Physically it is always possible to construct a system so that α^2 is negative, (by making (r/a) sufficiently large, for example). The difficulty arises because the Coulomb friction law is not always compatible with the equations of rigid-body dynamics.

Everything will be resolved if either the bearings or the rotating body are made elastic. When no rigid-body solution exists, the shaft seizes up, and no motion occurs. We note that the phenomenon does not occur in the case of a balanced rotor as it results in $F = 0$. It may be shown that the paradox also occurs when the c.g. is off the axis of rotation.

References

Armstrong-Helouvry, B., Dupont, P., and Canudas de Wit, C., 1994, "A Survey of Models, Analysis Tools and Compensation Methods for the Control of Machines with Friction," *Automatica*, Vol. 30, No. 7, pp. 1083–1138.

Lecornu, L., 1905, "Sur le frottement de glissement," *Comptes Rendu*, Vol. 140, pp. 635–637.

Klein, F., 1909, "Zu Painleve's Kritic der Coulombschen Reibungsgesetze," *Zeitschrift für Mathematic und Physik*, Vol. 58, pp. 186–191.

Wilms, E. V., and Cohen, H., 1981, "Planar Motion of a Rigid Body with a Friction Rotor," ASME Journal of Applied Mechanics, Vol. 48, pp. 205–206.

Wilms, E. V., and Cohen, H., 1983, "Limitations of the Coulomb Friction Model for Two Elementary Dynamical Systems," *International Journal of Mechanical Engineering Education*, Vol. 11, pp. 107–112.

Greenwood, D. T., 1965, *Principles of Dynamics*, Prentice-Hall, Englewood Cliffs, NJ.

Saddle-Point Principles for General Nonlinear Material Continua

A. Carini[11]

1 Introduction

A lot of techniques have been proposed for the theoretical and numerical treatment of inelastic (linear or nonlinear) boundary value problems, in particular for the elastic-plastic one. Some of these techniques rest on Colonnetti's idea (Colonnetti, 1918, 1950, 1955) of considering the material nonlinearities as imposed distortions in a supposedly linear elastic continuum. Colonnetti's approach to inelastic problems had numerous developments and applications in literature, particularly in elastoplasticity, by, among others, Ceradini (1966), Maier (1969), De Donato and Maier (1972), and Carter and Martin (1977). See also Koiter (1964) for some historical remarks on this topic.

Colonnetti's approach exploits the typical additive decomposition of the constitutive equations of many inelastic problems, into elastic and inelastic parts and, somehow, it is a precursor of the more recent *operator split* approach which is used in many fields, for instance in plasticity (Ortiz et al., 1983), in shakedown analysis (Zarka and Casier, 1979), and in micromechanics (Mura, 1987).

In previous works (Carini, 1996; Carini and De Donato, 1997) several *minimum principles* for the continuum problem with general nonlinear materials were shown. These principles rest on Colonnetti's idea and are based on the use of the solution of an *elastic auxiliary problem*, i.e., of the same problem in the elastic range with homogeneous boundary conditions in the presence of suitable distorsions.

The main characterizations of these minimum principles are (i) the energy meaning of the related functionals; (ii) their ability to specialize into well-known classical principles in case of linear elasticity and incremental elastoplasticity; (iii) their validity in all cases even in the lack of existence or uniqueness of the solution.

However, from a practical point of view, the main drawback which arises in the use of the above-mentioned functionals lies in the need to find the solution of the elastic auxiliary problem as already pointed out in Carini et al. (1995).

In this paper the above drawback is overcomed by the reformulation of the *minimum principles* as *saddle-point principles.*

After a brief preliminary description (in Section 2) of the continuum problem and a recall of the above-mentioned *minimum principles,* in Section 3 they are transformed reducing all terms of the functionals relevant to the auxiliary elastic problem to one term only. This allows us to recognize, in this term, the deformation energy of the elastic auxiliary problem and then to use classical extremum principles of elasticity for its evaluation. In this way, two new general *min-max* variational principles for the original problem are derived and later, in Section 4, an application is presented with reference to elastic-plastic material behavior.

2 Problem Formulation and Previous Variational Principles

A solid is considered to occupy a region Ω with a smooth external surface Γ in a triaxial orthogonal Cartesian reference system. Γ_u and Γ_p are the parts of surface Γ where displacements and surface tractions are imposed, respectively, while $\mathbf{x} = (x_1, x_2, x_3)$ denotes the position vector of a material point in Ω.

The external actions on the solid, i.e., the volume forces $F_i(\mathbf{x}; t)$ on Ω, the imposed displacements $v_i(\mathbf{x}; t)$ on Γ_u and the tractions $p_i(\mathbf{x}; t)$ on Γ_p, are given for any instant $t_0 \le t \le t_1$ of a known time interval $T = [t_0, t_1]$, through known time functions. We assume small strains and displacements, thus we consider the usual linear equilibrium and compatibility equations. Let's assume that the constitutive law is of the additive type, that is, the sum of two behaviors, the first one linear elastic and the second one inelastic, conceived as a deviation with respect to the linear elastic part.

