}_{11}u_1 + \bar{B}_{12}u_2 + L_1(y_1 - C_1\hat{x}_1)$$

in which $\hat{x}_1$ is the estimate of $x_1$ and $L_1$ is the Kalman filter gain. Defining the estimate error as $e_1 = x_1 - \hat{x}_1$ we have $e_1^+ = (A_1 - L_1C_1)e_1$.

**Proposition 12.** *For each $i \in \mathbb{I}_{[1,2]}$, $(A_i, C_i)$ is detectable if and only if for each $j \in \mathbb{I}_{[1,2]}$ $(A_{ij}, C_{ij})$ is detectable.*

*Proof.* The proof follows from the definition of $A_i$ and $C_i$ and the Hautus lemma. □

By Assumptions 3 and 11 with Proposition 12 there exists an $L_1$ such that $(A_1 - L_1C_1)$ is stable and therefore the estimator for subsystem 1 is stable. Defining $e_2$ similarly, the estimate error for the plant evolves

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^+ = \begin{bmatrix} A_{L1} & \\ & A_{L2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

in which $A_{Li} = A_i - L_iC_i$. We collect the estimate error of each subsystem together and write $e^+ = A_L e$.

### 4.3. Stability with Estimate Error

We consider the stability properties of the extended closed-loop system

$$\begin{pmatrix} \hat{x} \\ \mathbf{u} \\ e \end{pmatrix}^+ = \begin{pmatrix} F(\hat{x}, \mathbf{u}) + Le \\ g^p(\hat{x}, \mathbf{u}) \\ A_L e \end{pmatrix} \tag{4.2}$$

in which $F(\hat{x}, \mathbf{u}) = A\hat{x} + B_1u_1 + B_2u_2$ and $L = \text{diag}(L_1C_1, L_2C_2)$. Because $A_L$ is stable there exists a Lyapunov function $J(\cdot)$ with the following properties

$$\begin{aligned} \bar{a}\,|e|^{\sigma} &\le J(e) \le \bar{b}\,|e|^{\sigma} \\ J(e^+) - J(e) &\le -\bar{c}\,|e|^{\sigma} \end{aligned}$$

in which $\sigma > 0$, $\bar{a}, \bar{b} > 0$, and the constant $\bar{c} > 0$ can be chosen as large as desired by scaling $J(\cdot)$. For the remainder of this section, we choose $\sigma = 1$ in order to match the Lipschitz continuity of the plantwide objective function $V(\cdot)$. From the nominal properties of cooperative MPC, the origin of the nominal closed-loop system $x^+ = Ax + B_1u_1 + B_2u_2$ is exponentially stable on $\mathcal{X}_N$ if the suboptimal input trajectory $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ is computed using the actual state $x$, and the cost function $V(x, \mathbf{u})$ satisfies (2.2). We require the following feasibility assumption.

**Assumption 13.** The set $\mathcal{X}_N$ is compact, and there exist two sets $\hat{\mathcal{X}}_N$ and $\mathcal{E}$ containing the origin such that the following conditions hold: (i) $\hat{\mathcal{X}}_N \oplus \mathcal{E} \subseteq \mathcal{X}_N$, where $\oplus$ indicates the Minkowski sum; (ii) for each $\hat{x}(0) \in \hat{\mathcal{X}}_N$ and $\hat{e}(0) \in \mathcal{E}$, the solution of the extended closed-loop system (4.2) satisfies $\hat{x}(k) \in \mathcal{X}_N$ for all $k \ge 0$.

Consider the sum of the two Lyapunov functions

$$W(\hat{x}, \mathbf{u}, e) = V(\hat{x}, \mathbf{u}) + J(e)$$

We next show that $W(\cdot)$ is a Lyapunov for the perturbed system and establish exponential stability of the extended state origin $(\hat{x}, e) = (0, 0)$. From the definition of $W(\cdot)$ we have

$$\begin{aligned} a\,|(\hat{x}, \mathbf{u})|^2 + \bar{a}\,|e| \le W(\hat{x}, \mathbf{u}, e) &\le b\,|(\hat{x}, \mathbf{u})|^2 + \bar{b}\,|e| \quad \Longrightarrow \\ \tilde{a}(|(\hat{x}, \mathbf{u})|^2 + |e|) \le W(\hat{x}, \mathbf{u}, e) &\le \tilde{b}(|(\hat{x}, \mathbf{u})|^2 + |e|) \end{aligned} \tag{4.3}$$

in which $\tilde{a} = \min(a, \bar{a}) > 0$ and $\tilde{b} = \max(b, \bar{b}) > 0$. Next we compute the cost change

$$W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) = V(\hat{x}^+, \mathbf{u}^+) - V(\hat{x}, \mathbf{u}) + J(e^+) - J(e)$$

