

# Transactions of the Institute of Measurement and Control

<http://tim.sagepub.com>

---

## **Neural network solution for suboptimal control of non-holonomic chained form system**

Tao Cheng, Hanxu Sun, Zhihua Qu and Frank L. Lewis

*Transactions of the Institute of Measurement and Control* 2009; 31; 475 originally published online Aug 6, 2009;

DOI: 10.1177/0142331208094043

The online version of this article can be found at:  
<http://tim.sagepub.com/cgi/content/abstract/31/6/475>

---

Published by:



<http://www.sagepublications.com>

On behalf of:



[The Institute of Measurement and Control](#)

**Additional services and information for *Transactions of the Institute of Measurement and Control* can be found at:**

**Email Alerts:** <http://tim.sagepub.com/cgi/alerts>

**Subscriptions:** <http://tim.sagepub.com/subscriptions>

**Reprints:** <http://www.sagepub.com/journalsReprints.nav>

**Permissions:** <http://www.sagepub.co.uk/journalsPermissions.nav>

**Citations** <http://tim.sagepub.com/cgi/content/refs/31/6/475>

# Neural network solution for suboptimal control of non-holonomic chained form system

Tao Cheng<sup>1</sup>, Hanxu Sun<sup>2</sup>, Zhihua Qu<sup>3</sup> and Frank L. Lewis<sup>4</sup>

<sup>1</sup>School of Mechanical Engineering and Automation, Beihang University, Beijing 100083, China

<sup>2</sup>School of Automation, Beijing University of Posts and Telecommunications, Beijing 100876, China

<sup>3</sup>Department of Electrical and Computer Engineering, University of Central Florida, Orlando, FL 32816, USA

<sup>4</sup>Automation and Robotics Research Institute, The University of Texas at Arlington, TX 76118, USA

In this paper, we develop fixed-final time nearly optimal control laws for a class of non-holonomic chained form systems by using neural networks to approximately solve a Hamilton–Jacobi–Bellman equation. A certain time-folding method is applied to recover uniform complete controllability for the chained form system. This method requires an innovative design of a certain dynamic control component. Using this time-folding method, the chained form system is mapped into a controllable linear system for which controllers can systematically be designed to ensure exponential or asymptotic stability as well as nearly optimal performance. The result is a neural network feedback controller that has time-varying coefficients found by *a priori* offline tuning. The results of this paper are demonstrated in an example.

**Key words:** Constrained input systems; finite-horizon optimal control; Hamilton–Jacobi–Bellman; neural network control; non-holonomic systems.

---

**Address for correspondence:** Tao Cheng, School of Mechanical Engineering and Automation, Beihang University, Xueyuan Road, No. 37, Haidian District, Beijing 100083, China.

E-mail: tao2000000@hotmail.com

Figures 1–5 appear in colour online: <http://tim.sagepub.com>

## 1. Introduction

The constrained input optimization of dynamical systems has been the focus of many papers during the last few years. Several methods for deriving constrained control laws are found in Saberi *et al.* (1996), Sussmann *et al.* (1994) and Bernstein (1995). However, most of these methods do not consider optimal control laws for general constrained non-linear systems. Constrained-input optimization possesses challenging problems, and a great variety of versatile methods have been successfully applied in Athans and Falb (1966), Bernstein (1993), Dolphus and Schmitendorf (1995) and Saberi *et al.* (1996). Many problems can be formulated within the Hamilton–Jacobi–Bellman (HJB) and Lyapunov’s frameworks, but the resulting equations are difficult or impossible to solve, such as Lyshevski (1995, 1996, 1999a).

Successful neural networks (NN) controllers not based on optimal techniques have been reported in Chen and Liu (1994), Lewis *et al.* (1999), Sanner and Slotine (1991), Ge (1996), Polycarpou (1996), Rovithakis and Christodoulou (1994). It has been shown that NNs can effectively extend adaptive control techniques to non-linearly parameterized systems. NN applications to optimal control via the HJB equation were first proposed by Miller *et al.* (1990).

We were motivated by the important results in Abu-Khalaf and Lewis (2005), Beard (1995), Lyshevski (1995, 1996, 1998, 1999a, 1999b, 2001), and Lyshevski and Meyer (1995). However, Abu-Khalaf and Lewis (2005) focused on constrained policy iteration control with infinite horizon; Beard (1995) focuses on unconstrained policy iteration with finite-time horizon. In contrast to these works, we study finite-time horizon system with constrained control without policy iteration, establishing an innovative methodology that incorporates control constraints into the framework of the HJB philosophy. We use NN to approximately solve the time-varying HJB equation for constrained control non-linear systems. It is shown that using an NN approach, one can simply transform the problem into solving a non-linear ordinary differential equation (ODE) backwards in time. The coefficients of this ODE are obtained by the weighted residuals method. We provide uniform convergence results over a Sobolev space.

By Brockett’s theorem (Cheng *et al.*, 2005), non-holonomic systems cannot be asymptotically stabilized around a fixed point under any smooth (or even continuous) time-independent state feedback control law. In this paper, a smooth nearly-optimal time-varying control is designed to stabilize the chained form system. We show how to construct a time-folding transformation to recover linear controllability in the transformed time space. With a new dynamic control design for component  $u_1$ , a global non-linear time transformation is found to transform the chained form system into a controllable linear time-varying system.

## 2. Motivation and background

### 2.1 Background on fixed final time HJB optimal control

Consider an affine in the control non-linear dynamical system of the form

$$\dot{x} = f(x) + g(x)u(t) \quad (1)$$

where  $x \in \mathfrak{N}^n$ ,  $f(x) \in \mathfrak{N}^n$ ,  $g(x) \in \mathfrak{N}^{n \times m}$  and the input  $u(t) \in R^m$ . The dynamics  $f(x)$  and  $g(x)$  are assumed to be known and  $f(0) = 0$ . Assume that  $f(x) + g(x)u(t)$  is Lipschitz continuous on a set  $\Omega \subseteq \mathfrak{N}^n$  containing the origin, and that system (1) is stabilizable in the sense that there exists a continuous control on  $\Omega$  that asymptotically stabilizes the system. It is desired to find the constrained input control  $u(t)$  that minimizes a generalized functional

$$V(x(t_0), t_0) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [Q(x) + W(u)]dt \quad (2)$$

with  $Q(x)$ ,  $W(x)$  positive definite on  $\Omega$ , i.e.,  $\forall x \neq 0, x \in \Omega, Q(x) > 0$  and  $x = 0 \Rightarrow Q(x) = 0$ .

*Definition 1.* Admissible controls.

A control  $u$  is defined to be admissible with respect to (2) on  $\Omega_0$ , denoted by  $u \in \Psi(\Omega_0)$ , if  $u$  is continuous,  $u(0) = 0$ ,  $u$  stabilizes (1) on  $\Omega_0$ , and  $\forall x_0 = x(t_0) \in \Omega_0, V(x_0, t_0)$  is finite.

