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Research Article

On Certain Class of Analytic Functions Related to Cho-Kwon-Srivastava Operator

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Motivated by a multiplier transformation and some subclasses of meromorphic functions which were defined by means of the Hadamard product of the Cho-Kwon-Srivastava operator, we define here a similar transformation by means of the Ghanim and Darus operator. A class related to this transformation will be introduced and the properties will be discussed.

1. Introduction

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured unit disk $U = \{z : 0 < |z| < 1\}$. For $0 \leq \beta$, we denote by $S^*(\beta)$ and $k(\beta)$ the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in U (cf. e.g., [1–4]).

For functions $f_j(z)$ $(j = 1; 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.3}$$

Let us define the function $\tilde{\phi}(\alpha, \beta; z)$ by

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| z^n, \tag{1.4}$$

for $\beta \neq 0, -1, -2, \ldots$, and $\alpha \in \mathbb{C}/\{0\}$, where $(\lambda)n = \lambda(\lambda+1)_{n+1}$ is the Pochhammer symbol. We note that

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} {}_2F_1(1, \alpha, \beta; z), \tag{1.5}$$

where

$${}_2F_1(b, \alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{(b)_n (\alpha)_n}{(\beta)_n} \frac{z^n}{n!} \tag{1.6}$$

is the well-known Gaussian hypergeometric function.

Let us put

$$q_{\lambda,\mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} z^n, \quad \lambda > 0, \ \mu \geq 0. \tag{1.7}$$

Corresponding to the functions $\tilde{\phi}(\alpha, \beta; z)$ and $q_{\lambda,\mu}(z)$ and using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L(\alpha, \beta, \lambda, \mu) f(z)$ on $\sum$ by

$$\begin{aligned} L(\alpha, \beta, \lambda, \mu) f(z) &= \left(f(z) * \tilde{\phi}(\alpha, \beta; z) * q_{\lambda,\mu}(z) \right) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} a_n z^n. \end{aligned} \tag{1.8}$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [5, 6], Liu [7], Liu and Srivastava [8–10], and Cho and Kim [11].

For a function $f \in L(\alpha, \beta, \lambda, \mu) f(z)$, we define

$$I_{\alpha,\beta,\lambda}^{\mu,0} = L(\alpha, \beta, \lambda, \mu) f(z), \tag{1.9}$$

and, for $k = 1, 2, 3, \ldots$,

$$\begin{aligned} I^{\mu,k}_{\alpha,\beta,\lambda} f(z) &= z\left(I^{k-1} L(\alpha,\beta,\lambda,\mu) f(z)\right)' + \frac{2}{z} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} n^k \left|\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right| \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} a_n z^n. \end{aligned} \tag{1.10}$$

Note that if $n = \beta$, $k = 0$, the operator $I^{\mu,0}_{\alpha,n,\lambda}$ reduced to the one introduced by Cho et al. [12] for $\mu \in \mathbb{N}_0 = \mathbb{N} \cup 0$. It was known that the definition of the operator $I^{\mu,0}_{\alpha,n,\lambda}$ was motivated essentially by the Choi-Saigo-Srivastava operator [13] for analytic functions, which includes a simpler integral operator studied earlier by Noor [14] and others (cf. [15–17]). Note also the operator $I^{0,k}_{\alpha,\beta}$ has been recently introduced and studied by Ghanim and Darus [18] and Ghanim et al. [19], respectively. To our best knowledge, the recent work regarding operator $I^{\mu,0}_{\alpha,n,\lambda}$ was charmingly studied by Piejko and Sokól [20]. Moreover, the operator $I^{\mu,k}_{\alpha,\beta,\lambda}$ was then defined and studied by Ghanim and Darus [21]. In the same direction, we will study for the operator $I^{\mu,k}_{\alpha,\beta,\lambda}$ given in (1.10).

Now, it follows from (1.8) and (1.10) that

$$z\left(I^{\mu,k}_{\alpha,\beta,\lambda} f(z)\right)' = \alpha I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z) - (\alpha+1) I^{\mu,k}_{\alpha,\beta,\lambda} f(z). \tag{1.11}$$

Making use of the operator $I^{\mu,k}_{\alpha,\beta,\lambda} f(z)$, we say that a function $f(z) \in \Sigma$ is in the class $\Sigma^{\mu,k}_{\alpha,\beta,\lambda}(A,B)$ if it satisfies the following subordination condition:

$$\frac{z\left(I^{\mu,k}_{\alpha,\beta,\lambda} f(z)\right)'}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)} \prec 1 - \frac{(B-A)w(z)}{1+Bw(z)}, \quad z \in U;\ -1 \le B < A \le 1. \tag{1.12}$$

Furthermore, we say that a function $f(z) \in \Sigma^{\mu,k,+}_{\alpha,\beta,\lambda}(A,B)$ is a subclass of the class $\Sigma^{\mu,k}_{\alpha,\beta,\lambda}(A,B)$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n > 0,\ z \in U). \tag{1.13}$$

The main object of this paper is to present several inclusion relations and other properties of functions in the classes $\Sigma^{\mu,k}_{\alpha,\beta,\lambda}(A,B)$ and $\Sigma^{\mu,k,+}_{\alpha,\beta,\lambda}(A,B)$ which we have introduced here.