The Lyapunov function $V$ is quadratic in $(\hat{x}, \mathbf{u})$ and, hence, Lipschitz continuous on bounded sets. Therefore we have

$$\left| V(F(\hat{x}, \mathbf{u}) + Le, \mathbf{u}^+) - V(F(\hat{x}, \mathbf{u}), \mathbf{u}^+) \right| \le L_V\,|Le|$$

in which $L_V$ is the Lipschitz constant for $V$ with respect to its first argument. Using the system evolution we have

$$V(\hat{x}^+, \mathbf{u}^+) \le V(F(\hat{x}, \mathbf{u}), \mathbf{u}^+) + \bar{L}_V\,|e|$$

in which $\bar{L}_V = L_V\,|L|$. Subtracting $V(\hat{x}, \mathbf{u})$ from both sides gives

$$\begin{aligned} V(\hat{x}^+, \mathbf{u}^+) - V(\hat{x}, \mathbf{u}) &\le -c\,|(\hat{x}, u(0))|^2 + \bar{L}_V\,|e| \\ W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) &\le -c\,|(\hat{x}, u(0))|^2 + \bar{L}_V\,|e| - \bar{c}\,|e| \\ &\le -c\,|(\hat{x}, u(0))|^2 - (\bar{c} - \bar{L}_V)\,|e| \\ W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) &\le -\tilde{c}(|(\hat{x}, u(0))|^2 + |e|) \end{aligned} \tag{4.4}$$

in which we choose $\bar{c} > \bar{L}_V$ and $\tilde{c} = \min(c, \bar{c} - \bar{L}_V) > 0$. This choice is possible because $\bar{c}$ can be chosen arbitrarily large. Notice this step is what motivated the choice of $\sigma = 1$. Lastly, we require the constraint

$$|\mathbf{u}| \le d\,|\hat{x}|, \qquad \hat{x} \in \mathbb{B}_r \tag{4.5}$$

**Theorem 14** (Exponential stability of perturbed system)**.** *Given Assumptions 3, 11, 13, for each $\hat{x}(0) \in \hat{\mathcal{X}}_N$ and $e(0) \in \mathcal{E}$, there exist constants $\alpha > 0$ and $0 < \gamma < 1$, such that the solution of the perturbed system (4.2) satisfies, for all $k \ge 0$*

$$|(\hat{x}(k), e(k)| \le \alpha\,|(\hat{x}(0), e(0)|\,\gamma^k \tag{4.6}$$

*Proof.* Using the same arguments as for Lemma 2, we write:

$$W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) \le -\hat{c}(|(\hat{x}, \mathbf{u})|^2 + |e|) \tag{4.7}$$

in which $\hat{c} \ge \tilde{c} > 0$. Therefore $W(\cdot)$ is a Lyapunov function for the extended state $(\hat{x}, \mathbf{u}, e)$ with mixed norm powers. The standard exponential stability argument can be extended for the mixed norm power case to show that the origin of the extended closed-loop system (4.2) is exponentially stable, hence, for all $k \ge 0$

$$|(\hat{x}(k), \mathbf{u}(k), e(k))| \le \tilde{\alpha}\,|(\hat{x}(0), \mathbf{u}(0), e(0))|\,\gamma^k$$

in which $\tilde{\alpha} > 0$ and $0 < \gamma < 1$. Notice that Assumption 13 implies that $\mathbf{u}(k)$ exists for all $k \geq 0$ because $\hat{x}(k) \in \hat{\mathscr{X}}_N$.

We have, using the same arguments used in Lemma 2

$$\begin{aligned}|(\hat{x}(k), e(k))| &\leq |(\hat{x}(k), \mathbf{u}(k), e(k))| \leq \tilde{\alpha}\,|(\hat{x}(0), \mathbf{u}(0), e(0))|\,\gamma^k \\ &\leq \alpha\,|(\hat{x}(0), e(0))|\,\gamma^k\end{aligned}$$

in which $\alpha = \tilde{\alpha}(1 + \bar{d}) > 0$. □

**Corollary 15.** *Under the Assumptions of Theorem 14, for each $x(0)$ and $\hat{x}(0)$ such that $e(0) = x(0) - \hat{x}(0) \in \mathscr{E}$ and $\hat{x}(0) \in \hat{\mathscr{X}}_N$, the solution of the closed-loop state $x(k) = \hat{x}(k) + e(k)$ satisfies:*