The general inelastic problem is then here described by the following equations and it will be referred to as *problem P*:

Problem P:

$$\sigma_{ij,j} + F_i = 0 \quad \text{in} \quad \Omega \times T \tag{2.1}$$

$$\sigma_{ij} n_j = p_i \quad \text{on} \quad \Gamma_p \times T \tag{2.2}$$

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in} \quad \Omega \times T \tag{2.3}$$

$$u_i = v_i \quad \text{on} \quad \Gamma_u \times T \tag{2.4}$$

$$\sigma_{ij}(\mathbf{x}; t) = D_{ijhk}(\mathbf{x}; t)\epsilon_{hk}(\mathbf{x}; t) + \Psi^n_{ij}(\boldsymbol{\epsilon}(\mathbf{x}; t)) \tag{2.5}$$

$$\epsilon_{ij}(\mathbf{x}; t) = B_{ijhk}(\mathbf{x}; t)\sigma_{hk}(\mathbf{x}; t) + \Phi^n_{ij}(\boldsymbol{\sigma}(\mathbf{x}; t)) \tag{2.6}$$

$$u_i = \epsilon_{ij} = \sigma_{ij} = 0 \quad \text{on} \quad \bar{\Omega} \times (-\infty, t_0) \tag{2.7}$$

where $\bar{\Omega} = \Omega \cup \Gamma$. Relations (2.1) and (2.2) represent the indefinite and boundary equilibrium equations, respectively, while relations (2.3) and (2.4) represent the indefinite and boundary compatibility equations, respectively. Here, σ_{ij}, ϵ_{ij}, and u_i are the components of the stress tensor $\boldsymbol{\sigma}$, the strain tensor $\boldsymbol{\epsilon}$, and the displacement vector $\mathbf{u}$, respectively. Relations (2.5) and (2.6) represent the constitutive law in direct and invers form, respectively, where $D_{ijhk} = B^{-1}_{ijhk}$ is the time-dependent elastic moduli tensor with the usual simmetry and positive

[11] Department of Civil Engineering, University of Brescia, via Branze 38, 25123 Brescia, Italy.

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for publication in the ASME Journal of Applied Mechanics. Manuscript received by the ASME Applied Mechanics Division, July 15, 1996; final revision, May 21, 1997. Associate Technical Editor: W. K. Liu.

definiteness properties at any time t, while Ψ^n_{ij} and Φ^n_{ij} represent the "remaining" inelastic part operator of the constitutive law. Relations (2.7) represent the *initial conditions,* which are assumed as being homogeneous, for simplicity.

A stress distribution field $\sigma^*_{ij}(\mathbf{x}, t)$ is defined as *statically admissible* when it satisfies the following equilibrium equations:

$$\sigma^*_{ij,j} + F_i = 0 \quad \text{in} \quad \Omega \times T \tag{2.8}$$

$$\sigma^*_{ij} n_j = p_i \quad \text{on} \quad \Gamma_p \times T \tag{2.9}$$

while a stress distribution $\sigma^{**}_{ij}(\mathbf{x}, t)$ is defined as *self-equilibrated* when it satisfies the following *homogeneous* equilibrium equations:

$$\sigma^{**}_{ij,j} = 0 \quad \text{in} \quad \Omega \times T \tag{2.10}$$

$$\sigma^{**}_{ij} n_j = 0 \quad \text{on} \quad \Gamma_p \times T. \tag{2.11}$$

Analogously a strain field $\epsilon^0_{ij}(\mathbf{x}, t)$ is defined as *kinematically admissible* if it can be derived by means of the equation

$$\epsilon^0_{ij} = \tfrac{1}{2}(u^0_{i,j} + u^0_{j,i}) \quad \text{in} \quad \Omega \times T \tag{2.12}$$

from a displacement field $u^0_i(\mathbf{x}, t)$ satisfying the boundary conditions

$$u^0_i = v_i \quad \text{on} \quad \Gamma_u \times T \tag{2.13}$$

while a strain field $\epsilon^{00}_{ij}(\mathbf{x}, t)$ is here defined as *self-compatible* if it can be derived by means of the equation

$$\epsilon^{00}_{ij} = \tfrac{1}{2}(u^{00}_{i,j} + u^{00}_{j,i}) \quad \text{in} \quad \Omega \times T \tag{2.14}$$

from a displacement field $u^{00}_i(\mathbf{x}, t)$ vanishing on Γ_u, i.e., that satisfies the *homogeneous* boundary conditions

$$u^{00}_i = 0 \quad \text{on} \quad \Gamma_u \times T. \tag{2.15}$$

In the following the symbols ϵ^{ef}_{ij} and σ^{ef}_{ij} represent, respectively, the strains and the stresses in the solid under the external actions F_i, p_i, v_i and under the hypothesis of a linear elastic behavior with material properties corresponding to B_{ijhk}.

