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Research Article

Positive Solutions of Nonlocal Boundary Value Problem for High-Order Nonlinear Fractional q-Difference Equations

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We study the nonlinear q-difference equations of fractional order $(D_q^\alpha u)(t) + f(t, u(t)) = 0$, $0 < t < 1$, $(D_q^i u)(0) = 0$, $(D_q^\beta u)(1) = a(D_q^\beta u)(\eta)$, $0 \le i \le n-2$, where D_q^α is the fractional q-derivative of the Riemann-Liouville type of order α, $n-1 < \alpha \le n$, $\alpha > 2$, $1 \le \beta \le n-2$, and $0 \le a \le 1$. We obtain the existence and multiplicity results of positive solutions by using some fixed point theorems. Finally, we give examples to illustrate the results.

1. Introduction

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [1, 2]. It is rich in history and in applications as the reader can find in the work by Ernst [3]. For some recent existence results on q-difference equations, see [4–7] and the references therein.

The fractional q-difference calculus had its origin in the works by Al-Salam and Agarwal. Henceforth, fractional q-difference equations have gained considerable importance due to their application in various sciences, such as physics, chemistry, aerodynamics, biology, economics, control theory, mechanics, electricity, signal and image processing, biophysics, blood flow phenomena, and fitting of experimental data. It has been a significant development in difference equations involving fractional q-derivatives; see [8–11] and references therein. As well known, fractional differential equations boundary value problems is currently under strong research, see [12–21] and references therein. In particular, in recent years, fractional q-difference boundary value problem (BVP) was in its infancy, and many people begin to study the existence of positive solutions for this kind of BVP; see [22–27] and references therein. However, there are few related results available. Lots of work and development should be done in the future.

Recently, in [16], Li et al. considered the BVP of nonlinear fractional difference equation

$$\begin{gathered} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\xi), \end{gathered} \tag{1}$$

where $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 \le a \le 1$, $\xi \in (0, 1)$, and $f : [0,1] \times [0, \infty) \to [0, \infty)$ satisfies Caratheodory type conditions.

More recently, in [23], Ferreira considered the BVP of fractional q-difference equation

$$\begin{gathered} \left(D_q^{\alpha} y\right)(x) = -f(x, y(x)), \quad 0 < x < 1, \\ y(0) = \left(D_q y\right)(0) = 0, \qquad \left(D_q y\right)(1) = \beta \ge 0, \end{gathered} \tag{2}$$

where $2 < \alpha \le 3$ and $f : [0,1] \times R \to R$ is a nonnegative continuous function.

Motivated by the work above, in this paper, we will discuss the following BVP:

$$\begin{gathered} \left(D_q^{\alpha} u\right)(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ \left(D_q^{i} u\right)(0) = 0, \quad \left(D_q^{\beta} u\right)(1) = a\left(D_q^{\beta} u\right)(\eta), \quad 0 \le i \le n-2, \end{gathered} \tag{3}$$

where $n-1<\alpha\le n\ (n>2)$, $1\le\beta\le n-2$, $0<\eta<1$, $0\le a\le 1$, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ satisfies Caratheodory type conditions. We discuss the existence of positive solutions for BVP(3) and obtain multiplicity results which extend and improve the known results by using some fixed point theorems.

2. Preliminary Results

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Let $q\in(0,1)$ and define

$$[a]_q=\frac{1-q^a}{1-q},\quad a\in R. \tag{4}$$

The q-analogue of the power function $(a-b)^n$ with $n\in N_0$ is

$$(a-b)^{(0)}=1,\quad (a-b)^{(n)}=\prod_{k=0}^{n-1}\left(a-bq^k\right),\qquad n\in N,\quad a,b\in R. \tag{5}$$

More generally, if $\alpha\in R$, then

$$(a-b)^{(\alpha)}=a^\alpha\prod_{n=0}^{n-1}\frac{a-bq^n}{a-bq^{\alpha+n}}. \tag{6}$$

Note that, if $b=0$, then $a^{(\alpha)}=a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x)=\frac{(1-q)^{(x-1)}}{(1-q)^{x-1}},\quad x\in R\setminus\{0,-1,-2,\ldots\}, \tag{7}$$

and satisfies $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

Then, let us recall some basic concepts of q-calculus [28].

Definition 1. For $0<q<1$, we define the q-derivative of a real-value function f as

$$\left(D_qf\right)(x)=\frac{f(x)-f(qx)}{(1-q)x},\qquad \left(D_qf\right)(0)=\lim_{x\to 0}\left(D_qf\right)(x). \tag{8}$$

Note that $\lim_{q\to 1^-}D_qf(x)=f'(x)$.

Definition 2. The higher-order q-derivatives are defined inductively as

$$\left(D_q^0f\right)(x)=f(x),\qquad \left(D_q^nf\right)(t)=D_q\left(D_q^{n-1}f\right)(t),\quad n\in N. \tag{9}$$

Definition 3. The q-integral of a function f in the interval $[0,b]$ is given by

$$\left(I_qf\right)(x)=\int_0^x f(t)\,d_qt=x(1-q)\sum_{n=0}^{\infty}f\left(xq^n\right)q^n,\quad x\in[0,b]. \tag{10}$$

If $a\in[0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)\,d_qt=\int_0^b f(t)\,d_qt-\int_0^a f(t)\,d_qt. \tag{11}$$

Similarly as done for derivatives, an operator I_q^n can be defined; namely,

$$\left(I_q^0f\right)(x)=f(x),\qquad \left(I_q^nf\right)(x)=I_q\left(I_q^{n-1}f\right)(x),\quad n\in N. \tag{12}$$

Observe that

$$D_qI_qf(x)=f(x), \tag{13}$$

and if f is continuous at $x=0$, then $I_qD_qf(x)=f(x)-f(0)$.