$$|x(k)| \leq \tilde{\alpha}|x(0)|\gamma^k \tag{4.8}$$

*for some $\tilde{\alpha} > 0$ and $0 < \gamma < 1$.*

## 5. Coupled Constraints

In Remark 2, we commented that the constraint assumptions imply uncoupled constraints, because each input is constrained by a separate feasible region so that the full feasible space is defined $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{U} = \mathbb{U}_1 \times \mathbb{U}_2$. This assumption, however, is not always practical. Consider two subsystems sharing a scarce resource for which we control the distribution. There then exists an availability constraint spanning the subsystems. This constraint is *coupled* because each local resource constraint depends upon the amount requested by the other subsystem.

*Remark* 5. For plants with coupled constraints, implementing MPC problem (3.7) gives exponentially stable, yet suboptimal, feedback.

In this section, we relax the assumption so that $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{U}$ for any $\mathbb{U}$ compact, convex and containing the origin in its interior. Consider the decomposition of the inputs $\mathbf{u} = (\mathbf{u}_{U1}, \mathbf{u}_{U2}, \mathbf{u}_C)$ such that there exists a $\mathbb{U}_{U1}$,$\mathbb{U}_{U2}$, and $\mathbb{U}_C$ for which

$$\mathbb{U} = \mathbb{U}_{U1} \times \mathbb{U}_{U2} \times \mathbb{U}_C$$

and

$$\mathbf{u}_{U1} \in \mathbb{U}_{U1}, \quad \mathbf{u}_{U2} \in \mathbb{U}_{U2}, \quad \mathbf{u}_C \in \mathbb{U}_C$$

for which $\mathbb{U}_{U1}$, $\mathbb{U}_{U2}$, and $\mathbb{U}_C$ are compact and convex. We denote $\mathbf{u}_{Ui}$ the uncoupled inputs for subsystem $i$, $i \in \mathbb{I}_{[1,2]}$, and $\mathbf{u}_C$ the coupled inputs.

*Remark* 6. $\mathbb{U}_{U1}$,$\mathbb{U}_{U2}$, or $\mathbb{U}_C$ may be empty, and therefore such a decomposition always exists.

We modify the cooperative MPC problem (3.7) for the above decomposition. Define the augmented inputs $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2)$

$$\hat{\mathbf{u}}_1 = E_1 \begin{bmatrix} \mathbf{u}_{U1} \\ \mathbf{u}_{U2} \\ \mathbf{u}_C \end{bmatrix} \qquad \hat{\mathbf{u}}_2 = E_2 \begin{bmatrix} \mathbf{u}_{U1} \\ \mathbf{u}_{U2} \\ \mathbf{u}_C \end{bmatrix}$$

in which

$$E_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Define the augmented objective function

$$\hat{V}(x_1(0), x_2(0), \hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2) = V(x_1(0), x_2(0), \hat{E}_1\hat{\mathbf{u}}_1, \hat{E}_2\hat{\mathbf{u}}_2) \tag{5.1}$$

in which

$$\mathbf{u}_1 = \hat{E}_1\hat{\mathbf{u}}_1 \quad \mathbf{u}_2 = \hat{E}_2\hat{\mathbf{u}}_2$$

$$\hat{E}_1 = \begin{bmatrix} I & \\ & I_1 \end{bmatrix} \quad \hat{E}_2 = \begin{bmatrix} I & \\ & I_2 \end{bmatrix}$$

in which $(I_1, I_2)$ are diagonal matrices with either 0 or 1 diagonal entries and satisfy $I_1 + I_2 = I$. For simplicity, we summarize the previous relations as $\mathbf{u} = \hat{E}\hat{\mathbf{u}}$ with $\hat{E} = \mathrm{diag}(\hat{E}_1, \hat{E}_2)$. We solve the augmented cooperative MPC problem for $i \in \mathbb{I}_{[1,2]}$

$$\min_{\hat{\boldsymbol{v}}_i} \hat{V}(x_1(0), x_2(0), \hat{\boldsymbol{v}}_1, \hat{\boldsymbol{v}}_2) \tag{5.2a}$$

subject to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} \hat{E}_1\hat{v}_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} \hat{E}_2\hat{v}_2 \tag{5.2b}$$

$$\hat{\boldsymbol{v}}_i \in \mathbb{U}_{Ui} \times \mathbb{U}_C \tag{5.2c}$$

$$S^{u\,\prime}_{ji} x_{ji}(N) = 0 \quad j \in \mathbb{I}_{[1,2]} \tag{5.2d}$$

$$|\hat{\boldsymbol{v}}_i| \leq d_i\,|x_i(0)| \quad \text{if } x_i(0) \in \mathbb{B}_r \tag{5.2e}$$

$$\hat{\boldsymbol{v}}_j = \hat{\boldsymbol{v}}_j^p \quad j \in \mathbb{I}_{[1,2]} \setminus i \tag{5.2f}$$

The update (3.8) is used to determine the next iterate.