An infinitesimal equivalent to (2) is (Lewis and Syrmos, 1995)

$$-\frac{\partial V(x, t)}{\partial t} = L + \left( \frac{\partial V(x, t)}{\partial x} \right)^T (f(x) + g(x)u(t)) \quad (3)$$

where  $L = Q(x) + W(u)$ . This is a time-varying partial differential equation with  $V(x, t)$  the cost function for any given  $u(t)$  and is solved backward in time from  $t = t_f$ . By setting  $t_0 = t_f$  in (2) its boundary condition is seen to be

$$V(x(t_f), t_f) = \phi(x(t_f), t_f) \quad (4)$$

*Lemma 1.* If  $u$  is admissible, there exists a positive definite function  $V(x)$  so that it satisfies (3) and (4).

*Proof.* See Chen and Jagannathan (2005). ■

According to Bellman's optimality principle (Lewis and Syrmos, 1995), the optimal cost is given by

$$-\frac{\partial V(x, t)^*}{\partial t} = \min_{u(t)} \left( L + \left( \frac{\partial V(x, t)^*}{\partial x} \right)^T (f(x) + g(x)u(t)) \right) \quad (5)$$

which yields the optimal control

$$u^*(x, t) = -\frac{1}{2} R^{-1} g(x)^T \frac{dV(x, t)^*}{dx} \quad (6)$$

where  $V^*(x, t)$  is the optimal value function,  $R$  is positive definite and assumed to be symmetric for simplicity of analysis. Substituting (6) into (5) yields the well-known time-varying HJB equation (Lewis and Syrmos, 1995),

$$\frac{\partial V(x, t)^*}{\partial t} + \frac{\partial V(x, t)^*}{\partial x} f(x) + Q(x) - \frac{1}{4} \frac{\partial V(x, t)^*}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V(x, t)^*}{\partial x} = 0 \quad (7)$$

This equation and (6) provide the solution to fixed-final time optimal control for general non-linear systems. However, the close form solution for Equation (7) is in general impossible to find. In Cheng *et al.* (2005), we showed how to solve this equation approximately using NN.

## 2.2 HJB equation with constraints on the control system

Consider now the case when the control input is constrained by a saturated function  $\varphi(\cdot)$ , e.g.,  $\tanh$ , etc. To guarantee bounded controls, Abu-Khalaf and Lewis (2005) and Lyshevski (1998) introduced a generalized non-quadratic functional

$$\begin{aligned} W(u) &= 2 \int_0^u \varphi^{-T}(v) R \, dv \\ \varphi(v) &= [\phi(v_1) \cdots \phi(v_m)]^T \\ \varphi^{-1}(u) &= [\phi^{-1}(u_1) \cdots \phi^{-1}(u_m)] \end{aligned} \quad (8)$$

where  $v \in \mathfrak{R}^m$ ,  $\varphi \in \mathfrak{R}^m$ , and  $\varphi(\cdot)$  is a bounded one-to-one function that belongs to  $C^p(p \geq 1)$  and  $L_2(\Omega)$ . Moreover, it is a monotonic odd function with its first derivative bounded by a constant  $M$ . Note that  $W(u)$  is positive definite because  $\varphi^{-1}(u)$  is monotonic odd and  $R$  is positive definite.

When (8) is used, (2) becomes

$$V(x(t_0), t_0) = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} \left[ Q(x) + 2 \int_0^u \varphi^{-T}(v) R \, dv \right] dt \quad (9)$$

and (5) becomes

$$-\frac{\partial V(x, t)^*}{\partial t} = \min_{u(t)} \left( Q(x) + 2 \int_0^u \varphi^{-T}(v) R \, dv + \frac{\partial V(x, t)^*}{\partial x} (f(x, t) + g(x)u(t)) \right)$$

Minimizing the Hamiltonian of the optimal control problem with regard to  $u$  gives

$$g^T(x) \frac{\partial V(x, t)^*}{\partial x} + 2\varphi^{-1}(u^*(t)) = 0$$

so

$$u(x, t)^* = -\varphi \left( \frac{1}{2} R^{-1} g(x)^T \frac{\partial V(x, t)^*}{\partial x} \right), u \in U \subset \mathfrak{R}^m \quad (10)$$

This is constrained as required.

*Lemma 2.* The smooth bounded control law (10) guarantees at least a strong relative minimum for the performance cost (9) for all  $x \in X \subset \mathfrak{R}$  on  $[t_0, t_f]$ . Moreover, if an optimal control exists, it is unique and represented by (10).

*Proof.* See Lyshevski (1996). ■

When (10) is used, (5) becomes

$$\begin{aligned} HJB(V(x, t)^*) &= \frac{\partial V(x, t)^*}{\partial t} + \frac{\partial V(x, t)^{*T}}{\partial x} f(x) + 2 \int_0^u \phi^{-T}(v) R dv \\ &\quad - \frac{\partial V(x, t)^{*T}}{\partial x} \cdot g(x) \cdot \varphi \left( \frac{1}{2} R^{-1} g(x)^T \frac{\partial V(x, t)^*}{\partial x} \right) + Q(x) = 0 \end{aligned} \quad (11)$$

If this HJB equation can be solved for the value function  $V(x, t)$ , then (10) gives the optimal constrained control. This HJB equation cannot generally be solved. There is currently no method for rigorously solving for the value function of this constrained optimal control problem.

### 3. Non-linear fixed-final-time HJB solution by NN least-squares approximation

The HJB Equation (11) is difficult to solve for the cost function  $V(x, t)$ . In this section, NN are used to solve approximately for the value function in (11) over  $\Omega$  by approximating the cost function  $V(x, t)$  uniformly in  $t$ . The result is an efficient, practical and computationally tractable solution algorithm to find nearly optimal state feedback controllers for non-linear systems.

#### 3.1 NN approximation of the cost function $V(x, t)$

It is well known that a NN can be used to approximate smooth time-invariant functions on prescribed compact sets (Hornik *et al.*, 1990). Since the analysis required here is restricted to the region of asymptotical stability (RAS) of some initial stabilizing controller, NNs are natural for this application. In Sandberg (1998), it is shown that NNs with time-varying weights can be used to approximate uniformly continuous time-varying functions. We assume that  $V(x, t)$  is smooth and so uniformly continuous on a compact set. Therefore, one

can use the following equation to approximate  $V$  for  $t \in [t_0, t_f]$  on a compact set  $\Omega \subset \mathfrak{R}^n$

$$V_L(x, t) = \sum_{j=1}^L w_j(t) \sigma_j(x) = \mathbf{w}_L^T(t) \boldsymbol{\sigma}_L(x) \quad (12)$$

This is a NN with activation functions  $\sigma_j(x) \in C^1(\Omega)$ ,  $\sigma_j(0) = 0$ . The NN weights are  $w_j(t)$  and  $L$  is the number of hidden-layer neurons.  $\boldsymbol{\sigma}_L(x) \equiv [\sigma_1(x) \sigma_2(x) \dots \sigma_L(x)]^T$  is the vector of activation function,  $\mathbf{w}_L(t) \equiv [w_1(t) w_2(t) \dots w_L(t)]^T$  is the vector of NN weights.

