ON OPERATORS OF TRANSITION IN KREIN SPACES

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Abstract. The paper is devoted to investigation of operators of transition and the corresponding decompositions of Krein spaces. The obtained results are applied to the study of relationship between solutions of operator Riccati equations and properties of the associated operator matrix L. In this way, we complete the known result (see Theorem 5.2 in the paper of S. Albeverio, A. Motovilov, A. Skhalikov, Integral Equ. Oper. Theory 64 (2004), 455–486) and show the equivalence between the existence of a strong solution K ($\|K\| < 1$) of the Riccati equation and similarity of the J-self-adjoint operator L to a self-adjoint one.

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1. INTRODUCTION

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and with non-trivial fundamental symmetry J (i.e., $J = J^*$, $J^2 = I$, and $J \neq \pm I$).

The space $\mathfrak{H}$ endowed with the indefinite inner product (indefinite metric) $[\cdot,\cdot] := (J\cdot,\cdot)$ is called a Krein space $(\mathfrak{H}, [\cdot,\cdot])$. In what follows we will refer to [6] for general results of the Krein spaces theory.

The development of $\mathcal{PT}$-symmetric quantum mechanics (PTQM) achieved during the past decade (see [7] and the references therein) leads to a lot of new useful notions and motivates the further development of the Krein spaces theory [3,12]. In particular, the notion of $\mathcal{C}$-symmetry for pseudo-Hermitian Hamiltonians (which is one of the key concepts of PTQM) gives rise to the definition of $\mathcal{C}$-symmetry for operators acting in Krein spaces [2,3].

The property of $\mathcal{C}$-symmetry for an operator A is equivalent to its fundamental reducibility with respect to decomposition (2.11) [8], where $\mathfrak{L}_K$ and $\mathfrak{M}_K^*$ are maximal positive and maximal negative subspaces of the Krein space $(\mathfrak{H}, [\cdot,\cdot])$. Decompositions (2.11) are completely characterized by the collection of operators of transition, which are closely related to the concept of angular operators in the Krein space [6].

Operators of transition enable one to simplify many results of the Krein spaces theory which were initially formulated in terms of angular operators [9] and they can provide some useful operator framework for various investigations where geometric properties of the underlying Krein space have considerable importance.

In the present paper, we illustrate this point of view by considering well-known relationship [1, 4, 5, 13] between solutions of the operator Riccati equations (3.2) and properties of 2×2-block operator matrices L (3.1) with unbounded operator entries (Section 3). In particular, we prove the inverse statement to [4, Theorem 5.2] and, as a result, we establish the equivalence between the existence of a strong solution K ($\|K\| < 1$) of operator Riccati equation (3.3) and the similarity of the J-self-adjoint operator L to a self-adjoint one (Theorem 3.4).

Another aim of the present paper is to generalize operators of transition for the case of 'nonsymmetric' decompositions (2.4), which are more general than (2.11). In this case, the basic properties of operators of transitions remain true (Section 2). We believe that these results can be useful for the study of general (not necessarily J-self-adjoint) operator matrices L.

The following notations are used throughout the paper. $\mathcal{D}(A)$ and $\mathcal{R}(A)$ denote the domain and the range of a linear operator A. $A \upharpoonright_{\mathcal{D}}$ means the restriction of A onto a set $\mathcal{D}$. The symbol $\mathcal{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ denotes the set of bounded linear operators from a Hilbert space $\mathfrak{H}_0$ to a Hilbert space $\mathfrak{H}_1$. The notation $\rho(A)$ is used for the resolvent of A.