**Lemma 16.** *As $p \to \infty$ the cost $\hat{V}(x(0), \hat{\boldsymbol{v}}^p)$ converges to the optimal value $V^0(x(0))$, and the iterates $(\hat{E}_1\hat{\boldsymbol{v}}_1^p, \hat{E}_2\hat{\boldsymbol{v}}_2^p)$ converge to the Pareto optimal centralized solution $\mathbf{u}^0 = (\mathbf{u}_1^0, \mathbf{u}_2^0)$.*

Therefore, problem (5.2) gives optimal feedback and may be used for plants with coupled constraints.

## 6. *M* Subsystems

In this section, we show that the stability theory of cooperative control extends to any finite $M > 0$ number of subsystems.

For $M$ subsystems, the plantwide variables are defined

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_M \end{bmatrix} \quad B_i = \begin{bmatrix} \bar{B}_{1i} \\ \bar{B}_{2i} \\ \vdots \\ \bar{B}_{Mi} \end{bmatrix} \quad \forall i \in \mathbb{I}_{[1,M]}$$

$$V(x, \mathbf{u}) = \sum_{i \in \mathbb{I}_{[1,M]}} \rho_i V_i(x_i, \mathbf{u}_i) \quad A = \mathrm{diag}(A_1, \ldots, A_M)$$

Each subsystem solves the optimization

$$\min_{\boldsymbol{v}_i} V(x(0), \boldsymbol{v})$$

subject to

$$\begin{aligned}
x^+ &= Ax + \sum_{i \in \mathbb{I}_{[1,M]}} B_i v_i \\
\boldsymbol{v}_i &\in \mathbb{U}_i \\
S_{ji}^{u\,\prime} x_{ji}(N) &= 0 \quad j \in \mathbb{I}_{[1,M]} \\
|\boldsymbol{v}_i| \le d_i \sum_{j \in \mathbb{I}_{[1,M]}} \left| x_{ji}(0) \right| &\quad \text{if } x_{ji}(0) \in \mathbb{B}_r \quad j \in \mathbb{I}_{[1,M]} \\
\boldsymbol{v}_j &= \boldsymbol{v}_j^p \quad j \in \mathbb{I}_{[1,M]} \setminus i
\end{aligned}$$

The controller iteration is given by

$$\boldsymbol{v}^{p+1} = \sum_{i \in \mathbb{I}_{[1,M]}} w_i (\boldsymbol{v}_1^p, \ldots, \boldsymbol{v}_i^*, \ldots, \boldsymbol{v}_M^p)$$

in which $\boldsymbol{v}_i^* = \boldsymbol{v}_i^* \left( x(0), \boldsymbol{v}_j^p | j \in \mathbb{I}_{[1,M]} \setminus i \right)$. After $\bar{p}$ iterates, we set $\mathbf{u} \leftarrow \boldsymbol{v}^{\bar{p}}$ and inject $u(0)$ into the plant.

The warm start is generated by purely local information

$$\tilde{\mathbf{u}}_i^+ = \{u_i(1), u_i(2), \ldots, u_i(N-1), 0\} \quad \forall i \in \mathbb{I}_{[1,M]}$$

The plantwide cost function then satisfies for any $\bar{p} \ge 0$

$$\begin{aligned}
V(x^+, \mathbf{u}^+) &\le V(x, \mathbf{u}) - \sum_{i \in \mathbb{I}_{[1,M]}} \rho_i \ell_i(x_i, u_i) \\
|\mathbf{u}| &\le d|x| \quad x \in \mathbb{B}_r
\end{aligned}$$

Generalizing Assumption 3 to all $(i,j) \in \mathbb{I}_{[1,M]} \times \mathbb{I}_{[1,M]}$, we find that Theorem 10 applies and cooperative MPC of $M$ subsystems is exponentially stable.