In a previous paper (Carini, 1996) two minimum principles were shown. In particular the following two propositions were proved:

Proposition 1: *A statically admissible stress field σ^*_{ij} is a (or the) solution of problem P if and only if both the following conditions are satisfied:*

(i) *It minimizes (absolute minimum) the functional F^I_{ce}, where*

$$F^I_{ce}[\sigma^*_{ij}] = \int_T \left\{ \frac{1}{2}\int_\Omega \sigma^*_{ij} B_{ijhk} \sigma^*_{hk} d\Omega - \int_{\Gamma_u} v_i n_j \sigma^*_{ij} d\Gamma + \int_\Omega \Phi^n_{ij}(\boldsymbol{\sigma}^*)(\sigma^*_{ij} - \sigma^{ef}_{ij}) d\Omega + \frac{1}{2}\int_\Omega \sigma^d_{ij}(\boldsymbol{\sigma}^*) B_{ijhk} \sigma^d_{hk}(\boldsymbol{\sigma}^*) d\Omega \right\} dt. \tag{2.16}$$

(ii) *The minimum is equal to F^0_{ce}, where*

$$F^0_{ce} = \int_T \left\{ \frac{1}{2}\int_\Omega \sigma^{ef}_{ij} \epsilon^{ef}_{ij} d\Omega - \int_{\Gamma_u} v_i n_j \sigma^{ef}_{ij} d\Gamma \right\} dt. \tag{2.17}$$

Proposition 2: *A kinematically admissible displacement field u^0_i is a (or the) solution of problem P if and only if both the following conditions are satisfied:*

(i) *It minimizes (absolute minimum) the functional F^I_{tpe}, where*

$$F^I_{tpe}[u^0_i] = \int_T \left\{ \frac{1}{2}\int_\Omega \epsilon^0_{ij} D_{ijhk} \epsilon^0_{hk} d\Omega - \int_\Omega F_i u^0_i d\Omega - \int_{\Gamma_p} p_i u^0_i d\Gamma + \int_\Omega \Psi^n_{ij}(\boldsymbol{\epsilon}^0)(\epsilon^0_{ij} - \epsilon^{ef}_{ij}) d\Omega + \frac{1}{2}\int_\Omega \epsilon^s_{ij}(\boldsymbol{\epsilon}^0) D_{ijhk} \epsilon^s_{hk}(\boldsymbol{\epsilon}^0) d\Omega \right\} dt. \tag{2.18}$$

(ii) *The minimum is equal to F^0_{tpe}, where*

$$F^0_{tpe} = \int_T \left\{ \frac{1}{2}\int_\Omega \sigma^{ef}_{ij} \epsilon^{ef}_{ij} d\Omega - \int_\Omega F_i u^{ef}_i d\Omega - \int_{\Gamma_p} p_i u^{ef}_i d\Gamma \right\} dt. \tag{2.19}$$

It is worth noting that, in the functional (2.18), $\epsilon^0_{ij} = \epsilon^0_{ij}(\mathbf{u}^0) = \frac{1}{2}(u^0_{i,j} + u^0_{j,i})$. In (2.16) σ^d_{ij} is the solution of the following *imposed strain elastic auxiliary problem P^d*:

Problem P^d:

$$\sigma^d_{ij,j} = 0 \quad \text{in} \quad \Omega \times T \tag{2.20}$$

$$\sigma^d_{ij} n_j = 0 \quad \text{on} \quad \Gamma_p \times T \tag{2.21}$$

$$\epsilon^d_{ij} = \tfrac{1}{2}(u^d_{i,j} + u^d_{j,i}) \quad \text{in} \quad \Omega \times T \tag{2.22}$$

$$u^d_i = 0 \quad \text{on} \quad \Gamma_u \times T \tag{2.23}$$

$$\epsilon^d_{ij} = B_{ijhk} \sigma^d_{hk} + d_{ij}, \tag{2.24}$$

where d_{ij} is interpreted as the *imposed distorsions* given by $d_{ij} = -\Phi^n_{ij}(\boldsymbol{\sigma}^*)$ being $\boldsymbol{\sigma}^*$ any statically admissible stress field.

Conversely in (2.18) ϵ^s_{ij} is the solution of the following *imposed stress* elastic auxiliary problem P^s:

Problem P^s:

$$\sigma^s_{ij,j} = 0 \quad \text{in} \quad \Omega \times T \tag{2.25}$$

$$\sigma^s_{ij} n_j = 0 \quad \text{on} \quad \Gamma_p \times T \tag{2.26}$$

$$\epsilon^s_{ij} = \tfrac{1}{2}(u^s_{i,j} + u^s_{j,i}) \quad \text{in} \quad \Omega \times T \tag{2.27}$$

$$u^s_i = 0 \quad \text{on} \quad \Gamma_u \times T \tag{2.28}$$

$$\sigma^s_{ij} = D_{ijhk} \epsilon^s_{hk} + s_{ij}, \tag{2.29}$$

where s_{ij} is interpreted as the *imposed stresses* given by $s_{ij} = -\Psi^n_{ij}(\boldsymbol{\epsilon}^0)$ being $\boldsymbol{\epsilon}^0$ any kinematically admissible strain field. It is worth noting that both propositions 1 and 2 have been proved assuming B_{ijhk} and D_{ijhk} as time independent; however, it is easy to verify their validity also when B_{ijhk} and D_{ijhk} are time dependent.

3 New Variational Principles

A main computational drawback of the above principles (proposition 1 and 2) lies in the need to evaluate the last term of (2.16) and (2.18) through the solution of the elastic auxiliary problem for every admissible stress field $\boldsymbol{\sigma}^*$ (for *proposition 1*) or for every compatible strain field $\boldsymbol{\epsilon}^0$ (for *proposition 2*) or, in other words, in the need to find Green functions of the elastic problem. However, the last two mentioned terms are representative of elastic energies which can be evaluated, using classical elasticity principles, by maximization of suitable functionals in additional new variables.