2. OPERATORS OF TRANSITION

2.1. DEFINITION OF OPERATORS OF TRANSITION

Let $(\mathfrak{H}, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J. The corresponding orthogonal projections $P_\pm = \frac{1}{2}(I \pm J)$ determine *the fundamental decomposition* of $\mathfrak{H}$

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \qquad \mathfrak{H}_- = P_-\mathfrak{H}, \quad \mathfrak{H}_+ = P_+\mathfrak{H}. \tag{2.1}$$

A subspace $\mathfrak{L}$ of $\mathfrak{H}$ is called *hypermaximal neutral* if

$$\mathfrak{L} = \mathfrak{L}^{[\perp]} = \{x \in \mathfrak{H} \ : \ [x, y] = 0, \ \forall y \in \mathfrak{L}\}.$$

A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called *uniformly positive* (*uniformly negative*) if $[x, x] \geq a^2\|x\|^2$ $(-[x, x] \geq a^2\|x\|^2)$ for a fixed $a \in \mathbb{R}$ for all $x \in \mathfrak{L}$. The subspaces $\mathfrak{H}_\pm$ in (2.1) are examples of uniformly positive and uniformly negative subspaces and, moreover, they are maximal, i.e., $\mathfrak{H}_+$ $(\mathfrak{H}_-)$ is not proper subspace of uniformly positive (resp. negative) subspace.

For any $K \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$ and $Q \in \mathcal{B}(\mathfrak{H}_-, \mathfrak{H}_+)$ we put

$$\begin{aligned} \mathfrak{L}_K &= \{x = x_+ + Kx_+ \ : \ \forall x_+ \in \mathfrak{H}_+\}, \\ \mathfrak{M}_Q &= \{y = y_- + Qy_- \ : \ \forall y_- \in \mathfrak{H}_-\}. \end{aligned} \tag{2.2}$$

It is well known [6] that maximal uniformly positive (negative) subspaces of the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ are described by the first (second) formula in (2.2) with $\|K\| < 1$ ($\|Q\| < 1$). Hypermaximal neutral subspaces can be described by formulas (2.2) under the assumption that K (Q) is a unitary mapping of $\mathfrak{H}_+$ onto $\mathfrak{H}_-$ (of $\mathfrak{H}_-$ onto $\mathfrak{H}_+$).

It follows from (2.2) that $\mathfrak{L}_K = (I+T)\mathfrak{H}_+$ and $\mathfrak{M}_Q = (I+T)\mathfrak{H}_-$, where

$$T = KP_+ + QP_- \tag{2.3}$$

is a bounded operator in $\mathfrak{H}$. The operator T takes the form $T = \begin{pmatrix} 0 & Q \\ K & 0 \end{pmatrix}$ with respect to the fundamental decomposition (2.1).

Lemma 2.1. *The subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_Q$ are linearly independent and*

$$\mathfrak{H} = \mathfrak{L}_K \dot{+} \mathfrak{M}_Q \tag{2.4}$$

if and only if $I+T$ is a boundedly invertible operator in $\mathfrak{H}$ (i.e., $0 \in \rho(I+T)$).

The proof of Lemma 2.1 is quite obvious and similar statements (formulated in slightly different manner) are well known (see e.g., [4, Lemma 2.6]).

Definition 2.2. *If the relation* (2.4) *holds, then the operator T defined by* (2.3) *is called an operator of transition from the fundamental decomposition* (2.1) *to the decomposition* (2.4).

The collection of operators of transition admits a simple description which does not use the decomposition (2.4).

Proposition 2.3. *Let $T \in \mathcal{B}(\mathfrak{H})$ and let J be a fixed fundamental symmetry in $\mathfrak{H}$. Then T is an operator of transition with respect to the fundamental decomposition* (2.1) *if and only if the following conditions hold:*

$$JT = -TJ \qquad \text{and} \qquad 0 \in \rho(I+T). \tag{2.5}$$

Proof. If T is an operator of transition, then there exists a decomposition (2.4) and T is defined by (2.3). The formula (2.3) means that $JT = -TJ$. Furthermore, $0 \in \rho(I+T)$ by Lemma 2.1.