We now point out three formulas (${}_iD_q$ denotes the derivative with respect to variable i):

$$\begin{aligned}[a(t-s)]^{(\alpha)}&=a^\alpha(t-s)^{(\alpha)},\\ {}_tD_q(t-s)^{(\alpha)}&=[\alpha]_q(t-s)^{(\alpha-1)},\\ {}_xD_q\int_0^x f(x,t)\,d_qt&=\int_0^x {}_xD_qf(x,t)\,d_qt+f(qx,x).\end{aligned} \tag{14}$$

Remark 4. We note that if $\alpha\ge 0$ and $a\le b\le t$, then $(t-a)^{(\alpha)}\ge(t-b)^{(\alpha)}$ [19].

Definition 5 (see [9]). Let $\alpha\ge 0$ and let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0f)(x)=f(x)$ and

$$\left(I_q^\alpha f\right)(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x(x-qt)^{(\alpha-1)}f(t)\,d_qt,\qquad \alpha>0,\quad x\in[0,1]. \tag{15}$$

Definition 6 (see [11]). The fractional q-derivative of the Riemann-Liouville type of $\alpha\ge 0$ is defined by $(D_q^0f)(x)=f(x)$ and

$$\left(D_q^\alpha f\right)(x)=\left(D_q^mI_q^{m-\alpha}f\right)(x),\quad \alpha>0, \tag{16}$$

where m is the smallest integer greater than or equal to α.

Lemma 7 (see [9, 11]). *Let $\alpha,\beta\ge 0$ and let f be a function defined on $[0,1]$. Then, the next formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x)=(I_q^{\alpha+\beta}f)(x)$,
(2) $(D_q^\alpha I_q^\alpha f)(x)=f(x)$.

Remark 8. Assume that $g(t)\in[0,1]$ and α,β are two constants such that $\alpha>2\ge\beta\ge 1$. Then

$$D_q^\beta\int_0^t(t-qs)^{(\alpha-1)}g(s)\,d_qs=\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}g(s)\,d_q. \tag{17}$$

Proof. From Lemma 7, we can get

$$\left(D_q^{\beta} I_q^{\alpha} g\right)(t) = \left(D_q^{\beta} I_q^{\beta} I_q^{\alpha-\beta} g\right)(t) = \left(I_q^{\alpha-\beta} g\right)(t), \tag{18}$$

so

$$\begin{aligned} D_q^{\beta} \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s \\ = \frac{1}{\Gamma_q(\alpha-\beta)} \int_0^t (t-qs)^{(\alpha-\beta-1)} g(s)\, d_q; \end{aligned} \tag{19}$$

that is, (17) holds. The proof is completed. □

Lemma 9 (see [22]). *Let $\alpha > 0$ and let p be a positive integer. Then, the following equality holds:*

$$\begin{aligned} \left(I_q^{\alpha} D_q^{p} f\right)(x) = {} & \left(D_q^{p} I_q^{\alpha} f\right)(x) \\ & - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_q^{k} f\right)(0). \end{aligned} \tag{20}$$

Lemma 10. *Let $g(t) \in L[0,1]$; then the unique solution of*

$$\begin{gathered} \left(D_q^{\alpha} u\right)(t) + g(t) = 0, \quad 0 < t < 1, \\ \left(D_q^{i} u\right)(0) = 0, \quad \left(D_q^{\beta} u\right)(1) = a\left(D_q^{\beta} u\right)(\eta), \quad 0 \le i \le n-2, \end{gathered} \tag{21}$$

is

$$u(t) = \int_0^1 G(t, qs)\, g(s)\, d_q s, \tag{22}$$

where

$$G(t,s) = \frac{1}{\Lambda \Gamma_q(\alpha)} \times \begin{cases} (1-s)^{(\alpha-\beta-1)} t^{\alpha-1} \\ \quad -a(\eta-s)^{(\alpha-\beta-1)} t^{\alpha-1} \\ \quad -\Lambda(t-s)^{(\alpha-1)}, & 0 \le s \le \min(\eta, t) \le 1, \\ (1-s)^{(\alpha-\beta-1)} t^{\alpha-1} \\ \quad -a(\eta-s)^{(\alpha-\beta-1)} t^{\alpha-1}, & 0 \le t \le s \le \eta \le 1, \\ (1-s)^{(\alpha-\beta-1)} t^{\alpha-1} \\ \quad -\Lambda(t-s)^{(\alpha-1)}, & 0 \le \eta \le s \le t \le 1, \\ (1-s)^{(\alpha-\beta-1)} t^{\alpha-1}, & 0 \le \max(t, \eta) \le s \le 1, \end{cases} \tag{23}$$

where $\Lambda = 1 - a\eta^{\alpha-\beta-1}$.