The set  $\sigma_j(x)$  is selected to be independent. Then without loss of generality, they can be assumed to be orthonormal, i.e., select equivalent basis functions to  $\sigma_j(x)$  that are also orthonormal (Beard, 1995). The orthonormality of the set  $\{\sigma_j(x)\}_1^\infty$  on  $\Omega$  implies that if a function  $\psi(x, t) \in L_2(\Omega)$  then

$$\psi(x, t) = \sum_{j=1}^{\infty} \langle \psi(x, t), \sigma_j(x) \rangle_{\Omega} \sigma_j(x)$$

where  $\langle f, g \rangle_{\Omega} = \int_{\Omega} g \cdot f^T dx$  is inner product, and the series converges pointwise, i.e., for any  $\varepsilon > 0$  and  $x \in \Omega$ , one can choose  $N$  sufficiently large to guarantee that  $|\sum_{j=N+1}^{\infty} \langle \psi(x, t), \sigma_j(x) \rangle_{\Omega} \sigma_j(x)| < \varepsilon$  for all  $t \in [t_0, t_f]$  (Beard *et al.*, 1997).

Note that, since one requires  $\partial V(x, t)/\partial t$  in (11), the NN weights are selected to be time-varying. This is similar to methods such as assumed mode shapes in the study of flexible mechanical systems (Balas, 1978). However, here  $\boldsymbol{\sigma}_L(x)$  is an NN activation vector, not a set of eigenfunctions. That is, the NN approximation property significantly simplifies the specification of  $\boldsymbol{\sigma}_L(x)$ . For the infinite final time case, the NN weights are constant Abu-Khalaf and Lewis (2005). The NN weights will be selected to minimize a residual error in a least-squares sense over a set of points sampled from a compact set  $\Omega_0$  inside the RAS of the initial stabilizing control (Finlayson, 1972).

Note that

$$\frac{\partial V_L(x, t)}{\partial x} = \frac{\partial \boldsymbol{\sigma}_L^T(x)}{\partial x} \mathbf{w}_L(t) \equiv \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(t) \quad (13)$$

where  $\nabla \boldsymbol{\sigma}_L(x)$  is the Jacobian  $\partial \boldsymbol{\sigma}_L(x)/\partial x$ , and that

$$\frac{\partial V_L(x, t)}{\partial t} = \dot{\mathbf{w}}_L^T(t) \boldsymbol{\sigma}_L(x) \quad (14)$$

Therefore approximating  $V(x, t)$  by  $V_L(x, t)$  uniformly in  $t$  in the HJB Equation (11) results in

$$\begin{aligned} & -\dot{\mathbf{w}}_L^T(t)\boldsymbol{\sigma}_L(x) - \mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)f(x) - 2\int_0^u \boldsymbol{\varphi}^{-T}(v)R\,dv + \mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x) \cdot g(x) \cdot \boldsymbol{\varphi} \\ & \times \left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right) - Q(x) \\ & = e_L(x, t) \end{aligned} \quad (15)$$

or

$$HJB\left(V_L(x, t) = \sum_{j=1}^L w_j(t)\sigma_j(x)\right) = e_L(x, t) \quad (16)$$

where  $e_L(x, t)$  is a residual equation error. From (10) the corresponding constrained optimal control input is

$$u_L(x, t) = -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right) \quad (17)$$

To find the least-squares solution for  $\mathbf{w}_L(t)$ , the method of weighted residuals is used (Finlayson, 1972). The weight derivatives  $\dot{\mathbf{w}}_L(t)$  are determined by projecting the residual error onto  $\partial e_L(x, t)/\partial \dot{\mathbf{w}}_L(t)$  and setting the result to zero  $\forall x \in \Omega_0$  and  $\forall t \in [t_0, t_f]$  using the inner product, i.e.,

$$\left\langle \frac{\partial e_L(x, t)}{\partial \dot{\mathbf{w}}_L(t)}, e_L(x, t) \right\rangle_{\Omega} = 0 \quad (18)$$

From (15) we can get

$$\frac{\partial e_L(x, t)}{\partial \dot{\mathbf{w}}_L} = \boldsymbol{\sigma}_L(x) \quad (19)$$

Therefore we obtain

$$\begin{aligned} & \left\langle -\dot{\mathbf{w}}_L^T(t)\boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x) \right\rangle_{\Omega} + \left\langle -\mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)f(x), \boldsymbol{\sigma}_L(x) \right\rangle_{\Omega} + \left\langle -2\int_0^u \boldsymbol{\varphi}^{-T}(v)R\,dv, \boldsymbol{\sigma}_L(x) \right\rangle_{\Omega} \\ & + \left\langle \mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x) \cdot g(x) \cdot \boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right), \boldsymbol{\sigma}_L(x) \right\rangle_{\Omega} + \left\langle -Q(x), \boldsymbol{\sigma}_L(x) \right\rangle_{\Omega} = 0 \end{aligned} \quad (20)$$



So that

$$\begin{aligned} \dot{\mathbf{w}}_L(t) = & -\langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x) \rangle_{\Omega}^{-1} \cdot \langle \nabla \boldsymbol{\sigma}_L(x) f(x), \boldsymbol{\sigma}_L(x) \rangle_{\Omega} \cdot \mathbf{w}_L(t) \\ & - \langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x) \rangle_{\Omega}^{-1} \left\langle 2 \int_0^u \boldsymbol{\varphi}^{-T}(v) R \, dv, \boldsymbol{\sigma}_L(x) \right\rangle_{\Omega} \\ & + \langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}(x) \rangle_{\Omega}^{-1} \cdot \left\langle \mathbf{w}_L^T(t) \boldsymbol{\sigma}_L(x) \cdot g(x) \cdot \boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(t) \right), \boldsymbol{\sigma}_L(x) \right\rangle_{\Omega} \\ & - \langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x) \rangle_{\Omega}^{-1} \cdot \langle Q(x), \boldsymbol{\sigma}_L(x) \rangle_{\Omega} \end{aligned} \tag{21}$$

with boundary condition  $V(x, t_f) = \phi(x(t_f), t_f) = \mathbf{w}_L^T(t_f) \boldsymbol{\sigma}_L(x(t_f))$ . Note that, given a mesh of  $x(t_f)$  (see Section 3.3), the boundary condition allows one to determine  $\mathbf{w}_L(t_f)$ .

Therefore, the NN weights are simply found by integrating this non-linear ODE backwards in time.

We now show that this procedure provides a nearly optimal solution for the time-varying optimal control problem if  $L$  is selected large enough.

### 3.2 Uniform convergence in $t$ for time-varying function of the method of least-squares

In what follows, one shows convergence results as  $L$  increases for the method of least squares when NN are used to uniformly approximate the cost function in  $t$ .

The following assumptions are required.

*Assumption 1.* The system's dynamics and the performance integrands  $Q(x) + W(u)$  are such that are solutions of the cost function, which is continuous and differentiable, therefore, belonging to the Sobolev space  $V \in H^{1,2}(\Omega)$ . Here  $Q(x)$  and  $W(u)$  satisfy the requirement of existence of smooth solutions.