Conversely, if T satisfies (2.5), then the first relation in (2.5) leads to the presentation (2.3) of T with $KP_+ = T\restriction_{\mathfrak{H}_+}$ and $QP_- = T\restriction_{\mathfrak{H}_-}$; the second relation ensures the decomposition (2.4) (due to Lemma 2.1). Therefore, T is an operator of transition. $\square$

It is clear that there is a one-to-one correspondence between the set of all possible bounded operators T satisfying (2.5) and the set of all possible decompositions of the form (2.4).

Denote by $P_{\mathfrak{L}}$ and $P_{\mathfrak{M}}$ the projections onto $\mathfrak{L}_K$ and $\mathfrak{M}_Q$ with respect to the decomposition (2.4).

Lemma 2.4. *Let T be an operator of transition from* (2.1) *to* (2.4). *Then:*

$$P_+ = (I+T)^{-1}(P_{\mathfrak{L}} + TP_{\mathfrak{M}}), \qquad P_- = (I+T)^{-1}(P_{\mathfrak{M}} + TP_{\mathfrak{L}});$$

$$P_{\mathfrak{L}} = (I-T)^{-1}(P_+ - TP_-), \quad P_{\mathfrak{M}} = (I-T)^{-1}(P_- - TP_+),$$

where orthogonal projections $P_\pm = \frac{1}{2}(I \pm J)$ correspond to the fundamental decomposition (2.1).

Proof. The assertion of Lemma 2.4 was established in [9, Proposition 9.1] under the additional assumption that $\mathfrak{L}_K$ is a maximal uniformly positive subspace and $\mathfrak{M}_Q = \mathfrak{L}_K^{[\perp]}$. This proof can be directly extended to the general case. For convenience of readers, we outline principal steps.

For any $x \in \mathfrak{H}$, in view of (2.2) and (2.3), we have

$$(I+T)x = (I+T)(P_+ + P_-)x = l + m,$$

where $l = (I+T)P_+x \in \mathfrak{L}_K$ and $m = (I+T)P_-x \in \mathfrak{M}_Q$. Hence, $(I+T)P_+x = P_{\mathfrak{L}}(I+T)x$ and $(I+T)P_-x = P_{\mathfrak{M}}(I+T)x$. Since $0 \in \rho(I+T)$, we conclude

$$P_+ = (I+T)^{-1}(P_{\mathfrak{L}} + P_{\mathfrak{L}}T), \qquad P_- = (I+T)^{-1}(P_{\mathfrak{M}} + P_{\mathfrak{M}}T). \tag{2.6}$$

Let us show that $P_{\mathfrak{L}}T = TP_{\mathfrak{M}}$ and $P_{\mathfrak{M}}T = TP_{\mathfrak{L}}$. Since $P_{\mathfrak{L}} + P_{\mathfrak{M}} = I$, it is sufficient to verify the first relation. By (2.4), an arbitrary $z \in \mathfrak{H}$ has the decomposition $z = l + m$, where $l = (I+T)x_+$, $m = (I+T)x_-$, $x_\pm \in \mathfrak{H}_\pm$. Therefore,

$$P_{\mathfrak{L}}Tz = P_{\mathfrak{L}}T(l+m) = P_{\mathfrak{L}}(I+T)[Tx_+ + Tx_-] = (I+T)Tx_- = TP_{\mathfrak{M}}z.$$

Combining this with (2.6), we obtain the required expressions for $P_\pm$. Solving them with respect to $P_{\mathfrak{L}}$ and $P_{\mathfrak{M}}$ and taking into account that $I-T$ is a boundedly invertible operator (since $JT = -TJ$ and $0 \in \rho(I+T)$), we derive the formulas for $P_{\mathfrak{L}}$ and $P_{\mathfrak{M}}$. Lemma 2.4 is proved. □

Corollary 2.5. *The following identity holds:*

$$P_{\mathfrak{L}}P_+ - P_{\mathfrak{M}}P_- = P_+P_{\mathfrak{L}} - P_-P_{\mathfrak{M}}.$$

Proof. It follows from Lemma 2.4 and the identity $JT = -TJ$ that

$$\begin{aligned} P_+P_{\mathfrak{L}} - P_-P_{\mathfrak{M}} &= (I+T)^{-1}(P_{\mathfrak{L}} - P_{\mathfrak{M}}) = \\ &= (I+T)^{-1}(I-T)^{-1}(I+T)J = (I-T)^{-1}J = P_{\mathfrak{L}}P_+ - P_{\mathfrak{M}}P_-. \end{aligned}$$ □

The concept of operators of transition T enables one to characterize various specific decompositions (2.4) of $\mathfrak{H}$ by imposing additional restrictions onto T.