Proof. Let $u(t)$ be a solution of (21); in view of Lemma 7 and Lemma 9, (21) is equivalent to the integral equation

$$\begin{aligned} u(t) = {} & c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \\ & - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s, \end{aligned} \tag{24}$$

where $c_1, c_2, \ldots, c_n$ are some constants to be determined. The boundary conditions $(D_q^i u)(0) = 0$, $0 \le i \le n-2$, imply that $c_2 = c_3 = \cdots = c_n = 0$. Thus,

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s. \tag{25}$$

By Remark 8, we have

$$\begin{aligned} \left(D_q^{\beta} u\right)(t) = {} & c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} t^{\alpha-\beta-1} - \frac{1}{\Gamma_q(\alpha-\beta)} \\ & \times \int_0^t (t-qs)^{(\alpha-\beta-1)} g(s)\, d_q s. \end{aligned} \tag{26}$$

For $(D_q^{\beta} u)(1) = a(D_q^{\beta} u)(\eta)$,

$$\begin{aligned} c_1 = {} & \frac{1}{(1-a\eta^{\alpha-\beta-1})\Gamma_q(\alpha)} \left[\int_0^1 (1-qs)^{(\alpha-\beta-1)} g(s)\, d_q s \right. \\ & \left. - a \int_0^{\eta} (\eta-qs)^{(\alpha-\beta-1)} g(s)\, d_q s \right]. \end{aligned} \tag{27}$$

Hence,

$$\begin{aligned} u(t) = {} & \frac{t^{\alpha-1}}{\Lambda \Gamma_q(\alpha)} \left[\int_0^1 (1-qs)^{(\alpha-\beta-1)} g(s)\, d_q s \right. \\ & \left. - a \int_0^{\eta} (\eta-qs)^{(\alpha-\beta-1)} g(s)\, d_q s \right] \\ & - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s \\ = {} & \int_0^1 G(t, qs)\, g(s)\, d_q s. \end{aligned} \tag{28}$$

The proof is complete. □

Remark 11. For the special case where $a = 0$, it is easy to see that $G(t,s)$ can be written as

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \times \begin{cases} (1-s)^{(\alpha-\beta-1)} t^{\alpha-1} - (t-s)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-\beta-1)} t^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{29}$$

Lemma 12. *Green function G in Lemma 10 satisfies the following conditions:*

(i) $G(t, qs) \ge 0$ *for* $t, s \in [0,1]$;

(ii) $G(t, qs) \le G(1, qs)$ *for* $t, s \in [0,1]$;

(iii) $G(t, qs) \ge t^{\alpha-1} G(1, qs)$ *for* $t, s \in [0,1]$.

Proof. Let

$$\begin{aligned}
g_1(t,s) &= (1-s)^{(\alpha-\beta-1)}t^{\alpha-1} - a(\eta-s)^{(\alpha-\beta-1)}t^{\alpha-1} \\
&\quad - \Lambda(t-s)^{(\alpha-1)}, \quad 0 \le s \le \min(\eta,t) \le 1, \\
g_2(t,s) &= (1-s)^{(\alpha-\beta-1)}t^{\alpha-1} \\
&\quad - a(\eta-s)^{(\alpha-\beta-1)}t^{\alpha-1}, \quad 0 \le t \le s \le \eta \le 1, \\
g_3(t,s) &= (1-s)^{(\alpha-\beta-1)}t^{\alpha-1} \\
&\quad - \Lambda(t-s)^{(\alpha-1)}, \quad 0 \le \eta \le s \le t \le 1, \\
g_4(t,s) &= (1-s)^{(\alpha-\beta-1)}t^{\alpha-1}, \quad 0 \le \max(t,\eta) \le s \le 1.
\end{aligned} \tag{30}$$

We first prove part (i). For $t, s \in [0,1]$, from Remark 4, for $t \neq 0$,

$$\begin{aligned}
g_1(t,qs) &= (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - a\eta^{\alpha-\beta-1} \\
&\quad \times \left(1-\frac{qs}{\eta}\right)^{(\alpha-\beta-1)} t^{\alpha-1} - \Lambda\left(1-\frac{qs}{t}\right)^{(\alpha-1)} t^{\alpha-1} \\
&= \Lambda t^{\alpha-1}\left[(1-qs)^{(\alpha-\beta-1)} - \left(1-\frac{qs}{t}\right)^{(\alpha-1)}\right] \\
&\quad + a\eta^{\alpha-\beta-1}t^{\alpha-1}\left[(1-qs)^{(\alpha-\beta-1)}\right. \\
&\quad \left. - \left(1-\frac{qs}{\eta}\right)^{(\alpha-\beta-1)}\right] \\
&\ge \Lambda t^{\alpha-1}\left[(1-qs)^{(\alpha-\beta-1)} - (1-qs)^{(\alpha-1)}\right] \\
&\quad + a\eta^{\alpha-\beta-1}t^{\alpha-1}\left[(1-qs)^{(\alpha-\beta-1)} - (1-qs)^{(\alpha-\beta-1)}\right] \\
&\ge 0.
\end{aligned} \tag{31}$$

Since $g_1(t,qs) \ge 0$, it is easy to know $g_2(t,qs) \ge 0$, $g_3(t,qs) \ge 0$, and $g_4(t,qs) \ge 0$. Therefore, $G(t,qs) \ge 0$.