*Assumption 2.* We can choose a complete co-ordinate elements  $\{\sigma_j(x)\}_1^\infty \in H^{1,2}(\Omega)$  such that the solution  $V(x, t) \in H^{1,2}(\Omega)$  and  $\{\partial V(x, t)/\partial x_1, \dots, \partial V(x, t)/\partial x_n\}$  can be uniformly approximated in  $t$  by the infinite series built from  $\{\sigma_j(x)\}_1^\infty$ .

*Assumption 3.* The coefficients  $|w_j(t)|$  are uniformly bounded in  $t$  for all  $L$ .

The first two assumptions are standard in optimal control and NNs control literature. Completeness follows from Hornik *et al.* (1990).

*Lemma 3.* Convergence of approximate HJB equation. Given  $u \in \psi(\Omega)$ . Let  $V_L(x, t) = \sum_{j=1}^L w_j^T(t) \sigma_j(x)$  satisfy  $\langle HJB(V_L(x, t)), \boldsymbol{\sigma}_L(x) \rangle_{\Omega} = 0$  and  $\langle V_L(t_f), \boldsymbol{\sigma}_L(x) \rangle_{\Omega} = 0$ , and let  $V(x, t) = \sum_{j=1}^\infty c_j^T(t) \sigma_j(x)$  and  $\mathbf{c}_L(t) = [c_1(t) c_2(t) \dots c_L(t)]^T$  satisfy  $HJB(V(x, t)) = 0$  and  $V(x, t_f) = \phi(x(t_f), t_f)$ .

Then  $|HJB(V_L(x, t))| \rightarrow 0$  uniformly in  $t$  on  $\Omega_0$  as  $L$  increases.

*Proof.* The hypotheses imply that  $HJB(V_L(x, t))$  are in  $L_2(\Omega)$ . Note that

$$\begin{aligned} \langle HJB(V_L(x, t)), \sigma_j(x) \rangle_\Omega &= \sum_{k=1}^L w_k(t) \langle \sigma_k(x), \sigma_j(x) \rangle_\Omega + \sum_{k=1}^L w_k(t) \langle \nabla \sigma_k(x) f(x), \sigma_j(x) \rangle_\Omega \\ &+ \sum_{k=1}^L \left\langle 2 \int_0^{u_k} \boldsymbol{\varphi}^{-T}(v) R \, dv, \sigma_j \right\rangle_\Omega \\ &- \sum_{k=1}^L \left\langle w_k(t) \nabla \sigma_k(x) \cdot g(x) \cdot \boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(t) \right), \sigma_j(x) \right\rangle_\Omega + \langle Q(x), \sigma_j(x) \rangle_\Omega \end{aligned} \quad (22)$$

Since the set  $\{\sigma_j(x)\}_1^\infty$  are orthogonal,  $\langle \sigma_k(x), \sigma_j(x) \rangle_\Omega = 0$ .

Therefore

$$\begin{aligned} |HJB(V_L(x, t))| &= \left| \sum_{j=1}^\infty \langle HJB(V_L(x, t)), \sigma_j(x) \rangle_\Omega \sigma_j(x) \right| \\ &\leq \left| \sum_{j=L+1}^\infty \left( \sum_{k=1}^L w_k(t) \langle \nabla \sigma_k(x) f(x), \sigma_j(x) \rangle_\Omega \right) \cdot \sigma_j(x) \right| \\ &+ \left| \sum_{j=L+1}^\infty \left( \sum_{k=1}^L \left\langle 2 \int_0^{u_k} \boldsymbol{\varphi}^{-T}(v) R \, dv, \sigma_j(x) \right\rangle_\Omega \right) \cdot \sigma_j(x) \right| \\ &+ \left| \left( \sum_{j=L+1}^\infty \left( \sum_{k=1}^L \left\langle -w_k(t) \nabla \sigma_k(x) g(x) \boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T[x] \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(x) \right), \sigma_j(x) \right\rangle_\Omega \right) \right) \cdot \sigma_j(x) \right| \\ &+ \left| \sum_{j=L+1}^\infty \langle Q(x), \sigma_j(x) \rangle_\Omega \sigma_j(x) \right| = \left| \left( \sum_{k=1}^L w_k(t) \sum_{j=L+1}^\infty \langle \nabla \sigma_k(x) f(x), \sigma_j(x) \rangle_\Omega \cdot \sigma_j(x) \right) \right| \\ &+ \left| \left( \sum_{k=1}^L \sum_{j=L+1}^\infty \left\langle 2 \int_0^{u_k} \boldsymbol{\varphi}^{-T}(v) R \, dv, \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right) \right| \\ &+ \left| \left( \sum_{k=1}^L w_k(t) \sum_{j=L+1}^\infty \left\langle \nabla \sigma_k(x) g(x) \boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(t) \right), \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right) \right| \\ &+ \left| \sum_{j=L+1}^\infty \langle Q(x), \sigma_j(x) \rangle_\Omega \sigma_j(x) \right| \end{aligned} \quad (23)$$

then

$$\leq AB(x) + CD(x) + AE(x) + \left| \sum_{j=L+1}^\infty \langle Q(x), \sigma_j(x) \rangle_\Omega \sigma_j(x) \right|$$

where

$$\begin{aligned}
 A &= \max_{1 \leq k \leq L} |w_k(t)| \\
 B(x) &= \sup_{\substack{(t,x) \in \\ [t_0, T] \times \Omega}} \left| \sum_{k=1}^L \sum_{j=L+1}^{\infty} \langle \nabla \sigma_k(x) f(x), \sigma_j(x) \rangle_{\Omega} \cdot \sigma_j(x) \right| \\
 C &= 1 \\
 D(x) &= \sup_{\substack{(t,x) \in \\ [t_0, T] \times \Omega}} \left| \left( \sum_{k=1}^L \sum_{j=L+1}^{\infty} \left\langle 2 \int_0^{u_k} \boldsymbol{\varphi}^{-T}(v) R \, dv, \sigma_j(x) \right\rangle_{\Omega} \cdot \sigma_j(x) \right) \right| \\
 E(x) &= \sup_{\substack{(t,x) \in \\ [t_0, T] \times \Omega}} \left| \left( \sum_{k=1}^L \sum_{j=L+1}^{\infty} \left\langle \nabla \sigma_k(x) g(x) \boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(t) \right), \sigma_j(x) \right\rangle_{\Omega} \cdot \sigma_j(x) \right) \right|
 \end{aligned}$$

Assumptions 1, 2 and 3 imply that  $\Omega_0$  is compact and the functions  $\nabla \boldsymbol{\sigma}_L(x) f(x)$ ,  $2 \int_0^{u_k} \boldsymbol{\varphi}^{-T}(v) R \, dv$ ,  $\nabla \boldsymbol{\sigma}_L(x) g(x) \boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(t) \right)$ , and  $Q(x)$  are continuous on  $\Omega$  and are in  $L_2(\Omega)$ , and the coefficients  $|w_j(t)|$  are uniformly bounded for all  $L$ . So the orthonormality of the set  $\{\sigma_j(x)\}_1^{\infty}$  implies that  $B(x)$ ,  $D(x)$ ,  $E(x)$  and the fourth term on the right-hand side can be made arbitrarily small by an appropriate choice of  $L$ . Therefore

$$A \cdot B(x) + C \cdot D(x) + AE(x) \rightarrow 0 \quad \text{and} \quad \left| \sum_{j=L+1}^{\infty} \langle Q(x), \sigma_j(x) \rangle_{\Omega} \sigma_j(x) \right| \rightarrow 0$$