Proposition 2.6. *Let T be an operator of transition from* (2.1) *to* (2.4) (*i.e., T satisfies the conditions* (2.5)). *Then the following statements hold:*

(i) *the subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_Q$ in* (2.4) *are hypermaximal neutral $\iff$ T is a unitary operator in $\mathfrak{H}$;*

(ii) *the subspaces* $\mathfrak{L}_K$ *and* $\mathfrak{M}_Q$ *are* J*-orthogonal* (*i.e.*, $[x, y] = 0, \ \forall x \in \mathfrak{L}_K, \ y \in \mathfrak{M}_Q$) $\iff$ T *is self-adjoint in* $\mathfrak{H}$*;*

(iii) $\mathfrak{L}_K$ *and* $\mathfrak{M}_Q$ *are, respectively, uniformly positive and uniformly negative subspaces* $\iff$ T *is a strong contraction* ($\|T\| < 1$) *in* $\mathfrak{H}$*.*

Proof. (i) the subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_Q$ defined by (2.2) are hypermaximal neutral in the Krein space $(\mathfrak{H}, [\cdot, \cdot])$ $\iff$ $K : \mathfrak{H}_+ \to \mathfrak{H}_-$ and $Q : \mathfrak{H}_- \to \mathfrak{H}_+$ are unitary mappings [6]. These properties are equivalent to the unitarity of T (due to (2.3)).

(ii) J-orthogonality of $\mathfrak{L}_K$ and $\mathfrak{M}_Q$ are equivalent to the property $K^* = Q$ [6] $\iff$ T is self-adjoint in $\mathfrak{H}$ (since T is defined by (2.3)).

(iii) subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_Q$ are, respectively, uniformly positive and uniformly negative $\iff$ $\|K\| < 1$ and $\|Q\| < 1$ [6] $\iff$ T is a strong contraction. □

2.2. THE OPERATOR $\mathcal{C}$ AND ITS PROPERTIES

The bounded operator $\mathcal{C} = P_{\mathfrak{L}} - P_{\mathfrak{M}}$ describes the subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_Q$ in (2.4) as well:

$$\mathfrak{L}_K = \frac{1}{2}(I + \mathcal{C})\mathfrak{H}, \qquad \mathfrak{M}_Q = \frac{1}{2}(I - \mathcal{C})\mathfrak{H}. \tag{2.7}$$

It follows from Lemma 2.4 that

$$\mathcal{C} = P_{\mathfrak{L}} - P_{\mathfrak{M}} = (I - T)^{-1}J(I - T) = J(I + T)^{-1}(I - T). \tag{2.8}$$

It is clear that $\mathcal{C}$ is a bounded operator in $\mathfrak{H}$ and $\mathcal{C}^2 = I$.

Using Corollary 2.5 we can define the operator

$$U = P_{\mathfrak{L}}P_+ - P_{\mathfrak{M}}P_- = P_+P_{\mathfrak{L}} - P_-P_{\mathfrak{M}}. \tag{2.9}$$