Moreover, expressing the $M$ subsystem outputs as

$$y_i = \sum_{j \in \mathbb{I}_{[1,M]}} C_{ij} x_{ij} \quad i \in \mathbb{I}_{[1,M]}$$

and generalizing Assumption 11 for $(i,j) \in \mathbb{I}_{[1,M]} \times \mathbb{I}_{[1,M]}$, cooperative MPC for $M$ subsystems satisfies Theorem 14. Finally, for systems with coupled constraints, we can decompose the feasible space such that $\mathbb{U} = (\prod_{i \in \mathbb{I}_{[1,M]}} \mathbb{U}_{Ui}) \times \mathbb{U}_C$. Hence, the input augmentation scheme of Section 5 is applicable to plants of $M$ subsystems. Notice that, in general, this approach may lead to augmented inputs for each subsystem that are larger than strictly necessary to achieve optimal control. The most parsimonious augmentation scheme is described elsewhere [10].

## 7. Example

Consider a plant consisting entirely of three tanks connected in series through material flow. A pipe empties into the first tank, and the effluent of the last tank is split and partly redirected to the first tank (see Figure 1). The models

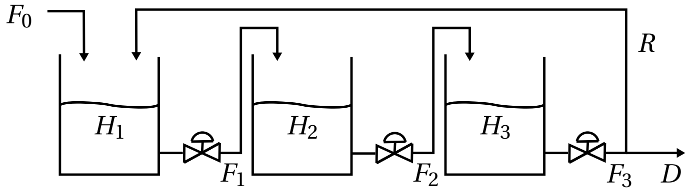


Figure 1: Three tanks in series with recycle.

for the tanks are

$$\begin{aligned}
\frac{dH_1}{dt} &= \frac{1}{S_1}(F_0 + R - F_1) \\
\frac{dH_2}{dt} &= \frac{1}{S_2}(F_1 - F_2) \\
\frac{dH_3}{dt} &= \frac{1}{S_3}(F_2 - F_3) \\
R &= \alpha F_3
\end{aligned}$$

in which $S_i$ is the cross-sectional area of tank $i$ and $\alpha$ is the fraction of $F_3$ recycled. All steady-state flows satisfy the equations

$$\begin{aligned}
F_1 &= \frac{F_0}{1-\alpha} \\
F_2 &= F_1 \\
F_3 &= F_2
\end{aligned}$$

Given a inlet flow of $F_0 = 10$ with $\alpha = 0.5$, we have the steady-state flows $F_1^s = F_2^s = F_3^s = 20$. The steady-state heights are any $H_1^s = H_2^s = H_3^s = \bar{H}^s$. Here, we choose $\bar{H}^s = 10$. Defining deviation variables around this steady state and transforming it into discrete time we summarize this model as

$$\begin{aligned}
x^+ &= Ax + Bu \\
y &= Cx
\end{aligned} \tag{7.1}$$

in which

$$x = \begin{bmatrix} H_1 - H_1^s \\ H_2 - H_2^s \\ H_3 - H_3^s \end{bmatrix} \quad u = \begin{bmatrix} F_1 - F_1^s \\ F_2 - F_2^s \\ F_3 - F_3^s \end{bmatrix} \quad y = \begin{bmatrix} H_1 - H_1^s \\ H_2 - H_2^s \\ H_3 - H_3^s \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad B = \begin{bmatrix} -\frac{\Delta}{S_1} & & \alpha\frac{\Delta}{S_1} \\ \frac{\Delta}{S_2} & -\frac{\Delta}{S_2} & \\ & \frac{\Delta}{S_3} & -\frac{\Delta}{S_3} \end{bmatrix} \quad C = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Notice the system is composed completely of integrating modes, i.e., $|\lambda| = 1$ for all $\lambda \in \text{eig}(A)$.

### 7.1. *Distributed control*

One distributed control strategy is to control each of the tanks independently. We select the pairing $(y_i, u_i) = (H_i, F_i)$, and construct a model of the form (4.1). Consider the dis-