So  $|HJB(V_L(x, t))| \rightarrow 0$  uniformly in  $t$  on  $\Omega_0$  as  $L$  increases. ■

*Lemma 4. Convergence of NN weights. Given  $u \in \Psi(\Omega)_0$  and supposing the hypotheses of Lemma 3 hold, then*

$$\|\mathbf{w}_L(t) - \mathbf{c}_L(t)\|_2 \rightarrow 0 \quad \text{uniformly in } t \text{ as } L \text{ increases}$$

*Proof.* See Cheng *et al.* (2005). ■

*Lemma 5. Convergence of approximate value function. Under the hypotheses of Lemma 3, one has*

$$\|V_L(x, t) - V(x, t)\|_{L_2(\Omega)} \rightarrow 0 \quad \text{uniformly in } t \text{ on } \Omega \text{ as } L \text{ increases.}$$

*Proof.* See Cheng *et al.* (2005). ■

*Lemma 6. Convergence of value function gradient. Under the hypotheses of Lemma 3,*

$$\left\| \frac{\partial V_L(x, t)}{\partial x} - \frac{\partial V(x, t)}{\partial x} \right\|_{L_2(\Omega)} \rightarrow 0 \text{ uniformly in } t \text{ on } \Omega_0 \text{ as } L \text{ increases.}$$

*Proof.* See Cheng et al. (2005). ■

*Lemma 7. Convergence of control inputs. If the conditions of Lemma 3 are satisfied and*

$$u_L(x, t) = -\varphi\left(\frac{1}{2}R^{-1}g^T(x)\frac{\partial V_L(x, t)}{\partial x}\right)$$

$$u(x, t) = -\varphi\left(\frac{1}{2}R^{-1}g^T(x)\frac{\partial V(x, t)}{\partial x}\right)$$

*Then*

$$\|u_L(x, t) - u(x, t)\|_{L_2(\Omega)} \rightarrow 0 \text{ in } t \text{ on } \Omega_0 \text{ as } L \text{ increases.}$$

*Proof.* See Cheng et al. (2005). ■

*Lemma 8. Convergence of state trajectory. Let  $x_L(t)$  be the state using control (12), suppose the hypotheses of Lemma 3 hold. Then*

$$x(t) - x_L(t) \rightarrow 0 \text{ uniformly in } t \text{ on } \Omega_0 \text{ as } L \text{ increases.}$$

*Proof.* See Cheng et al. (2005). ■

The above lemmas demonstrate uniform convergence in  $t$  in the mean in Sobolev space  $H^{1,2}(\Omega)$ . In fact, the next result shows even stronger convergence properties, namely uniform convergence in both  $x$  and  $t$ .

*Lemma 9. Uniform convergence. Since a local Lipschitz condition holds on (29), then*

$$\sup_{x \in \Omega} |V_L(x, t) - V(x, t)| \rightarrow 0,$$

$$\sup_{x \in \Omega} \left| \frac{\partial V_L(x, t)}{\partial x} - \frac{\partial V(x, t)}{\partial x} \right| \rightarrow 0 \text{ and } \sup_{x \in \Omega} |u_L(x, t) - u(x, t)| \rightarrow 0$$

*The final result shows that if the number  $L$  of hidden layer units is large enough, the proposed solution method yields an admissible control.*

*Lemma 10. Admissibility of  $u_L(x, t)$ . If the conditions of Lemma 3 are satisfied, then  $\exists L_0 : L \geq L_0, u_L \in \Psi(\Omega_0)$ .*

## 3.3 Optimal algorithm based on NN approximation

Solving the integration in (20) is expensive computationally, since evaluation of the  $L_2$  inner product over  $\Omega_0$  is required. This can be addressed using the collocation method (Finlayson, 1972). The integrals can be well approximated by discretization. A mesh of points over the integration region can be introduced on  $\Omega_0$  of size  $\Delta x$ . The terms of (21) can be rewritten as follows

$$A = \left[ \sigma_L(x)|_{x_1} \dots \sigma_L(x)|_{x_p} \right]^T \quad (24)$$

$$B = \left[ \sigma_L(x)f(x)|_{x_1} \dots \sigma_L(x)f(x)|_{x_p} \right]^T \quad (25)$$

$$C = \left[ 2 \int_0^{u_L} \boldsymbol{\varphi}^{-T}(v)R \, dv|_{x_1} \dots 2 \int_0^{u_L} \boldsymbol{\varphi}^{-T}(v)R \, dv|_{x_p} \right]^T \quad (26)$$

$$D = \left[ \nabla \sigma_L(x)g(x)\boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \sigma_L^T(x) \mathbf{w}_L(t) \right) \right. \\ \left. \left|_{x_1} \dots \nabla \sigma_L(x)g(x)\boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \sigma_L^T(x) \mathbf{c}_L(t) \right) \right|_{x_p} \right]^T \quad (27)$$

$$E = \left[ Q(x)|_{x_1} \dots Q(x)|_{x_p} \right]^T \quad (28)$$

where  $p$  in  $x_p$  represents the number of points of the mesh. Reducing the mesh size, we have

$$\left\langle -\dot{\mathbf{w}}_L^T(t) \sigma_L(x), \sigma_L(x) \right\rangle_{\Omega} = \lim_{\|\Delta x\| \rightarrow 0} -(A^T A) \cdot \dot{\mathbf{w}}_L(t) \cdot \Delta x \quad (29)$$

$$\left\langle -\mathbf{w}_L^T(t) \nabla \sigma_L(x) f(x), \sigma_L(x) \right\rangle_{\Omega} = \lim_{\|\Delta x\| \rightarrow 0} -(A^T B) \cdot \mathbf{w}_L(t) \cdot \Delta x \quad (30)$$

$$\left\langle -2 \int_0^{u_L} \boldsymbol{\varphi}^{-T}(v) R \, dv, \sigma_L(x) \right\rangle_{\Omega} = \lim_{\|\Delta x\| \rightarrow 0} -A^T C \cdot \Delta x \quad (31)$$

$$\left\langle -\mathbf{w}_L^T(t) \nabla \sigma_L(x), g(x) \boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \sigma_L^T(x) \mathbf{w}_L(t) \right), \sigma_L(x) \right\rangle_{\Omega} = \lim_{\|\Delta x\| \rightarrow 0} A^T D \mathbf{w}_L(t) \cdot \Delta x \quad (32)$$

$$\left\langle -Q(x), \sigma_L(x) \right\rangle_{\Omega} = \lim_{\|\Delta x\| \rightarrow 0} -(A^T E) \cdot \Delta x \quad (33)$$

This implies that (20) can be converted to

$$-A^T A \dot{\mathbf{w}}_L(t) - A^T B \mathbf{w}_L(t) - A^T C + A^T D \mathbf{w}_L(t) - A^T E = 0 \quad (34)$$

then

$$\dot{\mathbf{w}}_L(t) = -(A^T A)^{-1} A^T B \mathbf{w}_L(t) - (A^T A)^{-1} A^T + (A^T A)^{-1} A^T D \mathbf{w}_L(t) - (A^T A)^{-1} A^T E \quad (35)$$

This is a non-linear ODE that can easily be integrated backwards using final condition  $\mathbf{w}_L(t_f)$  to find the least-squares optimal NN weights. Then, the nearly optimal value function is given by

$$V_L(x, t) = \mathbf{w}_L^T(t) \boldsymbol{\sigma}_L(x)$$

and the nearly optimal control by

$$u_L(x, t) = -\boldsymbol{\varphi} \left( \frac{1}{2} R^{-1} g^T(x) \nabla \boldsymbol{\sigma}_L^T(x) \mathbf{w}_L(t) \right) \quad (36)$$

Note that, in practice, we use a numerically efficient least-squares relative to solve (35) without matrix inversion.