Next, we prove part (ii). Fix $s \in [0,1]$, and

$$\begin{aligned}
{}_tD_q g_1(t,qs) &= [\alpha-1]_q(1-qs)^{(\alpha-\beta-1)}t^{\alpha-2} \\
&\quad - a\eta^{\alpha-\beta-1}[\alpha-1]_q\left(1-\frac{qs}{\eta}\right)^{(\alpha-\beta-1)} t^{\alpha-2} \\
&\quad - \Lambda[\alpha-1]_q\left(1-\frac{qs}{t}\right)^{(\alpha-2)} t^{\alpha-2} \\
&= [\alpha-1]_q t^{\alpha-2}\left\{\Lambda\left[(1-qs)^{(\alpha-\beta-1)}\right.\right. \\
&\quad \left. -\left(1-\frac{qs}{t}\right)^{(\alpha-2)}\right] + a\eta^{\alpha-\beta-1} \\
&\quad \times \left[(1-qs)^{(\alpha-\beta-1)}\right. \\
&\quad \left.\left. -\left(1-\frac{qs}{\eta}\right)^{(\alpha-\beta-1)}\right]\right\} \\
&\ge [\alpha-1]_q t^{\alpha-2}\left\{\Lambda\left[(1-qs)^{(\alpha-\beta-1)}\right.\right. \\
&\quad \left.\left. -(1-qs)^{(\alpha-2)}\right]\right\} \\
&\ge 0.
\end{aligned} \tag{32}$$

That is, $g_1(t,qs)$ is increasing function of t. By the same way, we can conclude that $g_2(t,qs)$, $g_3(t,qs)$, and $g_4(t,qs)$ are increasing functions of t for fixed $s \in [0,1]$. Thus, $G(t,qs) \le G(1,qs)$ for $t, s \in [0,1]$.

Finally, we prove part (iii). Suppose that $0 \le qs \le \min\{t,\eta\} \le 1$; then

$$\begin{aligned}
&\frac{G(t,qs)}{G(1,qs)} \\
&= \frac{(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - a(\eta-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - \Lambda(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-\beta-1)} - a(\eta-qs)^{(\alpha-\beta-1)} - \Lambda(1-qs)^{(\alpha-1)}} \\
&= \frac{t^{\alpha-1}\left[(1-qs)^{(\alpha-\beta-1)} - a(\eta-qs)^{(\alpha-\beta-1)} - \Lambda(1-qs/t)^{(\alpha-1)}\right]}{(1-qs)^{(\alpha-\beta-1)} - a(\eta-qs)^{(\alpha-\beta-1)} - \Lambda(1-qs)^{(\alpha-1)}} \\
&\ge t^{\alpha-1}.
\end{aligned} \tag{33}$$

For other circumstances, we also get $G(t,qs) \ge t^{\alpha-1}G(1,qs)$ and this completes the proof. □

Remark 13. Let $0 < \tau < 1$; then $0 < \tau^{\alpha-1} < 1$ and

$$\min_{t\in[\tau,1]} G(t,qs) \ge \tau^{\alpha-1}G(1,qs) \quad \text{for } s \in [0,1]. \tag{34}$$

Lemma 14 (see [29]). *Let X be a Banach space with $C \subset X$ being closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ is complete continuous. Then either*

(i) *T has a fixed point in $\overline{U}$, or*

(ii) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T u$.*

Lemma 15 (see Krasnoselskii's [30]). *Let E be a Banach space, and $K \in E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator. In addition, suppose that either*

(H_1) *$\|Tu\| \le \|u\|$, for all $u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, for all $u \in K \cap \partial\Omega_2$ or*

(H_2) *$\|Tu\| \le \|u\|$, for all $u \in K \cap \partial\Omega_2$ and $\|Tu\| \ge \|u\|$, for all $u \in K \cap \partial\Omega_1$*

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 16 (see [31]). *Let P be a cone in a real Banach space E, $P_c = \{x \in P : \|x\| < c\}$, θ is a nonnegative continuous concave functional on P such that $\theta \le \|x\|$, for all $x \in \overline{P_c}$, and $P(\theta, b, d) = \{x \in P : b \le \theta(x), \|x\| \le d\}$. Suppose that $T : \overline{P_c} \to \overline{P_c}$ is completely continuous and there exist positive constants $0 < a < b < d \le c$ such that*

- (H_1) *$\{x \in P(\theta, b, d) : \theta(x) > b\} \neq \emptyset$ and $\theta(Tx) > b$ for $x \in P(\theta, b, d)$,*
- (H_2) *$\|Tx\| < a$ for $x \in \overline{P_a}$,*
- (H_3) *$\theta(Tx) > b$ for $x \in P(\theta, b, d)$ with $\|Tx\| > d$.*

Then T has at least three fixed points x_1, x_2, and x_3 with

$$\|x\| < a, \quad b < \theta(x_2), \quad a < \|x_3\| \quad \text{with } \theta(x_3) < b. \tag{35}$$

Remark 17. If $d = c$, then (H_1) implies (H_3).

3. Main Result

In this section, we will consider the question of positive solutions for BVP (3). At first, we prove some lemmas required for the main result.