3.1 A Min-Max Extended Complementary Energy Principle. Following this path of reasoning and making reference to *proposition 1* it is possible to find functional F^d_{ce} to be maximized, in order to obtain the last term of (2.16), simply by using the principle of virtual work as follows (at any instant t):

$$\frac{1}{2}\int_{\Omega}\sigma_{ij}^{d}(\boldsymbol{\sigma}^*)B_{ijhk}\sigma_{hk}^{d}(\boldsymbol{\sigma}^*)d\Omega=-\frac{1}{2}\int_{\Omega}\sigma_{ij}^{d}(\boldsymbol{\sigma}^*)B_{ijhk}\sigma_{hk}^{d}(\boldsymbol{\sigma}^*)d\Omega+\int_{\Omega}\Phi_{ij}^{n}(\boldsymbol{\sigma}^*)\sigma_{ij}^{d}d\Omega. \quad (3.1)$$

It is easy to recognize the right-end side of (3.1) as the (changed in sign) complementary energy value at the solution of the elastic problem (at instant t) with imposed distorsions $d_{ij}=-\Phi_{ij}^{n}(\boldsymbol{\sigma}^*)$. This means that, by virtue of the minimum principle of complementary energy, we can write

$$\frac{1}{2}\int_{\Omega}\sigma_{ij}^{d}(\boldsymbol{\sigma}^*)B_{ijhk}\sigma_{hk}^{d}(\boldsymbol{\sigma}^*)d\Omega=\max_{\sigma_{ij}^{d**}}\{F_{ce}^{d}[\sigma_{ij}^{d**},\sigma_{ij}^{*}]\} \quad (3.2)$$

where

$$F_{ce}^{d}[\sigma_{ij}^{d**},\sigma_{ij}^{*}]=\int_{T}\left\{-\frac{1}{2}\int_{\Omega}\sigma_{ij}^{d**}B_{ijhk}\sigma_{hk}^{d**}d\Omega+\int_{\Omega}\Phi_{ij}^{n}(\boldsymbol{\sigma}^*)\sigma_{ij}^{d**}d\Omega\right\}dt, \quad (3.3)$$

σ_{ij}^{d**} being any *self-equilibrated* stress field. Then, by substitution of Eq. (3.2), the functional (2.16) is transformed into the following new functional:

$$\mathcal{F}_{ce}[\sigma_{ij}^{*},\sigma_{ij}^{d**}]=\int_{T}\left\{\frac{1}{2}\int_{\Omega}\sigma_{ij}^{*}B_{ijhk}\sigma_{hk}^{*}d\Omega-\int_{\Gamma_u}v_in_j\sigma_{ij}^{*}d\Gamma+\int_{\Omega}\Phi_{ij}^{n}(\boldsymbol{\sigma}^*)(\sigma_{ij}^{*}-\sigma_{ij}^{ef})d\Omega-\frac{1}{2}\int_{\Omega}\sigma_{ij}^{d**}B_{ijhk}\sigma_{hk}^{d**}d\Omega+\int_{\Omega}\Phi_{ij}^{n}(\sigma_{ij}^{*})\sigma_{ij}^{d**}d\Omega\right\}dt \quad (3.4)$$

and the following statement can be asserted:

Proposition 3: *A statically admissible stress field σ_{ij}^{*} and a self-equilibrated stress field σ_{ij}^{d**} are a (or the) solution of problem P and problem P^d, respectively, if and only if both the following conditions are satisfied:*

(i) *They make stationary (minimum respect to σ_{ij}^{*} and maximum respect to σ_{ij}^{d**}) the functional (3.4);*

(ii) *The saddle-point value of the functional (3.4) is equal to F_{ce}^{0} Eq. (2.17).*

It is worth noting that whereas on the one hand the new functional does not require the preliminar solution of the elastic auxiliary problem or the evaluation of the stress Green function due to distorsions, on the other hand the new formulation is larger in terms of the number (double) of unknown fields required to solve the problem.

3.2 A Min-Max Extended Total Potential Energy Principle. In an analogous way it is possible to derive, from proposition 2, a new min-max principle through the determination of functional F_{tpe}^{s} to be maximized in order to obtain the last term of (2.18). This may be made simply by using the principle of virtual work as follows (at any instant t):

$$\frac{1}{2}\int_{\Omega}\epsilon_{ij}^{s}(\boldsymbol{\epsilon}^0)D_{ijhk}\epsilon_{hk}^{s}(\boldsymbol{\epsilon}^0)d\Omega=-\frac{1}{2}\int_{\Omega}\epsilon_{ij}^{s}(\boldsymbol{\epsilon}^0)D_{ijhk}\epsilon_{hk}^{s}(\boldsymbol{\epsilon}^0)d\Omega+\int_{\Omega}\Psi_{ij}^{n}(\boldsymbol{\epsilon}^0)\epsilon_{ij}^{s}d\Omega. \quad (3.5)$$