#### 4. NN algorithm for chained form system with time-folding method

Brockett's theorem indicates that non-holonomic systems cannot be asymptotically stabilized around a fixed point under any smooth (or even continuous) time-independent state feedback control law. In this section, a smooth nearly-optimal time-varying control is designed to stabilize the chained form system using a time-folding method (Qu *et al.*, 2006a, b), with a new dynamic control design, a global non-linear time transformation is found to transform the chained form system into a controllable linear time-varying system.

##### 4.1 Chained form system description

Consider the following two-input, three-dimensional non-holonomic chained form system:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 \end{aligned} \quad (37)$$

where  $x = [x_1 \dots x_n]^T \in \mathfrak{R}^n$  is the state,  $u = [u_1 \quad u_2]^T \in \mathfrak{R}^2$  is the control input. The objective of this paper is to present time-varying and continuous feedback controls that globally stabilize the system (37) and are optimal with respect to certain performance indices. It is straightforward to extend the proposed results to  $m$ -input non-holonomic systems that can be transformed into the chained form.

The chained form system (37) can be decomposed into the following two interconnected subsystems:

$$\dot{x}_1 = u_1 \quad (38)$$

$$\dot{z} = u_2 A_1 z + B_1 u_2 \quad (39)$$

where  $z = [z_1 \ z_2]^T = [x_2 \ x_3]^T$ , and  $A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

#### 4.2 Dynamic control design

In this subsection, two dynamic feedback control components  $u_1$  and  $u_2$  will sequentially be designed to form the proposed asymptotically stabilizing control. As the first step, dynamic feedback control  $u_1$  is chosen to be of the following form:

$$\hat{u}_1 = \lambda(t)u_1 \quad (40)$$

$$\hat{u}_2 = \lambda(t)u_2 = (t+a)u_2 \quad (41)$$

$$\lambda(t) = t+a \quad (42)$$

where  $\hat{u}_1$  and  $\hat{u}_2$  are transformed controls, and  $a$  is constant. From (39), letting  $\tau = \ln(t+a)$ , then

$$\frac{d\hat{z}}{d\tau} = \gamma(\tau, a)A_1\hat{z} + B_1\hat{u}_2 \quad (43)$$

where  $\hat{z}(\tau) = z(t)$ ,  $\gamma(\tau, a)$  is a scale factor.

With the above transformation, the control should be changed to:

$$u = -\frac{1}{2}\varphi(\lambda(t)R^{-1}g^T\nabla\sigma_L^T(x)\mathbf{w}_L(t)) \quad (44)$$

Here

$$g = \begin{bmatrix} t+a & 0 \\ 0 & 1 \\ 0 & (t+a)x_1 \end{bmatrix}$$

From Qu *et al.* (2006a, b), we can clearly get the following lemma regarding controllability.

*Lemma 11.* Suppose that component  $u_1(t)$  is designed to be uniformly right continuous, uniformly bounded and uniformly non-vanishing. Then system (37) is uniformly completely controllable.

*Proof.* See Qu *et al.* (2006a, b). ■

*Lemma 12.* Consider the solution to the following differential Riccati equation: for some  $P_2(\infty) > 0$  and for any given  $0 < q_2(t)$  and  $0 < r_2(t)$ ,

$$0 = \dot{P}_2(t) + P_2(t)F_2(t) + F_2^T(t)P_2(t) + C_2^T q_2(t)C_2 - P_2(t)B_2 r_2^{-1}(t)B_2^T P_2(t)$$

If both pairs  $\{F_2(t), B_2\}$  and  $\{F_2^T(t), C_2^T\}$  are uniformly completely controllable, solution  $P_2(t)$  exists and is uniformly bounded,  $V(\xi, t) = \xi^T P(t)\xi$  is positive definite

*Proof.* See Qu et al. (2006a, b). ■

## 5. Control design and simulation

We now show the power of our NN control technique using a time-folding method for finding nearly optimal fixed-final time controllers to a mobile robot, which is a non-holonomic system (Kolmanovsky and McClamroch, 1995). Its kinematics model can be transformed into a chained form (37) with  $n = 3$ . It is known (Brockett, 1983) that a continuous time-invariant feedback control law that minimizes the cost does not exist. Our method will yield a time-varying gain.

For a non-holonomic system, define performance index

$$V(x(t_0), t_0) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} (Q(x) + W(u))dt$$

Here  $Q$  and  $R$  are chosen as identity matrices. To solve for the value function of the related optimal control problem, we selected the smooth approximating function

$$\begin{aligned} V(x_1, x_2, x_3) = & w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + w_4 x_1 x_2 + w_5 x_1 x_3 + w_6 x_2 x_3 \\ & + w_7 x_1^4 + w_8 x_2^4 + w_9 x_3^4 + w_{10} x_1^2 x_2^2 + w_{11} x_1^2 x_3^2 + w_{12} x_2^2 x_3^2 \\ & + w_{13} x_1^2 x_2 x_3 + w_{14} x_1 x_2^2 x_3 + w_{15} x_1 x_2 x_3^2 + w_{16} x_1^3 x_2 \\ & + w_{17} x_1^3 x_3 + w_{18} x_1 x_2^3 + w_{19} x_1 x_3^3 + w_{20} x_2 x_3^3 + w_{21} x_2^3 x_3 \end{aligned} \quad (45)$$