Lemma 2.7. *The operator* U *satisfies the relations*

$$U = (I - T)^{-1}J, \qquad UJ = \mathcal{C}U, \qquad U\mathcal{C} = JU \tag{2.10}$$

and its restrictions $U\restriction_{\mathfrak{H}_+}$, $U\restriction_{\mathfrak{H}_-}$, $U\restriction_{\mathfrak{L}_K}$, *and* $U\restriction_{\mathfrak{M}_Q}$ *determine boundedly invertible mappings of* $\mathfrak{H}_+$, $\mathfrak{H}_-$, $\mathfrak{L}_K$, *and* $\mathfrak{M}_Q$ *onto* $\mathfrak{L}_K$, $\mathfrak{M}_Q$, $\mathfrak{H}_+$, *and* $\mathfrak{H}_-$, *respectively.*

Proof. The first relation in (2.10) follows from the proof of Corollary 2.5. Using (2.9) one concludes

$$UJ = U(P_+ - P_-) = P_{\mathfrak{L}}P_+ + P_{\mathfrak{M}}P_- = (P_{\mathfrak{L}} - P_{\mathfrak{M}})U = \mathcal{C}U.$$

The relation $U\mathcal{C} = JU$ is proved in the same manner.

The first identity in (2.10) and (2.7) mean that $U : \mathfrak{H}_+ \to \mathfrak{L}_K$ and $U : \mathfrak{H}_- \to \mathfrak{M}_Q$. Further, since $U = (I - T)^{-1}J$, one concludes that $0 \in \rho(U)$. Combining this with the decompositions (2.1) and (2.4) we arrive at the conclusion that the operators $U\restriction_{\mathfrak{H}_\pm}$ are boundedly invertible mappings of $\mathfrak{H}_\pm$ onto $\mathfrak{L}_K$ and $\mathfrak{M}_Q$, respectively. The other cases can be considered by analogy. □

Assume that $\mathfrak{L}_K$ in (2.4) is a maximal uniformly positive subspace of the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ and the subspace $\mathfrak{M}_Q$ is J-orthogonal to $\mathfrak{L}_K$. Then $\mathfrak{M}_Q$ turns out to be a maximal uniformly negative subspace of $(\mathfrak{H}, [\cdot,\cdot])$ and $Q = K^*$ [6]. According to items (ii) and (iii) of Proposition 2.6, the operator of transition T from (2.1) to the decomposition

$$\mathfrak{H} = \mathfrak{L}_K[\dot{+}]\mathfrak{M}_{K^*} \tag{2.11}$$

($[+]$ means the orthogonality with respect to the indefinite metric $[\cdot,\cdot]$) is a self-adjoint strong contraction in $\mathfrak{H}$. The collection of operators of transition T with these properties (i.e., $T = T^*$ and $\|T\| < 1$) is in one-to-one correspondence with the set of all possible decompositions (2.11) of $\mathfrak{H}$, where subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_{K^*} = \mathfrak{L}_K^{[\perp]}$ are, respectively, maximal uniformly positive and maximal uniformly negative.

Lemma 2.8. *The operator $\mathcal{C}$ in (2.8) admits the representation $\mathcal{C} = Je^Y$, where Y is a bounded self-adjoint operator in $\mathfrak{H}$ such that $JY = -YJ$ if and only if the corresponding operator of transition T in (2.8) is a self-adjoint strong contraction.*

Proof. If T is a self-adjoint strong contraction in $\mathfrak{H}$, then its spectrum is contained in $\mathrm{I} = (-1, 1)$ and formula (2.8) can be rewritten as $\mathcal{C} = Je^Y$, where

$$Y = f(T), \qquad f(\lambda) = \ln\frac{1-\lambda}{1+\lambda}$$

is a bounded self-adjoint operator in $\mathfrak{H}$. Since $JT = -TJ$, the projection valued measure E_δ associated with T satisfies the relation $JE_\delta = E_{-\delta}J$ for an arbitrary Borel set δ [10]. Using this relation and taking into account that $f(\lambda) = \ln\frac{1-\lambda}{1+\lambda}$ is an odd function on I we obtain

$$JY = J\int_{\mathrm{I}} f(\lambda)dE_\lambda = \int_{\mathrm{I}} f(\lambda)dE_{-\lambda}J = -YJ.$$