Let $E = C[0,1]$ be the Banach space endowed with the norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$. Let $\tau = q^n$ for a given $n \in N$, and define the cone $P \subset E$ by

$$P = \left\{u \in E : u(t) \ge 0, \min_{t\in[\tau,1]} u(t) \ge \tau^{\alpha-1}\|u\|\right\}. \tag{36}$$

Let the nonnegative continuous concave functional θ on the cone P be defined by

$$\theta(x) = \min_{\tau\le t\le 1} |u(t)|. \tag{37}$$

In this paper, we assume that $f : [0,1] \times [0,+\infty) \to [0,+\infty]$ satisfies the following conditions of Caratheodory type:

- (D_1) $f(t,u)$ is Lebesgue measurable with respect to t on $[0,1]$;
- (D_2) $f(t,u)$ is continuous with respect to u on $[0,+\infty)$.

Theorem 18. *Assume that the conditions (D_1) and (D_2) hold. Suppose further that there exists a real-valued function $h(t) \in L[0,1]$ such that $|f(t,u) - f(t,v)| \le h(t)|u - v|$ for almost every $t \in [0,1]$ and all $u, v \in [0,+\infty)$. If*

$$0 < \int_0^1 G(1,qs)\,h(s)\,d_qs < 1, \tag{38}$$

then there exist unique positive solutions of BVP (3) on $[0,1]$.

Proof. Consider the operator $T : P \to P$ defined by

$$Tu(t) := \int_0^1 G(t,qs)\,f(s,u(s))\,d_qs < 1. \tag{39}$$

For any $u, v \in P$, we have

$$\begin{aligned}|Tu(t) - Tv(t)| &= \left|\int_0^1 G(t,qs)\,(f(s,u(s)) - f(s,v(s)))\,d_qs\right| \\ &\le \int_0^1 G(t,qs)\,|(f(s,u(s)) - f(s,v(s)))|\,d_qs \\ &\le \int_0^1 G(t,qs)\,h(s)\,|u(s) - v(s)|\,d_qs \\ &\le \int_0^1 G(1,qs)\,h(s)\,|u(s) - v(s)|\,d_qs \\ &\le \|u - v\| \int_0^1 G(1,qs)\,h(s)\,d_qs < \|u - v\|.\end{aligned} \tag{40}$$

This implies that T is a contraction mapping. By the Banach contraction mapping principle, we deduce that T has a unique fixed point which is obviously a solution of BVP (3). The proof is complete. □

Corollary 19. *Assume that the conditions (D_1) and (D_2) hold. Suppose further that there exists a positive constant $L \in (0, 1/A)$ with $|f(t,u) - f(t,v)| \le L|u - v|, t \in [0,1], (u,v) \in [0,+\infty)$, where*

$$A = \int_0^1 G(1,qs)\,d_qs; \tag{41}$$

then there exists a unique positive solution of BVP (3) on $[0,1]$.

Corollary 20. *Assume that the conditions (D_1) and (D_2) hold. Suppose further that there exists a real-valued function $h(t) \in L[0,1]$ such that $|f(t,u) - f(t,v)| \le h(t)|u - v|$, $t \in [0,1]$, $(u,v) \in [0,+\infty)$. If*

$$0 < \int_0^1 h(s)\,d_qs < \Lambda\Gamma_q(\alpha), \tag{42}$$

then there exists a unique positive solution of BVP (3) on $[0,1]$.

Corollary 21. *Assume that the conditions (D_1) and (D_2) hold. Suppose further that there exists a positive constant $L \in (0, \Lambda\Gamma_q(\alpha))$ with $|f(t,u) - f(t,v)| \le L|u - v|$, $t \in [0,1]$, $(u,v) \in [0,+\infty)$; then there exists a unique positive solution of BVP (3) on $[0,1]$.*

Next, we discuss multiple solutions of BVP (3).

Lemma 22. *Assume that the conditions (D_1) and (D_2) hold. Suppose further that there exist two nonnegative real-valued functions $m, n \in L[0,1]$ such that $|f(t,u)| \le n(t) + m(t)u$ for almost every $t \in [0,1]$ and all $u, v \in [0,+\infty)$. Then the operation $T : P \to P$ defined by (39) is completely continuous.*

Proof. We will divide the proof into three parts.

(I) We show that $T : P \to P$ is continuous.

For any $u_n, u \in P$, $n = 1, 2, \ldots$, with $\lim_{n\to\infty}\|u_n - u\| = 0$, we have

$$\lim_{n\to\infty} u_n(t) = u(t), \quad t \in [0,1]. \tag{43}$$

Thus

$$\lim_{n\to\infty} f(t, u_n(t)) = f(t, u(t)), \quad t \in [0,1]. \tag{44}$$

So, we can obtain that

$$\sup_{s\in[0,1]} |f(s, u_n(s)) - f(s, u(s))| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{45}$$

On the other hand, we have

$$\begin{aligned} |(Tu_n)(t) - (Tu)(t)| &= \left|\int_0^1 G(t,qs)(f(s,u_n(s)) - f(s,u(s)))\,d_qs\right| \\ &\le \sup_{s\in[0,1]} |f(s,u_n(s)) - f(s,u(s))| \\ &\quad \times \int_0^1 G(t,qs)\,d_qs. \end{aligned} \tag{46}$$

It implies that

$$\begin{aligned} \|Tu_n - Tu\| \le \sup_{s\in[0,1]} &|f(s,u_n(s)) - f(s,u(s))| \\ &\times \int_0^1 G(t,qs)\,d_qs. \end{aligned} \tag{47}$$

Therefore, $\|Tu_n - Tu\| \to 0$ as $n \to \infty$. This means that T is continuous.

(II) We will prove that T maps bounded sets into bounded sets in P.