It is easy to recognize the right-end side of (3.5) as the (changed in sign) total potential energy value at the solution of the elastic problem (at instant t) with imposed stresses $s_{ij}=-\Psi_{ij}^{n}(\boldsymbol{\epsilon}^0)$. This means that, by virtue of the minimum principle of total potential energy, we can write

$$\frac{1}{2}\int_{\Omega}\epsilon_{ij}^{s}(\boldsymbol{\epsilon}^0)D_{ijhk}\epsilon_{hk}^{s}(\boldsymbol{\epsilon}^0)d\Omega=\max_{u_i^{s00}}\{F_{tpe}^{s}[u_i^{s00},u_i^{0}]\} \quad (3.6)$$

where

$$F_{tpe}^{s}[u_i^{s00},u_i^{0}]=\int_{T}\left\{-\frac{1}{2}\int_{\Omega}\epsilon_{ij}^{s00}D_{ijhk}\epsilon_{hk}^{s00}d\Omega+\int_{\Omega}\Psi_{ij}^{n}(\boldsymbol{\epsilon}^0)\epsilon_{ij}^{s00}d\Omega\right\}dt, \quad (3.7)$$

ϵ_{ij}^{s00} being any *self-compatible* strain field (that is any strain field which can be derived from a displacement field u_i^{s00} satisfying the homogeneous boundary conditions $u_i^{s00}=0$ on Γ_u). Then, by substitution of Eq. (3.6), the functional (2.18) can be transformed into the following new functional:

$$\mathcal{F}_{tpe}[u_i^{0},u_i^{s00}]=\int_{T}\left\{\frac{1}{2}\int_{\Omega}\epsilon_{ij}^{0}D_{ijhk}\epsilon_{hk}^{0}d\Omega-\int_{\Omega}F_iu_i^{0}d\Omega-\int_{\Gamma_p}p_iu_i^{0}d\Gamma+\int_{\Omega}\Psi_{ij}^{n}(\boldsymbol{\epsilon}^0)(\epsilon_{ij}^{0}-\epsilon_{ij}^{ef})d\Omega-\frac{1}{2}\int_{\Omega}\epsilon_{ij}^{s00}D_{ijhk}\epsilon_{hk}^{s00}d\Omega+\int_{\Omega}\Psi_{ij}^{n}(\boldsymbol{\epsilon}^0)\epsilon_{ij}^{s00}d\Omega\right\}dt \quad (3.8)$$

and the following statement can be asserted:

Proposition 4: *A kinematically admissible displacement field u_i^{0} and a displacement field u_i^{s00} vanishing on Γ_u are a (or the) solution of problem P and problem P^s, respectively, if and only if both the following conditions are satisfied:*

(i) *they make stationary (minimum respect to u_i^{0} and maximum respect to u_i^{s00}) the functional (3.8);*

(ii) *The saddle-point value of the functional (3.8) is equal to F_{tpe}^{0} Eq. (2.19).*

It is worth noting that, in the functional (3.8), $\epsilon_{ij}^{0}=\epsilon_{ij}^{0}(\mathbf{u}^0)=\frac{1}{2}(u_{i,j}^{0}+u_{j,i}^{0})$ and that $\epsilon_{ij}^{s00}=\epsilon_{ij}^{s00}(\mathbf{u}^{s00})=\frac{1}{2}(u_{i,j}^{s00}+u_{j,i}^{s00})$.

In this case too, the same remark pointed out at the end of proposition 3 applies with reference to the fact that whereas, on the one hand, the new functional does not require the preliminar solution of the elastic auxiliary problem or the evaluation of the strain Green function due to imposed stresses, on the other hand, the new formulation is larger in terms of the number (double) of unknown fields required to solve the problem.

Remarks. In order to make a comparison with existing variational principles for the nonlinear continuum problem, let's summarize the main properties of the above two min-max principles:

(a) Each functional has a physical meaning amenable to an energy (which is useful for its construction).

(b) All the variables involved have a physical meaning (stress, strain, etc.).

(c) The specialization of the constitutive law (2.5) or (2.6) to the elastic case ($\Psi_{ij}^{n}=\Phi_{ij}^{n}=0$) leads the functionals to the classical ones of the theory of elasticity.

When a comparison of the above properties is made with those of other existing variational principles for the nonlinear continuum problem, it so happens that some or all of the above-mentioned properties (a) to (c) do not apply or, rather, some drawbacks arise. In particular the following can be said.

1 With reference to the variational formulations derived using the linear or nonlinear method of adding the adjoint operator (see Morse and Feshbach, 1953), the following drawbacks occur: (a) the additional variables required by the method usually have no physical meaning; (b) in the case of the linear constitutive law the nonlinear (quadratic) part of the functional is not defined in sign, which means that the solution corresponds to a *non-oriented* saddle-point of the functional, that is, the solution does not correspond to the minimum of the functional with respect to a subset of the variables nor does it correspond to the maximum with respect to the remaining variables.

2 With reference to the variational formulations derived using the least square method, the following drawbacks occur: (a) the functionals lack physical meaning, (b) the functionals lead (when a discretized solution procedure is adopted) to equation systems with worse (i.e., larger) values of the conditioning index than those of the corresponding discretized equation systems derived from the above saddle-point principles (proposition 3 and 4) (see Carini et al., 1995); in fact, the present formulation (propositions 3 and 4) could be regarded as a *preconditioned* least-square formulation, interpreting the inverse of the elastic operator as a *preconditioning* operator (see Carini et al., 1995).