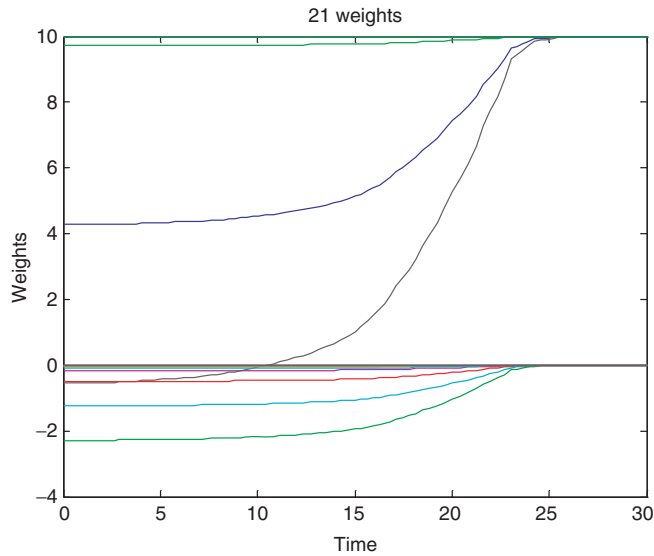
The selection of the NN is usually a natural choice guided by engineering experience and intuition. This is an NN with polynomial activation functions, and hence  $V(0) = 0$ . This is a power series NN with 21 activation functions containing powers of the state variable of the system up to the fourth order. The number of neurons required is chosen to guarantee the uniform convergence of the algorithm. In this example,

$$\mathbf{w}_L(t_f) = [10; 10; 10; 0; 0; 0; 10; 10; 10; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0]$$

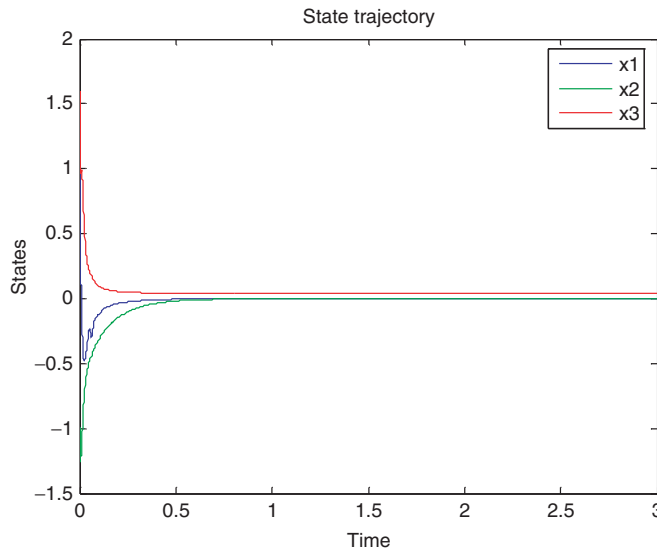
and  $t_f = 30$  seconds.



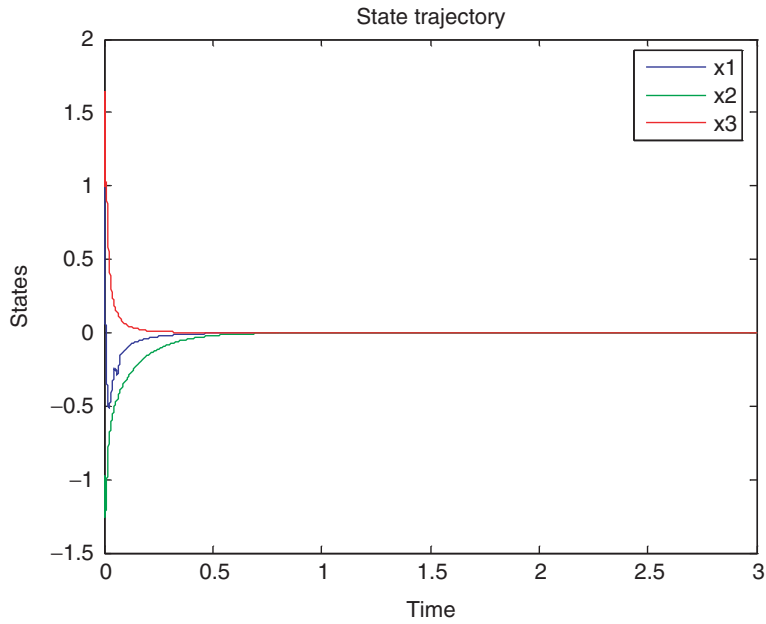
In the simulation, the initial condition of the state is set to be  $x(t_0) = [1 \quad -1 \quad \pi/2]^T$ . Figure 1 indicates that weights converge to constants when they are integrated backwards. Figures 2 and 3 show that the system's state response, including  $x_1$ ,  $x_2$  and  $x_3$ , are all bounded. It can be seen that the state  $x_3$ 's steady value can be controlled by changing  $a$



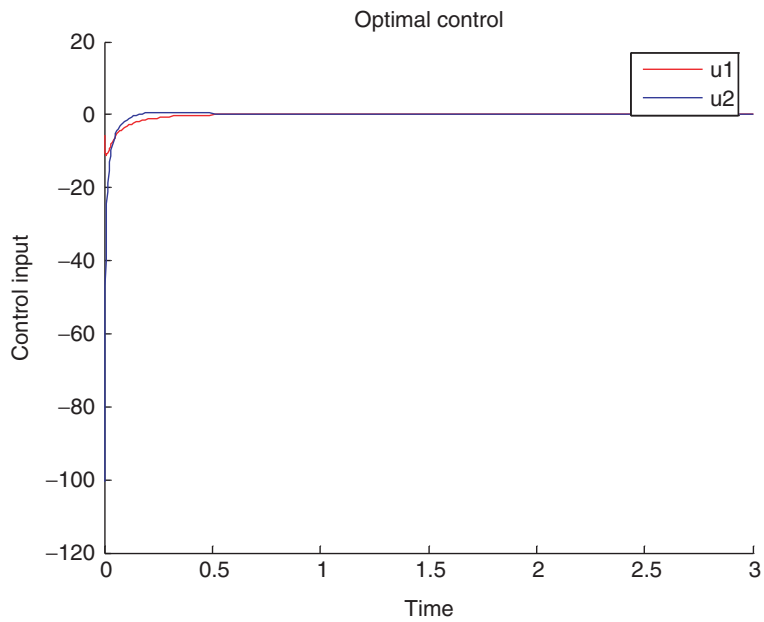
**Figure 1** Non-linear system weights



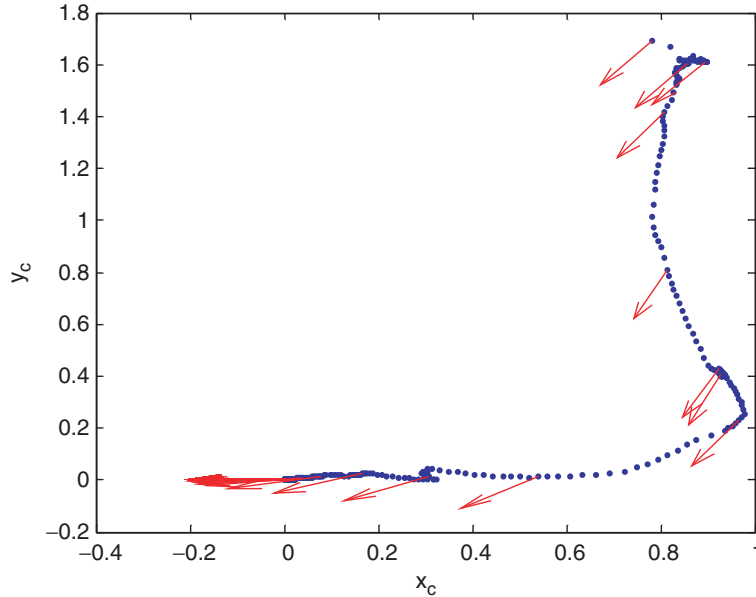
**Figure 2** State trajectories under the time folding control ( $a = 0.5$ )



**Figure 3** State trajectories under the time folding control ( $a = 0.61$ )



**Figure 4** Optimal NN control law



**Figure 5** Trajectories in co-ordinates  $(x_c, y_c)$

in Equation (42). When  $a = 0.61$ ,  $x_3$  does converge to the origin. Figure 4 shows the nearly-optimal control converges to zero. Consider a state and control transformation defined by

$$x_1 = -x_c \cos \theta - y_c \sin \theta$$

$$x_2 = \theta$$

$$x_3 = -x_c \sin \theta + y_c \cos \theta$$

System response in transformed co-ordinates  $(x_c, y_c)$  is shown in Figure 5.