For any $\xi > 0$, there exists a positive constant $l > 0$ such that for each $u \in B_\xi = \{u \in P : \|u\| \le \xi\}$, we have $\|Tu\| < l$. By the definition of T, for each $t \in [0,1]$, we get

$$\begin{aligned} |Tu(t)| &= \left|\int_0^1 G(t,qs) f(s,u(s))\,d_qs\right| \\ &\le \int_0^1 G(t,qs)(n(s) + m(s)u(s))\,d_qs \\ &\le \int_0^1 G(1,qs)n(s)\,d_qs + \|u\| \int_0^1 G(1,qs)m(s)\,d_qs \\ &\le \int_0^1 G(1,qs)n(s)\,d_qs + \xi \int_0^1 G(1,qs)m(s)\,d_qs := l. \end{aligned} \tag{48}$$

That is, $\|Tu\| \le l$.

(III) We will show that T maps bounded sets into equicontinuous sets of P.

Let $B_\xi \subset P$ be a bounded set, $t_1, t_2 \in [0,1]$ and $t_1 < t_2$. For any $u \in B_\xi$, we have

$$\begin{aligned} &|Tu(t_2) - Tu(t_1)| \\ &\quad = \left|\int_0^1 (G(t_2,qs) - G(t_1,qs)) f(s,u(s))\,d_qs\right| \\ &\quad \le \int_0^1 |G(t_2,qs) - G(t_1,qs)|\,|f(s,u(s))|\,d_qs \\ &\quad \le \int_0^1 |G(t_2,qs) - G(t_1,qs)|\,(n(s) + m(s)u(s))\,d_qs \\ &\quad \le \int_0^1 |G(t_2,qs) - G(t_1,qs)|\,(n(s) + \xi m(s))\,d_qs. \end{aligned} \tag{49}$$

Since $G(t,qs)$ is continuous on $[0,1]\times[0,1]$, then $G(t,qs)$ is uniformly continuous in $[0,1]\times[0,1]$. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$, whenever $|t_2 - t_1| < \delta$, and we have

$$|G(t_2,qs) - G(t_1,qs)| < \frac{\varepsilon}{1 + \int_0^1 (n(s) + \xi m(s))\,d_qs}. \tag{50}$$

So $|Tu(t_2) - Tu(t_1)| < \varepsilon$; that is, $\{Tu : u \in B_\xi\}$ is equicontinuous.

By Arzela-Ascoli theorem, we can conclude that T is completely continuous. This completes the proof. ☐

Remark 23. If $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous, T is also completely continuous.

Theorem 24. *Assume that all the assumptions of Lemma 22 hold. If*

$$0 < \int_0^1 G(1,qs)m(s)\,d_qs < 1, \tag{51}$$

then BVP (3) *has at least one positive solution.*

Proof. Let $U = \{u \in P : \|u\| < r\}$, where

$$r = \frac{\int_0^1 G(1,qs)n(s)\,d_qs}{1 - \int_0^1 G(1,qs)m(s)\,d_qs} > 0 \tag{52}$$

and $T : \overline{U} \to P, Tu(t) := \int_0^1 G(t,qs)f(s,u(s))d_qs$. From Lemma 22, T is completely continuous.

Assume that there exist $u \in P$ and $\lambda \in (0,1)$ such that $u = \lambda Tu$; we claim that $\|u\| \ne r$:

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,qs) f(s,u(s))\,d_qs \\ &\le \lambda \int_0^1 G(t,qs)(n(s) + m(s)u(s))\,d_qs \\ &\le \lambda \left(\int_0^1 G(t,qs)n(s)\,d_qs + \|u\| \int_0^1 G(t,qs)m(s)\,d_qs\right), \end{aligned} \tag{53}$$

and then

$$\|u\| \le \lambda \left(\int_0^1 G(1, qs)\, n(s)\, d_q s + r \int_0^1 G(1, qs)\, m(s)\, d_q s \right) < r. \tag{54}$$

That is, $u \notin \partial U$. By Lemma 14, T has a fixed point $u \in \overline{U}$. Therefore, BVP (3) has at least a positive solution. The proof is complete. □

In the following, we set

$$M = \left(\int_0^1 G(1, qs)\, d_q s \right)^{-1}, \qquad N = \left(\int_\tau^1 \tau^{\alpha-1} G(1, qs)\, d_q s \right)^{-1}. \tag{55}$$

Theorem 25. *Assume that all the assumptions of Lemma 22 hold. If there exist two positive constants $r_2 > r_1 > 0$ such that*

$$f(t, u) \ge N r_1 \quad \text{for } (t, u) \in [0, 1] \times [0, r_1], \tag{56}$$

$$f(t, u) \le M r_2 \quad \text{for } (t, u) \in [0, 1] \times [0, r_2], \tag{57}$$

then BVP (3) has at least one positive solution.

Proof. Because $T : P \to P$ is completely continuous, we just only show that $u = Tu$ has a solution $u(t) > 0$ for $t \in [0, 1]$.

Let $\Omega_1 = \{u \in E : \|u\| < r_1\}$. For any $u \in P \cap \partial\Omega_1$, we know $\tau^{\alpha-1} r_1 \le u(t) \le r_1$ on $[\tau, 1]$. Using (56) and (34), we have

$$\begin{aligned} |Tu(t)| &= \left| \int_0^1 G(t, qs)\, f(s, u(s))\, d_q s \right| \\ &\ge \int_\tau^1 G(t, qs)\, f(s, u(s))\, d_q s \\ &\ge N r_1 \int_\tau^1 G(t, qs)\, \tau^{\alpha-1} d_q s = r_1 = \|u\|, \end{aligned} \tag{58}$$

which implies that $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$.