Conversely, if $\mathcal{C} = Je^Y \ \ (JY = -YJ)$, then $\mathcal{C}$ is determined by (2.8) with $T = (I - e^Y)(I + e^Y)^{-1}$. This means that $T = T^*$ and $\|T\| < 1$. Since $Je^Y = e^{-Y}J$ we get $JT = -TJ$. □

3. OPERATOR RICCATI EQUATION

3.1. PRELIMINARIES

Let A_0 and A_1 be densely defined closed operators acting in the Hilbert spaces $\mathfrak{H}_0(\equiv \mathfrak{H}_+)$ and $\mathfrak{H}_1(\equiv \mathfrak{H}_-)$, respectively and let $B \in \mathcal{B}(\mathfrak{H}_-, \mathfrak{H}_+)$, $C \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$. Then the operator matrix

$$L = \begin{pmatrix} A_0 & B \\ C & A_1 \end{pmatrix}, \qquad \mathcal{D}(L) = \mathcal{D}(A_0) \oplus \mathcal{D}(A_1) \tag{3.1}$$

is a densely defined closed operator on the Hilbert space $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$.

The operator-matrix L is determined with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. Considering this decomposition as a fundamental one (see (2.1)) we can interpret $\mathfrak{H}$ as a Krein space $(\mathfrak{H}, [\cdot,\cdot])$ with the fundamental symmetry $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

The following operator Riccati equations are naturally associated with the operator matrix L:

$$KA_0 - A_1K + KBK = C, \qquad QA_1 - A_0Q + QCQ = B. \tag{3.2}$$

An operator $K \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$ is called a strong solution of the first Riccati equation in (3.2) if

$$\mathcal{R}(K\restriction_{\mathcal{D}(A_0)}) \subset \mathcal{D}(A_1) \quad \text{and} \quad KA_0x - A_1Kx + KBKx = Cx, \quad \forall x \in \mathcal{D}(A_0).$$

A strong solution $Q \in \mathcal{B}(\mathfrak{H}_-, \mathfrak{H}_+)$ of the second Riccati equation is defined in a similar way [4, 13].

The next result is well known (see, e.g., [4, Lemma 2.4]).

Lemma 3.1. *An operator $K \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$ $(Q \in \mathcal{B}(\mathfrak{H}_-, \mathfrak{H}_+))$ is a strong solution of the first (second) Riccati equation if and only if the subspace $\mathfrak{L}_K$ $(\mathfrak{M}_Q)$ in* (2.2) *is invariant for the operator L.*

Proposition 3.2. *With the notation as before, the following statements are equivalent:*

(i) *the operator L can be decomposed*

$$L = L_+ \dot{+} L_-, \qquad L_+ = L\restriction_{\mathfrak{L}_K}, \quad L_- = L\restriction_{\mathfrak{M}_Q}$$

with respect to the decomposition (2.4)*;*

(ii) *the operators K and Q in* (2.2) *are strong solutions of the Riccati equations* (3.2) *and $0 \in \rho(I + T)$, where $T = \begin{pmatrix} 0 & Q \\ K & 0 \end{pmatrix}$;*

(iii) *the operator ULU commutes with J, where U is defined by* (2.9).

Proof. The equivalence of (i) and (ii) follows from Lemmas 2.1, 3.1.

The statement (i) is equivalent to the commutation relation $L\mathcal{C} = \mathcal{C}L$, where $\mathcal{C}$ is defined by (2.8). In that case, using (2.10), one gets $JULU = U\mathcal{C}LU = UL\mathcal{C}U = ULUJ$. Conversely, if $JULU = ULUJ$, then $U\mathcal{C}LU = UL\mathcal{C}U$ (due to (2.10)) and, hence $\mathcal{C}L = L\mathcal{C}$ (since $0 \in \rho(U)$). Therefore, (ii) $\iff$ (iii). □

3.2. THE CASE OF J-SELF-ADJOINT OPERATOR L

An operator L in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ is called *J-self-adjoint* if L is self-adjoint with respect to the indefinite metric $[\cdot,\cdot]$. It is clear that L is J-self-adjoint if and only if $L^*J = JL$, where L^* is the standard adjoint operator in the Hilbert space $\mathfrak{H}$.