Table 1: Performance comparison

| | Cost | Performance loss (%) |
|---|---|---|
| Centralized MPC | 210.38 | 0.00 |
| Cooperative MPC (10 iterations) | 213.92 | 1.69 |
| Cooperative MPC (1 iteration) | 229.06 | 8.88 |
| Noncooperative MPC | 252.00 | 19.78 |
| Decentralized MPC | 274.51 | 30.48 |

tributed model

$$
\begin{aligned}
\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}^+ &= \begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} + \begin{bmatrix} -\frac{\Delta}{S_1} & & \\ & 0 & \\ & & \alpha\frac{\Delta}{S_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
y_1 &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} \\
\begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}^+ &= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} + \begin{bmatrix} \frac{\Delta}{S_2} & & \\ & -\frac{\Delta}{S_2} & \\ & & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
y_2 &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} \\
\begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix}^+ &= \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} + \begin{bmatrix} 0 & & \\ & \frac{\Delta}{S_3} & \\ & & -\frac{\Delta}{S_3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
y_3 &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix}
\end{aligned}
\tag{7.2}
$$

in which $A_{12}$, $A_{23}$, and $A_{31}$ are zero and satisfy the stabilizability and detectability assumptions.

*Remark* 7. Given the initial condition $x_{ij}(0) = H_i(0) - H_i^s$ $\forall i, j \in \mathbb{I}_{[1,3]} \times \mathbb{I}_{[1,3]}$, the models (7.1) and (7.2) have the same input/output behavior, i.e., the response of height $H_i$ to flow $F_j$ for all $i, j \in \mathbb{I}_{[1,3]} \times \mathbb{I}_{[1,3]}$.

Therefore model (7.2) is an adequate model for implementation in distributed control.

### *7.2. Simulation*

Consider the performance of distributed control of model (7.2). We choose the parameters $S_1 = S_2 = S_3 = 1$ and a sample time $\Delta = 0.1$. The tuning parameters are specified

$$Q_i = C_i' C_i + 0.001 I \quad R_i = I \quad \forall i \in \mathbb{I}_{[1,3]}$$

The inputs have been constrained so that there is no negative flow, i.e., $F_i - F_i^s \geq 5$ for all $i \in \mathbb{I}_{[1,3]}$. We simulate a setpoint change in all of the tanks at $t = 5$ so that all levels should increase 5 units. In Figure 2, the performance of the distributed control strategies are compared to the centralized control benchmark. For this example, noncooperative control is an improvement over decentralized control (see Table 1). Cooperative control with only a single iteration is significantly better than noncooperative control, however, and approaches centralized control as more iteration is allowed.

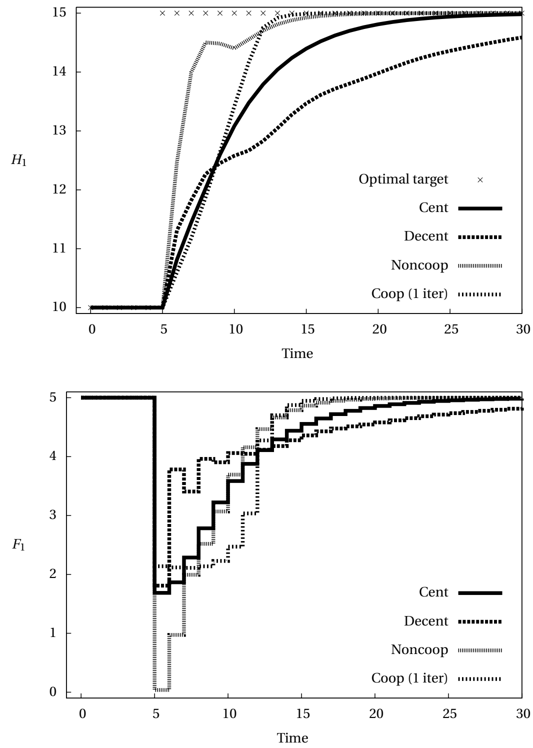


Figure 2: Performance of three tanks simulation. First tank input and output are shown. All tanks perform similarly.

## 8. Conclusion

In this paper we present a novel cooperative controller in which the subsystem controllers optimize the same objective function in parallel without the use of a coordinator. The control algorithm is equivalent to a suboptimal centralized controller, allowing the distributed optimization to be terminated at any iterate before convergence. At convergence, the feedback is Pareto optimal. We show exponential stability for the nominal case and for perturbation by a stable state estimator. For plants with sparsely coupled constraints, the controller can be extended by repartitioning the decision variables to maintain Pareto optimality.

We make no restrictions on the strength of the dynamic coupling in the network of subsystems, offering flexibility in plantwide control design. Moreover, the cooperative controller can improve performance of plants over traditional decentralized control and noncooperative control, especially for plants with strong open-loop interactions between subsystems. A simple example is given showing this performance improvement.