Let $\Omega_2 = \{u \in E : \|u\| < r_2\}$. For any $u \in P \cap \partial\Omega_2$, we get $0 \le u(t) \le r_2$ on $[0, 1]$. Using (57), we have

$$\begin{aligned} |Tu(t)| &= \left| \int_0^1 G(t, qs)\, f(s, u(s))\, d_q s \right| \\ &\le M r_2 \int_0^1 G(1, qs)\, d_q s = r_2 = \|u\|; \end{aligned} \tag{59}$$

that is, $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_2$.

In view of Lemma 15, T has a fixed point $u_0 \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is the solution of BVP (3). □

Theorem 26. *Assume that all assumptions of Lemma 22 hold. If there exist constants $0 < A < B < C$ such that*

$$\begin{aligned} &(G_1)\ f(t, u) \le MA, \ (t, u) \in [0, 1] \times [0, A], \\ &(G_2)\ f(t, u) \le MC, \ (t, u) \in [0, 1] \times [0, C], \\ &(G_3)\ f(t, u) \ge NB, \ (t, u) \in [\tau, 1] \times [B, C], \end{aligned} \tag{60}$$

hold, then BVP (3) has at least three positive solutions u_1, u_2, and u_3 with

$$\begin{gathered} \|u_1\| < A, \qquad B < \theta(u_2) < \|u_2\| \le C, \\ A < \|u_3\|, \qquad \theta(u_3) \le B. \end{gathered} \tag{61}$$

Proof. First, if $u \in \overline{P}_C = \{u \in P : \|u\| < C\}$, then $\|u\| \le C$. So $0 \le u(t) \le C$, $t \in [0, 1]$. By (G_2), we have

$$\begin{aligned} |Tu(t)| &= \left| \int_0^1 G(t, qs)\, f(s, u(s))\, d_q s \right| \\ &\le MC \int_0^1 G(1, qs)\, d_q s = C, \end{aligned} \tag{62}$$

which implies that $\|Tu\| \le C, u \in \overline{P}_C$. Hence, $T : \overline{P}_C \to \overline{P}_C$. In view of Lemma 22, $T : \overline{P}_C \to \overline{P}_C$ is completely continuous.

By using the analogous argument, from (G_1), we can get that if $u \in \overline{P}_A$, then $\|Tu\| < A$.

Set $u(t) = (B + C)/2$, $t \in [0, 1]$, so $u(t) = (B + C)/2 \in P(\theta, B, C)$, $\theta(u) = (B + C)/2 > B$. Therefore, $\{u \in P(\theta, B, C) | \theta(u) > B\} \ne \emptyset$.

On the other hand, if $u \in P(\theta, B, C)$, then $B \le u(t) \le C$, $t \in [\tau, 1]$. By (G_3), we have

$$\begin{aligned} \theta(Tu) &= \min_{t \in [\tau, 1]} |Tu(t)| \\ &= \min_{t \in [\tau, 1]} \left| \int_0^1 G(t, qs)\, f(s, u(s))\, d_q s \right| \\ &\ge \min_{t \in [\tau, 1]} \int_0^1 G(t, qs)\, NB d_q s, \\ &\ge NB \int_\tau^1 \tau^{\alpha-1} G(1, qs)\, d_q s = B, \end{aligned} \tag{63}$$

which implies that $\theta(Tu) > B$, for $u \in P(\theta, B, C)$.

By Lemma 16, BVP(3) has at least three positive solutions u_1, u_2, and u_3 with

$$\begin{gathered} \|u_1\| < A, \qquad B < \theta(u_2) < \|u_2\| < C, \\ A < \|u_3\|, \qquad \theta(u_3) < B. \end{gathered} \tag{64}$$

The proof is complete. □

4. Example

Example 27. Consider the following BVP:

$$\begin{gathered} \left(D_{1/2}^{4.5} u\right)(t) + \frac{e^t u}{1 + e^t} + t^2 + 1 = 0, \quad 0 < t < 1, \\ \left(D_{1/2}^{i} u\right)(0) = 0, \quad \left(D_{1/2}^{1.5} u\right)(1) = \frac{1}{100} \left(D_{1/2}^{1.5} u\right)\left(\frac{9}{10}\right), \\ 0 \le i \le 3. \end{gathered} \tag{65}$$

Let $f(t,u) = e^t u/(1+e^t) + t^2 + 1$, $(t,u) \in [0,1] \times [0,+\infty)$, $h(t) = e^t/(1+e^t)$, so $|f(t,u) - f(t,v)| \leq h(t)|u-v|$, for $(t,u), (t,v) \in [0,1] \times [0,+\infty)$.

By simple calculation, we get

$$0 < \int_0^1 G(1,qs)\, h(s)\, d_q s \leq \frac{1}{\Lambda \Gamma_q(\alpha)} = 0.8471 \cdots < 1. \tag{66}$$

All conditions of Theorem 18 are satisfied. Thus, BVP (65) has a unique positive solution.