Lemma 3.3. *The operator L defined by* (3.1) *is J-self-adjoint in the Krein space* $(\mathfrak{H}, [\cdot,\cdot])$ *if and only if A_0 and A_1 are self-adjoint operators in the Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, respectively and $C = -B^*$.*

The proof of the Lemma immediately follows from the relation $L^*J = JL$ and (3.1) if one takes into account that the initial inner product $(\cdot,\cdot)$ of $\mathfrak{H}$ coincides with $[\cdot,\cdot]$ on $\mathfrak{H}_+$ and with $-[\cdot,\cdot]$ on $\mathfrak{H}_-$.

Theorem 3.4. *Assume that the operator matrix L defined by* (3.1) *is a J-self-adjoint operator in the Krein space* $(\mathfrak{H}, [\cdot,\cdot])$. *Then L is similar to a self-adjoint operator in $\mathfrak{H}$ if and only if the Riccati equation*

$$KA_0 - A_1K + KBK = -B^* \tag{3.3}$$

has a strong solution $K \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$ such that $\|K\| < 1$.

Proof. The implication: *strong solution $\Rightarrow$ the similarity* was proved in [4, Theorem 5.2]. We just repeat its principal stages.

Indeed, in that case, the second Riccati equation in (3.2) takes the form

$$QA_1 - A_0Q - QB^*Q = B \tag{3.4}$$

and it coincides with the adjoint $K^*A_1 - A_0K^* - K^*B^*K^* = B$ of (3.3). Therefore, if $K \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$ is a strong solution of (3.3), then $Q = K^* \in \mathcal{B}(\mathfrak{H}_-, \mathfrak{H}_+)$ is a strong solution of (3.4) (see [4, Remark 2.2]). Moreover $\|Q\| = \|K^*\| < 1$ since $\|K\| < 1$. This means that the corresponding operator $T = KP_+ + K^*P_-$ defined by (2.3) is self-adjoint and $\|T\| < 1$. According to Proposition 2.6, T is the operator of transition from (2.1) to (2.11).

Since K and K^* are strong solutions of the Riccati equations (3.3) and (3.4) the subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_{K^*}$ in (2.11) are invariant with respect to L (Lemma 3.1). This is equivalent to the commutation relation

$$\mathcal{C}L = L\mathcal{C}, \tag{3.5}$$

where $\mathcal{C}$ is defined by (2.8).

Since T is a self-adjoint strong contraction, the operator $\mathcal{C}$ admits the presentation $\mathcal{C} = Je^Y$ (Lemma 2.8). This allows one to rewrite (3.5) as follows $L^*e^Y = e^YL$ (since $L^*J = JL$). The latter relation means that L is similar to the self-adjoint operator $B = e^{Y/2}Le^{-Y/2}$ in $\mathfrak{H}$.

Conversely, assume that L is similar to a self-adjoint operator in $\mathfrak{H}$. Then there exists a J-orthogonal decomposition $\mathfrak{H} = \mathfrak{L}[\dot{+}]\mathfrak{M}$, where the maximal uniformly positive subspace $\mathfrak{L}$ and the maximal uniformly negative subspace $\mathfrak{M}$ are invariant with respect to L (see e.g., [3, Theorem 3] or [11, Theorem 6.1]). In that case subspaces $\mathfrak{L}$ and $\mathfrak{M}$ are determined by (2.2), where the corresponding operators K and Q satisfy the relations

$$\|K\| < 1, \quad \|Q\| < 1, \quad Q = K^*.$$

It follows from Lemma 3.1 that K is a strong solution of the Riccati equation (3.3). $\square$

Corollary 3.5. *If the Riccati equation* (3.3) *has a strong solution* $K \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$ *such that* $\|K\| < 1$, *then* L *is a self-adjoint operator in the Hilbert space* $\mathfrak{H}$ *with respect to the new inner product* $(\cdot,\cdot)_{\mathcal{C}} = [\mathcal{C}\cdot,\cdot]$.