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## A. Further Proofs

*Proof of Lemma 6.* By assumption, the initial guess is feasible. Because $\mathbb{U}_1$ and $\mathbb{U}_2$ are convex, the convex combination (3.8) with $p=0$ implies $(\boldsymbol{v}_1^1,\boldsymbol{v}_2^1)$ is feasible. Feasibility for $p>1$ follows by induction. □

*Proof of Lemma 7.* For every $p \geq 0$, the cost function satisfies the following

$$\begin{aligned}
V(x(0),\boldsymbol{v}^{p+1}) &= V(x(0), w_1(\boldsymbol{v}_1^*,\boldsymbol{v}_2^p) + w_2(\boldsymbol{v}_1^p,\boldsymbol{v}_2^*)) \\
&\leq w_1 V(x(0),(\boldsymbol{v}_1^*,\boldsymbol{v}_2^p)) + w_2 V(x(0),(\boldsymbol{v}_1^p,\boldsymbol{v}_2^*)) \quad \text{(A.1a)}\\
&\leq w_1 V(x(0),(\boldsymbol{v}_1^p,\boldsymbol{v}_2^p)) + w_2 V(x(0),(\boldsymbol{v}_1^p,\boldsymbol{v}_2^p)) \quad \text{(A.1b)}\\
&\leq V(x(0),\boldsymbol{v}^p)
\end{aligned}$$

The first equality follows from (3.8). The inequality (A.1a) follows from convexity of $V(\cdot)$. The next inequality (A.1b) follows from the optimality of $\boldsymbol{v}_i^*$ $\forall i \in \mathbb{I}_{[1,2]}$, and the final line follows from $w_1+w_2=1$. Because the cost is bounded below, it converges. □

*Proof of Lemma 8.* We give a proof that requires only closedness (not compactness) of $\mathbb{U}_i$, $i\in\mathbb{I}_{[1,2]}$. From Lemma 7, the cost converges, say to $\underline{V}$. Since $V$ is quadratic and strongly convex, its sublevel sets $\text{lev}_{\leq a}(V)$ are compact and bounded for all $a$. Hence, all iterates belong to the compact set $\text{lev}_{\leq V(\boldsymbol{v}^0)}(V)\cap\mathbb{U}$, so there is at least one accumulation point. Let $\bar{\boldsymbol{v}}$ be any such accumulation point, and choose a subsequence $\mathscr{P}\subset\{1,2,3,\ldots\}$ such that $\{\boldsymbol{v}^p\}_{p\in\mathscr{P}}$ converges to $\bar{\boldsymbol{v}}$. We obviously have that $V(x(0),\bar{\boldsymbol{v}})=\underline{V}$, and moreover that

$$\lim_{p\in\mathscr{P},\,p\to\infty} V(x(0),\boldsymbol{v}^p) = \lim_{p\in\mathscr{P},\,p\to\infty} V(x(0),\boldsymbol{v}^{p+1}) = \underline{V} \quad \text{(A.2)}$$

By strong convexity of $V$ and compactness of $\mathbb{U}_i$, $i\in\mathbb{I}_{[1,2]}$, the minimizer of $V(x(0),\cdot)$ is attained at a unique point $\mathbf{u}^0=(\mathbf{u}_1^0,\mathbf{u}_2^0)$. By taking limits in (A.1) as $p\to\infty$ for $p\in\mathscr{P}$, and using $w_1>0$, $w_2>0$, we can deduce easily that

$$\lim_{p\in\mathscr{P},\,p\to\infty} V(x(0),(\boldsymbol{v}_1^*(\boldsymbol{v}_2^p),\boldsymbol{v}_2^p)) = \underline{V} \quad \text{(A.3a)}$$

$$\lim_{p\in\mathscr{P},\,p\to\infty} V(x(0),(\boldsymbol{v}_1^p,\boldsymbol{v}_2^*(\boldsymbol{v}_1^p))) = \underline{V} \quad \text{(A.3b)}$$

We suppose for contradiction that $\underline{V}\neq V(x(0),\mathbf{u}^0)$ and thus $\bar{\boldsymbol{v}}\neq\mathbf{u}^0$. Because $V(x(0),\cdot)$ is convex, we have

$$\nabla V(x(0),\bar{\boldsymbol{v}})'(\mathbf{u}^0-\bar{\boldsymbol{v}}) \leq \Delta V := V(x(0),\mathbf{u}^0) - V(x(0),\bar{\boldsymbol{v}}) < 0$$

where $\nabla V(x(0),\cdot)$ denotes the gradient of $V(x(0),\cdot)$. It follows immediately that either

$$\nabla V(x(0),\bar{\boldsymbol{v}})'\begin{bmatrix}\mathbf{u}_1^0-\bar{\boldsymbol{v}}_1\\ 0\end{bmatrix} \leq (1/2)\Delta V \ \text{ or} \quad \text{(A.4a)}$$

$$\nabla V(x(0),\bar{\boldsymbol{v}})'\begin{bmatrix}0\\ \mathbf{u}_2^0-\bar{\boldsymbol{v}}_2\end{bmatrix} \leq (1/2)\Delta V \quad \text{(A.4b)}$$