## 6. Conclusion

We use NNs to solve approximately a time-varying HJB equation to design effective controls for non-holonomic chained form systems. A certain time-folding method is applied to recover uniform complete controllability for the chained form system. Then, NNs are used to solve approximately an associated HJ equation. Full conditions for convergence have been derived. A simulation example has been carried out to show the effectiveness of the proposed method.

## References

- Abu-Khalaf, M.** and **Lewis, F.L.** 2005: Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach. *Automatica* 41, 779–91.
- Athans, M.** and **Falb, P.L.** 1966: *Optimal control: an introduction to the theory and its applications*. McGraw-Hill.
- Balas, M.J.** 1978: Modal control of certain flexible dynamic systems. *SIAM Journal of Control and Optimization* 16, 450–62.
- Beard, R.** 1995: Improving the closed-loop performance of nonlinear systems. PhD thesis, Rensselaer Polytechnic Institute, Troy, New York.
- Beard, R., Saridis, G.** and **Wen, J.** 1997: Galerkin approximations of the generalized Hamilton–Jacobi–Bellman equation. *Automatica* 33, 2159–77.
- Bernstein, D.S.** 1995: Optimal nonlinear, but continuous, feedback control of systems with saturating actuators. *International Journal of Control* 62, 1209–16.
- Bernstein, D.S.** 1993: Nonquadratic cost and nonlinear feedback control. *International Journal of Robust and Nonlinear Control* 3, 211–29.
- Brockett, R.W.** 1983: Asymptotic stability and feedback stabilization. In Brockett, R.W., Millman, R.S. & Sussmann, H.J., editors, *Differential geometric control theory*. Birkhauser.
- Chen, F.C.** and **Liu, C.C.** 1994: Adaptively controlling nonlinear continuous-time systems using multiplayer neural networks. *IEEE Transactions on Automatic Control* 39, 1306–10.
- Chen, Z.** and **Jagannathan, S.** 2005: Neural network-based nearly optimal Hamilton–Jacobi–Bellman solution for affine nonlinear discrete-time systems. *Proceeding of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005*, Seville, 12–15 December.
- Cheng, T., Lewis, F.L.** and **Abu-Khalaf, M.** 2005: A neural network solution for fixed-final time optimal control of nonlinear systems. Submitted.
- Dolphus, R.M.** and **Schmitendorf, W.E.** 1995: Stability analysis for a class of linear controllers under control constraints. *Dynamics and Control* 5, 191–203.
- Finlayson, B.A.** 1972: *The method of weighted residuals and variational principles*. Academic Press.
- Ge, S.S.** 1996: Robust adaptive NN feedback linearization control of nonlinear systems. *International Journal of Systems Science* 27, 1327–38.
- Hornik, K., Stinchcombe, M.** and **White, H.** 1990: Universal approximation of an unknown mapping and its derivatives using multilayer feedforward networks. *Neural Networks* 3, 551–60.
- Kolmanovsky, I.** and **McClamroch, N.H.** 1995: Developments in nonholonomic control problems. *IEEE Control Systems* 20–36.
- Lewis, F.L., Jagannathan, S.** and **Yesildire, A.** 1999: *Neural network control of robot manipulators and nonlinear systems*. Taylor & Francis.
- Lewis, F.L.** and **Syrmos, V.L.** 1995: *Optimal control*. John Wiley & Sons.
- Lyshevski, S.E.** 1995: Robust nonlinear control of uncertain systems with state and control constraints. *Proceedings of the 34th Conference on Decision & Control*, New Orleans, LA, December.
- Lyshevski, S.E.** 1996: Constrained optimization and control of nonlinear systems: new results in optimal control. *Proceedings of the 35th Conference on Decision & Control*, Kobe, December.
- Lyshevski, S.E.** 1998: Optimal control of nonlinear continuous-time systems: design of bounded controllers via generalized nonquadratic functionals. *Proceedings of the American Control Conference*, June, 205–209.
- Lyshevski, S.E.** 1999a: Optimal tracking control of nonlinear dynamic systems with control bounds. *Proceedings of the 38th Conference on Decision & Control*, Phoenix, AZ, December.
- Lyshevski, S.E.** 1999b: Optimization of a class of nonholonomic dynamic systems. *Proceedings of the American Control Conference*, San Diego, CA, June.
- Lyshevski, S.E.** 2001: *Control systems theory with engineering applications*. Birkhauser.
- Lyshevski, S.E.** and **Meyer, A.U.** 1995: Control system analysis and design upon the Lyapunov method. *Proceedings of the American Control Conference*, June, 3219–23.
- Miller, W.T., Sutton, R.** and **Werbos, P.** 1990: *Neural networks for control*. The MIT Press.

- Polycarpou, M.M.** 1996: Stable adaptive neural control scheme for nonlinear systems. *IEEE Transactions on Automatic Control* 41, 447–51.
- Qu, Z., Wang, J., Hull, R.A. and Martin, J.** 2006a: Continuous control designs for stabilizing chained systems: a global state scaling transformation and a time scaling method. *45th IEEE Conference on Decision and Control*, San Diego, CA.
- Qu, Z., Wang, J., Plaisted, C.E. and Hull, R.A.** 2006b: Global-stabilizing near-optimal control design for nonholonomic chained systems. *IEEE Transactions on Automatic Control*. Accepted.
- Rovithakis, G.A. and Christodoulou, M.A.** 1994: Adaptive control of unknown plants using dynamical neural networks. *IEEE Transactions on Systems, Man, and Cybernetics* 24, 400–12.
- Saberi, A., Lin, Z. and Teel, A.R.** 1996: Control of linear systems with saturating actuators. *IEEE Transaction on Automatic Control* AC-41, 368–78.
- Sandberg, E.W.** 1998: Notes on uniform approximation of time-varying systems on finite time intervals. *IEEE Transactions on Circuits and Systems-1: Fundamental Theory and Applications* 45, August.
- Sanner, R.M. and Slotine, J.J.E.** 1991: Stable adaptive control and recursive identification using radial gaussian networks. *Proceedings of the IEEE Conference on Decision & Control*, 2116–23, Brighton.
- Sussmann, H., Sontag, E.D. and Yang, Y.** 1994: A general result on the stabilization of linear systems using bounded controls. *IEEE Transactions on Automatic Control* 39, 2411–25.