Proof. Since $\|K\| < 1$, the corresponding operator of transition T is a self-adjoint strong contraction. This means that $\mathcal{C} = Je^Y$ (Lemma 2.8) and, hence, $(\cdot,\cdot)_{\mathcal{C}} = [\mathcal{C}\cdot,\cdot]_J = (e^Y\cdot,\cdot) = (e^{Y/2}\cdot, e^{Y/2}\cdot)$ is a new inner product in $\mathfrak{H}$ which is equivalent to the initial inner product $(\cdot,\cdot)$. The operator L is self-adjoint in $\mathfrak{H}$ with respect to $(\cdot,\cdot)_{\mathcal{C}}$ since $B = e^{Y/2}Le^{-Y/2}$ is self-adjoint in $\mathfrak{H}$ with respect to $(\cdot,\cdot)$ (see the proof of Theorem 3.4). □

Theorem 3.6. *Let* L *be defined by* (3.1) *and let* $K \in \mathcal{B}(\mathfrak{H}_+, \mathfrak{H}_-)$ *satisfy the condition* $0 \in \rho(I - K^*K)$. *Then the following assertions are equivalent:*

(i) *the operator* L *is a* J*-self-adjoint operator in the Krein space* $(\mathfrak{H}, [\cdot,\cdot])$ *and* K *is a strong solution of the Riccati equation* (3.3);
(ii) *the operator* ULU *commutes with* J *and, simultaneously, it is self-adjoint in* $\mathfrak{H}$.

Proof. It is clear that the operator $T = KP_+ + K^*P_-$ satisfies the relation

$$I - T^2 = (I - T)(I + T) = (I - K^*K) \oplus (I - KK^*),$$

where the decomposition is taken with respect to (2.1). This means that $0 \in \rho(I + T)$ (see, e.g., [4, Lemma 2.6 and Remark 2.7]). Therefore, T is the operator of transition from (2.1) to the decomposition $\mathfrak{H} = \mathfrak{L}_K[\dot{+}]\mathfrak{M}_{K^*}$.

Assume that L is a J-self-adjoint operator and K is a strong solution of the Riccati equation (3.3). Then, repeating the proof of Theorem 3.4 we conclude that the subspaces $\mathfrak{L}_K$ and $\mathfrak{M}_{K^*}$ are invariant with respect to L. By Proposition 3.2 this means that the operator ULU commutes with J. Further, using the first relation in (2.10) and taking into account the identity $JL = L^*J$, one gets

$$\begin{aligned}(ULU)^* &= J(I-T)^{-1}L^*J(I-T)^{-1} = \\ &= J[(I-T)^{-1}JL(I-T)^{-1}J]J = J[ULU]J = ULUJ^2 = ULU.\end{aligned} \tag{3.6}$$

Thus, ULU is a self-adjoint operator in $\mathfrak{H}$.

Conversely, if ULU commutes with J and it is self-adjoint, then

$$\begin{aligned}J(I-T)^{-1}L^*J(I-T)^{-1} &= (ULU)^* = ULU = \\ &= ULUJ^2 = J[ULU]J = J(I-T)^{-1}JL(I-T)^{-1}\end{aligned}$$

and, hence, $L^*J = JL$. Thus, L is a J-self-adjoint operator. Using Proposition 3.2, we arrive at the conclusion that K is a strong solution of the Riccati equation (3.3). □

Remark 3.7. The condition (ii) means that ULU is decomposed into two *self-adjoint* operators acting in the subspaces $\mathfrak{H}_\pm$ of the fundamental decomposition (2.1). However, in general, L is not similar to a self-adjoint operator (since ULU is not a similarity transformation).

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