3 When using Tonti's general method for the variational formulation of any nonlinear problem (see Tonti, 1984) the above properties (a), (b), (c) cannot in general be guaranteed while, on the contrary, they are satisfied by the functionals of propositions 3 and 4. This derived by the use (see Carini and De Donato, 1997), as integrating operator, of the inverse of the linear elastic operator in deriving propositions 1 and 2 (from which proposition 3 and 4 are obtained). This emphasizes the peculiar role of the choice of a particular integrating operator in deriving, from Tonti's general method, variational formulations satisfying properties (a), (b), and (c).

4 For time-dependent problems and with reference to variational formulations which use convolutive bilinear forms (see, for instance, Gurtin, 1964; Tonti, 1973; Rafalski, 1969a, 1969b; Reiss and Haug, 1978), the following can be pointed out: (a) these variational formulations are valid only for linear problems with constant physical properties; (b) in contrast to these formulations, the present one applies to generally nonlinear time dependent problems; (c) the original formulation by Gurtin and Tonti is not an extremal formulation. In fact the solution corresponds to a *non-oriented* saddle-point of the functional. The subsequent formulation by Rafalski (1969a, b) and Reiss and Haug (1978), need integrations over unbounded intervals.

5 Finally, it is worth noting that, for the elastic-plastic problem formulated in terms of variational formulations based on nonlinear programming techniques (see, for instance, Comi et al., 1991; Romano et al., 1993), the existence of a potential (generally nonsmooth) is always required. In the author's opinion it is possible to overcome this drawback using the present formulation, as partially showed in Carini (1996) and in the following Section 4.

4 An Application to Incremental Elastoplasticity

As an application of the total potential energy principle (proposition 4) to elastoplasticity, in the following we consider the elastoplastic continuum problem in the presence of an incremental constitutive law with hardening and/or softening and non associated flow rule. The set of relations describing the above behavior is the following:

$$\dot{\sigma}_{ij} = D_{ijhk}\dot{\epsilon}^{e}_{hk} \tag{4.1}$$

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^{e}_{ij} + \dot{\epsilon}^{p}_{ij} \tag{4.2}$$

$$\dot{\epsilon}^{p}_{ij} = \frac{\partial\psi(\sigma_{ij}, \alpha_\rho)}{\partial\sigma_{ij}}\dot{\lambda} \tag{4.3}$$

$$\dot{\alpha}_\rho = l_\rho(\sigma_{ij}, \alpha_\rho)\dot{\lambda} \tag{4.4}$$

$$\phi = \phi(\sigma_{ij}, \alpha_\rho) \le 0 \tag{4.5}$$

$$\dot{\lambda} = 0 \quad \text{in} \quad \Omega_e(\text{i.e., if } \phi < 0) \tag{4.6}$$

$$\dot{\phi} \le 0, \quad \dot{\lambda} \ge 0, \quad \dot{\phi}\dot{\lambda} = 0 \quad \text{in} \quad \Omega_p(\text{i.e., if } \phi = 0) \tag{4.7}$$

where Eq. (4.1) establishes the relation between the incremental stress $\dot{\sigma}_{ij}$ and the elastic incremental strains $\dot{\epsilon}^{e}_{ij}$; Eq. (4.2) states the additivity of the elastic and plastic strains; Eqs. (4.3) and (4.4) state the evolution of the inelastic kinematic variables $\dot{\epsilon}^{p}_{ij}$ and $\dot{\alpha}_\rho$ as a function of the plastic multipliers $\dot{\lambda}$ and of the plastic potential ψ being l_ρ a given nonlinear function of σ_{ij} and α_ρ. Equation (4.5) defines the elastic domain while the set of Eqs. (4.6)–(4.7) expresses the loading-unloading criterion. In a more compact form, all the Eqs. (4.1)–(4.7) may be written as follows:

$$\dot{\sigma}_{ij} = \bar{D}_{ijhk}\dot{\epsilon}_{hk} \tag{4.8}$$

where

$$\bar{D}_{ijhk} = D_{ijhk} - \frac{D_{ijmn}\dfrac{\partial\psi}{\partial\sigma_{mn}}\dfrac{\partial\phi}{\partial\sigma_{pq}}D_{pqhk}}{\dfrac{\partial\phi}{\partial\sigma_{mn}}D_{mnrs}\dfrac{\partial\psi}{\partial\sigma_{rs}} - \dfrac{\partial\phi}{\partial\alpha_\rho}l_\rho}$$

$$\text{if} \quad \phi = 0 \quad \text{and} \quad \dot{\phi} = 0 \tag{4.9}$$

while $\bar{D}_{ijhk} = D_{ijhk}$,

$$\text{if} \quad \phi < 0 \quad \text{or} \quad \phi = 0 \quad \text{and} \quad \dot{\phi} < 0 \tag{4.10}$$

being $\bar{D}_{ijhk} \neq \bar{D}_{hkij}$ as a consequence of the non-associated character of the constitutive law.