Example 28. Consider the following BVP:

$$\begin{gathered} \left(D_{1/2}^{4.5} u\right)(t) + \frac{1}{4}(\sin t + 1)\frac{u}{1+u^2} + t^2 + 1 = 0, \quad 0 < t < 1, \\ \left(D_{1/2}^{i} u\right)(0) = 0, \quad \left(D_{1/2}^{1.5} u\right)(1) = \frac{1}{100}\left(D_{1/2}^{1.5} u\right)\left(\frac{9}{10}\right), \\ 0 \leq i \leq 3. \end{gathered} \tag{67}$$

Let $n(t) = t^2 + 1$, $m(t) = (1/4)(\sin t + 1)$. It is easy to check that $f(t,u) \leq n(t) + m(t)u$ for $(t,u) \in [0,1] \times [0,+\infty)$. Since

$$0 < \int_0^1 G(1,qs)\, m(s)\, d_q s \leq \frac{1}{\Lambda \Gamma_q(\alpha)} \cdot \frac{1}{2} = 0.4235 \cdots < 1, \tag{68}$$

by Theorem 24, BVP (67) has at least one positive solution.

Example 29. Consider the following BVP:

$$\begin{gathered} \left(D_{1/2}^{2.5} u\right)(t) + \frac{u}{4} + \frac{\sin^2 t}{5} + 1 = 0, \quad 0 < t < 1, \\ \left(D_{1/2}^{i} u\right)(0) = 0, \quad \left(D_{1/2} u\right)(1) = \frac{1}{2}\left(D_{1/2} u\right)\left(\frac{3}{4}\right), \\ 0 \leq i \leq 1. \end{gathered} \tag{69}$$

Let $f(t,u) = u/4 + \sin^2 t/5 + 1$, $(t,u) \in [0,1] \times [0,+\infty)$. By calculation, we get

$$\begin{gathered} M = \left(\int_0^1 G(1,qs)\, d_q s\right)^{-1} \geq \Lambda \Gamma_q(\alpha) = 1.4809 \ldots, \\ N = \left(\int_\tau^1 \tau^{\alpha-1} G(1,qs)\, d_q s\right)^{-1} \leq 2.414 \ldots. \end{gathered} \tag{70}$$

Choosing $r_1 = 1/3$, $r_2 = 1$, we have

$$\begin{gathered} f(t,u) = \frac{u}{4} + \frac{\sin^2 t}{5} + 1 \leq 1.45 \leq M r_2, \\ (t,u) \in [0,1] \times [0, r_2], \\ f(t,u) = \frac{u}{4} + \frac{\sin^2 t}{5} + 1 \geq 1 \geq N r_1, \\ (t,u) \in [0,1] \times [0, r_1]. \end{gathered} \tag{71}$$

By Theorem 25, BVP (69) has at least one positive solution u such that $1/3 \leq \|u\| \leq 1$.

Example 30. Consider the following BVP:

$$\begin{gathered} \left(D_{1/2}^{2.5} u\right)(t) + f(t,u) = 0, \quad 0 < t < 1, \\ \left(D_{1/2}^{i} u\right)(0) = 0, \quad \left(D_{1/2} u\right)(1) = \frac{1}{2}\left(D_{1/2} u\right)\left(\frac{3}{4}\right), \\ 0 \leq i \leq 1. \end{gathered} \tag{72}$$

Here,

$$f(t,u) = \begin{cases} \dfrac{t}{4} + 4u^4, & (t,u) \in [0,1] \times [0,1], \\ \dfrac{t+7}{4} + u, & (t,u) \in [0,1] \times [1,+\infty). \end{cases} \tag{73}$$

By Example 29, we have

$$\begin{gathered} M = \left(\int_0^1 G(1,qs)\, d_q s\right)^{-1} \geq \Lambda \Gamma_q(\alpha) = 1.4809 \ldots, \\ N = \left(\int_\tau^1 \tau^{\alpha-1} G(1,qs)\, d_q s\right)^{-1} \leq 2.414 \ldots. \end{gathered} \tag{74}$$

Choosing $A = 1/2$, $B = 1$, $C = 5$, we have

$$\begin{gathered} f(t,u) = \frac{t}{4} + 4u^4 \leq 0.5 \leq MA, \quad (t,u) \in [0,1] \times \left[0, \frac{1}{2}\right], \\ f(t,u) = \frac{t+7}{4} + u \leq 7 \leq MC, \quad (t,u) \in [0,1] \times [0,5], \\ f(t,u) = \frac{t+7}{4} + u \geq 2.875 \geq NB, \quad (t,u) \in \left[\frac{1}{2}, 1\right] \times [1,5]. \end{gathered} \tag{75}$$

By Theorem 26, BVP (72) has at least three solutions u_1, u_2, and u_3 such that

$$\|u_1\| < \frac{1}{2}, \qquad 1 < \theta(u_2), \qquad \frac{1}{2} < \|u_3\|, \qquad \theta(u_3) < 1. \tag{76}$$

5. Conclusion

In this paper, we obtain the existence and multiplicity results of positive solutions of BVP for high-order fractional q-difference equations by some fixed point theorems, which enrich the theories for fractional q-difference equations, and provide the theoretical guarantee for the application of fractional q-difference equations in every field. In the future, we will use bifurcation theory, critical point theory, variational method, and other methods to continue our works in this area.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

Each of the authors, Changlong Yu and Jufang Wang, contributed to each part of this work equally and read and approved the final version of the paper.

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