Suppose first that (A.4a) holds. Using the fact that $V$ is quadratic, we have that

$$\begin{aligned}
&V(x(0),(\boldsymbol{v}_1^p+\epsilon(\mathbf{u}_1^0-\boldsymbol{v}_1^p),\boldsymbol{v}_2^p))\\
&=V(x(0),\boldsymbol{v}^p)+\epsilon\nabla V(x(0),\boldsymbol{v}^p)'\begin{bmatrix}\mathbf{u}_1^0-\boldsymbol{v}_1^p\\0\end{bmatrix}\\
&\quad+\frac{1}{2}\epsilon^2\begin{bmatrix}\mathbf{u}_1^0-\boldsymbol{v}_1^p\\0\end{bmatrix}'\nabla^2 V(x(0),\boldsymbol{v}^p)\begin{bmatrix}\mathbf{u}_1^0-\boldsymbol{v}_1^p\\0\end{bmatrix}\\
&\leq\underline{V}+(1/4)\epsilon\Delta V+\beta\epsilon^2 \quad \text{(A.5)}
\end{aligned}$$

for all $p\in\mathscr{P}$ sufficiently large, for some $\beta$ independent of $\epsilon$ and $p$. By fixing $\epsilon$ to a suitably small value (certainly less than 1), we have both that the right-hand side of (A.5) is strictly less than $\underline{V}$ and that $\boldsymbol{v}_1^p+\epsilon(\mathbf{u}_1^0-\boldsymbol{v}_1^p)\in\mathbb{U}_1$. By taking limits in (A.5) and using (A.3) and the fact that $\boldsymbol{v}_1^*(\boldsymbol{v}_2^p)$ is optimal for $V(x(0),(\cdot,\boldsymbol{v}_2^p))$ in $\mathbb{U}_1$, we have

$$\begin{aligned}
\underline{V} &= \lim_{p\in\mathscr{P},\,p\to\infty} V(x(0),(\boldsymbol{v}_1^*(\boldsymbol{v}_2^p),\boldsymbol{v}_2^p))\\
&\leq \lim_{p\in\mathscr{P},\,p\to\infty} V(x(0),(\boldsymbol{v}_1^p+\epsilon(\mathbf{u}_1^0-\boldsymbol{v}_1^p),\boldsymbol{v}_2^p))\\
&<\underline{V}
\end{aligned}$$

giving a contradiction. By identical logic, we obtain the same contradiction from (A.4b). We conclude that $\underline{V}=V(x(0),\mathbf{u}^0)$ and thus $\bar{\boldsymbol{v}}=\mathbf{u}^0$. Since $\bar{\boldsymbol{v}}$ was an arbitrary accumulation point of the sequence $\{\boldsymbol{v}^p\}$, and since this sequence is confined to a compact set, we conclude that the whole sequence converges to $\mathbf{u}^0$. □

*Proof of Corollary 15.* We first note that: $|x(k)| \leq |\hat{x}(k)| + |e(k)| \leq \sqrt{2}|(\hat{x}(k),e(k))|$. From Theorem 14 we can write:

$$|x(k)| \leq \sqrt{2}\alpha\,|(\hat{x}(0),e(0))|\,\gamma^k \leq \bar{\alpha}\,|\hat{x}(0)+e(0)|\,\gamma^k$$

with $\bar{\alpha}=\sqrt{2}\alpha$, which concludes the proof by noticing that $x(0)=\hat{x}(0)+e(0)$. □

*Proof of Lemma 16.* Because $\hat{V}(\cdot)$ is convex and bounded below, the proof follows from Lemma 8 and from noticing that the point $\mathbf{u}^0 = (\hat{E}_1\hat{\mathbf{u}}_1^0, \hat{E}_2\hat{\mathbf{u}}_2^0)$, with $\hat{\mathbf{u}}_i^0 = \lim_{p\to\infty} \hat{\boldsymbol{v}}_i, \quad i \in \mathbb{I}_{[1,2]}$, is Pareto optimal. $\square$