In order to write Eq. (4.8) in the split form (2.5), the following position can be adopted:

$$\bar{D}_{ijhk} = D_{ijhk} - cD^{n}_{ijhk} \tag{4.11}$$

with

$$D^{n}_{ijhk} \neq D^{n}_{hkij} \tag{4.12}$$

$$\begin{cases} c = 1 & \text{if} \quad \phi = 0 \quad \text{and} \quad \dot{\phi} = 0 \\ c = 0 & \text{if} \quad \phi < 0 \quad \text{or} \quad \phi = 0 \quad \text{and} \quad \dot{\phi} < 0 \end{cases} \tag{4.13}$$

leading finally to the following expression of the inelastic part of the constitutive law:

$$\Psi^{n}_{ij}(\dot{\epsilon}_{hk}) = -cD^{n}_{ijhk}\dot{\epsilon}_{hk}. \tag{4.14}$$

The application of the extended total potential energy principle (proposition 4) to the above elastic-plastic behavior finally leads to the following extended functional:

$$\begin{aligned}\mathcal{F}_{tpe}[\dot{u}^{0}_{i}, \dot{u}^{s00}_{i}] = {} & \tfrac{1}{2}\int_\Omega \dot{\epsilon}^{0}_{ij}D_{ijhk}\dot{\epsilon}^{0}_{hk}d\Omega - \int_\Omega \dot{F}_i\dot{u}^{0}_{i}d\Omega \\ & - \int_{\Gamma_p}\dot{p}_i\dot{u}^{0}_{i}d\Gamma - \int_{\Omega_p}c\dot{\epsilon}^{0}_{ij}D^{n}_{ijhk}(\dot{\epsilon}^{0}_{hk} - \dot{\epsilon}^{ef}_{hk})d\Omega \\ & - \tfrac{1}{2}\int_\Omega \dot{\epsilon}^{s00}_{ij}D_{ijhk}\dot{\epsilon}^{s00}_{hk}d\Omega - \int_{\Omega_p}c\dot{\epsilon}^{0}_{ij}D^{n}_{ijhk}\dot{\epsilon}^{s00}_{hk}d\Omega.\end{aligned} \tag{4.15}$$

An analogous functional was stated by Telega (1980), using the *nonlinear method of adding the adjoint operator*. However, it is worth noting that while both the present and Telega's approaches have the doubling of the unknown variables in common, the main differences between the two approaches may be summarized in the following two points:

(a) In the present approach the solution corresponds to the min-max of the functional (4.15) while with Telega's approach, the solution corresponds only to the stationarity of its analogous functional.

(b) As already said, in the present approach all variables of the functional (4.15) have a physical meaning; this is not true for all the variables of Telega's functional (in particular for the variables of the adjoint operator).

References

Carini, A., 1996, "Colonnetti's minimum principle extension to generally nonlinear materials," *Int. J. Solids Structures*, Vol. 33, pp. 121–144.

Carini, A., and De Donato, O., 1997, "A Comprehensive Energy Formulation for General Nonlinear Material Continua," ASME JOURNAL OF APPLIED MECHANICS, Vol. 64, pp. 353–360.

Carini, A., Gelfi, P., and Marchina, E., 1995, "An energetic formulation for the linear viscoelastic problem. Part I: Theoretical results and first calculations," *Int. J. Numer. Methods Eng.*, Vol. 38, pp. 37–62.

Carter, P., and Martin, J. B., 1977, "A note on the derivation of the kinematic rate theorem of plasticity from the free energy minimum principle," *Journal de Mécanique*, Vol. 16, pp. 791–802.

Ceradini, G., 1966, "A maximum principle for the analysis of elastic-plastic systems," *Meccanica*, Vol. 1, pp. 77–82.

Colonnetti, G., 1918, "Sul problema delle coazioni elastiche," *Rend. Accad. Lincei*, Vol. 27, Serie 5a, 2. sem.

Colonnetti, G., 1950, "Elastic equilibrium in the presence of permanent set," *Quart. Appl. Math.*, Vol. 7, pp. 353–362.

Colonnetti, G., 1955, *L'equilibre des corps déformables*, Dunod, Paris.

Comi, C., Corigliano, A., and Maier, G., 1991, "Extremum properties of finite-step solutions in elastoplasticity with non-linear mixed hardening," *Int. J. Solids Structures*, Vol. 27, pp. 965–981.

De Donato, O., and Maier, G., 1972, "Mathematical programming method for the inelastic analysis of reinforced concrete frames allowing for limited rotation capacity," *Int. J. Numer. Methods Eng.*, Vol. 4, pp. 307–329.

Gurtin, M. E., 1964, "Variational principles for linear initial-value problems," *Quart. Appl. Math.*, Vol. 22, pp. 252–256.

Koiter, W. T., 1964, "General theorems for elastic-plastic solids," *Progress in Solid Mechanics*, I. N. Sneddon and R. Hill, eds., Vol. 1, Chapter IV, pp. 167–221.

Maier, G., 1969, "Some theorems for plastic strain rates and plastic strains," *Journal de Mécanique*, Vol. 8, pp. 5–19.

Morse, P. M., and Feshbach, H., 1953, *Methods of theoretical physics*, Vol. I, McGraw-Hill, New York.

Mura, T., 1987, *Micromechanics of defects in solids*, Martinus Nijhoff, Dordrecht, The Netherlands.

Ortiz, M., Pinsky, P. M. and Taylor, L., 1983, "Operator split methods for the numerical solution of the elastoplastic dynamic problem," *Comput. Meths. Appl. Mech. Engrg.*, Vol. 39, pp. 137–157.

Rafalski, P., 1969, "Orthogonal projection method. I. Heat conduction boundary problem," *Bull. Acad. Polon. Sci., Sér. Sci. Techn.*, Vol. 17, pp. 63–67.

Rafalski, P., 1969, "Orthogonal projection method. II. Thermoelastic problem," *Bull. Acad. Polon. Sci., Sér. Sci. Techn.*, Vol. 17, pp. 69–74.

Reiss, R., and Haug, E. J., 1978, "Extremum principles for linear initial-value problems of mathematical physics," *Int. J. Engng. Sci.*, Vol. 16, pp. 231–251.

Romano, G., Rosati, L., and Marotti de Sciarra, F., 1993, "A variational theory for finite-step elastoplastic problems," *Int. J. Solids Structures*, Vol. 30, pp. 2317–2334.

Telega, J. J., 1980, "Variational principles for rate boundary-value problems in nonassociated plasticity," *ZAMM*, Vol. 60, pp. 71–82.

Tonti, E., 1973, "On the variational formulation for linear initial value problems," *Annali di Matematica Pura ed Applicata*, Vol. 45, pp. 331–359.

Tonti, E., 1984, "Variational formulation for every nonlinear problem," *Int. J. Engng. Sci.*, Vol. 22, pp. 1343–1371.

Zarka, J., and Casier, J., 1979, "Elastic-plastic response of a structure to cyclic loading: practical rules," *Mechanics Today*, Vol. 6, Nemat-Nasser, ed., Pergamon Press, New York, pp. 93–198.

Impact Coefficients and Tangential Impacts

R. M. Brach[12]

Introduction

The last decade has seen a flurry of papers on the topic of rigid-body impact theory. Despite the apparent simplicity of the topic, some challenging problems exist and there have been some interesting and, at times, controversial developments. For example, some dispute has arisen on how best to define and use coefficients, particularly coefficients associated with the process of restitution normal to the contact surface in the presence of Coulomb friction. At least five coefficients, e, have been proposed and discussed, one is a kinematic coefficient (defined as the ratio of normal velocity components) attributed to Newton. Another is a kinetic coefficient (ratio of normal contact impulses) attributed to Poisson. A third is an energetic coefficient (ratio of work done by normal contact impulses) defined by Stronge (1990). A fourth coefficient is by Ivanov (1992), (defined as a ratio of kinetic energy losses). Batlle and Cordona (1997) analyze three-dimensional tangential impacts and define another suitable for that problem. Coefficients associated with directions other than normal and processes other than compression have been defined and used (see Brach, 1991), but most of the controversy, so far, is with the normal coefficients.

Newton's coefficient has been used for collinear impacts and frictionless oblique impacts for nearly 300 years. Its use has been generalized and extended to three-dimensional collisions by Brach (1991, 1997). This approach has been extended to pseudo-rigid bodies by Cohen and Mac Sithigh (1996). Whereas the coefficient has an upper bound of unity for certain special conditions, it is now recognized that this is not true in general since a value of 1 sometimes can lead to a solution that violates energy conservation. Poisson's coefficient has likewise been known for many years; some, such as Wang and Mason (1992), claim that its use guarantees energy conservation but Newton's law does not. Poisson's coefficient also is bounded by 0 and 1 only for certain types of collisions and body orientations. The lack of unit bounds has prompted questions of "energetic consistency." In discussing the process of normal deformation, Stronge says that in a consistent theory the part of the energy dissipation during restitution cannot be larger than the corresponding part during compression. He then defines a coefficient using a work constraint such that $W_n^R \le e^2 W_n^A$, where each W_n is the work done by the normal impulse during rebound and approach, respectively. Stronge also demands an energetically consistent (normal) coefficient of restitution to be one that is independent of the tangential contact process (friction). It remains to be determined under what conditions Stronge's coefficient has a unit upper bound.

Another question is sometimes raised concerning a dependence of coefficients on initial conditions. Some hold the notion that "impact coefficients" should be similar to material constants and should not depend on initial conditions. In the case of impact coefficients, dependence on initial conditions is not a drawback but a necessity. Newton's and other coefficients are quantities that represent nonlinear dynamic material behavior. If impact coefficients did *not* depend on initial conditions, they would have no utility. A requirement that impact coefficients be independent of each other is a lofty but impractical goal. When complicated, nonlinear three-dimensional contact processes are considered in the presence of inertial coupling, arbitrary initial conditions and perhaps nonpoint contact, the expectation of coefficient independence is unrealistic.

Newton's coefficient is said to lack energetical consistency; but other deficiencies are also claimed. The existence of a tangential impact, or Impact Without Collision (IW/OC), associated with Painlevé's Paradox (see Brogliato, 1996) has been given attention recently. Wang and Mason (1992) and Batlle (1993) use a tangential impact as an example of where the use of Newton's coefficient not only is said to be energetically inconsistent but is impossible to apply. The basis of this claim appears not to be that tangential impacts do not require normal deformation but rather that the definition of $e = -V_{cn}/v_{cn}$ is undefined for $v_{cn} = 0$. It is the purpose of this paper to address these issues and to determine whether or not the use of Newton's

[12] Department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, IN 46556.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Aug. 30, 1996; final revision, May 30, 1997. Associate Technical Editor: S. W. Shaw.