2. Main Results

We begin by recalling the following result (popularly known as Jack's Lemma), which we will apply in proving our first inclusion theorem.

Lemma 2.1 (see [Jack's Lemma] [22]). *Let the (nonconstant) function $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then*

$$z_0 w'(z_0) = \gamma w(z_0), \tag{2.1}$$

where γ is a real number and $\gamma \geq 1$.

Theorem 2.2. *If*

$$\alpha > \frac{(A-B)}{1+B} \quad (-1 < B < A \leq 1), \tag{2.2}$$

then

$$\Sigma^{\mu,k}_{\alpha+1,\beta,\lambda}(A,B) \subset \Sigma^{\mu,k}_{\alpha,\beta,\lambda}(A,B). \tag{2.3}$$

Proof. Let $f \in \Sigma^{\mu,k}_{\alpha+1,\,\beta,\lambda}(A,B)$, and suppose that

$$\frac{z\left(I^{\mu,k}_{\alpha,\beta,\lambda} f(z)\right)'}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)} = 1 - \frac{(B-A)w(z)}{1+Bw(z)}, \tag{2.4}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$. By using (2.4) and (1.11), we have

$$\alpha \frac{I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)} = \frac{\alpha + [\alpha B - (A-B)]w(z)}{1+Bw(z)}. \tag{2.5}$$

Upon differentiating both sides of (2.5) with respect to z logarithmically and using the identity (1.11), we obtain

$$\frac{z\left(I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z)\right)'}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)} = 1 - \frac{(B-A)w(z)}{1+Bw(z)} - \frac{(A-B)zw'(z)}{[1+Bw(z)](\alpha + [\alpha B - (A-B)]w(z))}. \tag{2.6}$$

We suppose now that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (z \in U) \tag{2.7}$$

and apply Jack's Lemma, we thus find that

$$z_0 w'(z_0) = \gamma w(z_0) \quad (\gamma \geq 1). \tag{2.8}$$

By writing

$$w(z_0) = e^{i\theta} \quad (0 \leq \theta < 2\pi) \tag{2.9}$$

and setting $z = z_0$ in (2.6), we find after some computations that

$$\begin{aligned} \left| \frac{z_0 \left(I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z_0) \right)' + I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z_0)}{B z_0 \left(I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z_0) \right)' + A I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z_0)} \right|^2 - 1 &= \left| \frac{(\alpha+\gamma) + [\alpha B - (A-B)] e^{i\theta}}{\alpha + [\alpha B - \gamma - (A-B)] e^{i\theta}} \right|^2 - 1 \\ &= \frac{2\gamma(1+\cos\theta)[\alpha(B+1) - (A-B)]}{\left| \alpha + [\alpha B - \gamma - (A-B)] e^{i\theta} \right|^2}. \end{aligned} \tag{2.10}$$

Set

$$g(\theta) = 2\gamma(1+\cos\theta)[\alpha(B+1) - (A-B)]. \tag{2.11}$$

Then, by hypothesis, we have

$$\begin{gathered} g(0) = 4\gamma[\alpha(B+1) - (A-B)] \geq 0, \\ g(\pi) = 0, \end{gathered} \tag{2.12}$$

which, together, imply that

$$g(\theta) \geq 0 \quad (0 \leq \theta < 2\pi). \tag{2.13}$$

View of (2.13) and (2.10) would obviously contradict our hypothesis that

$$f \in \Sigma^{\mu,k}_{\alpha+1,\beta,\lambda}(A,B). \tag{2.14}$$

Hence, we must have

$$|w(z)| < 1 \quad (z \in U), \tag{2.15}$$

and we conclude from (2.4) that

$$f \in \Sigma^{\mu,k}_{\alpha,\beta,\lambda}(A,B). \tag{2.16}$$

The proof of Theorem 2.2 is thus complete. □

3. Properties of the Class $f \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$

Throughout this section, we assume further that $\alpha,\ \beta\ > 0$ and

$$A + B \leq 0 \quad (-1 < B < A \leq 1). \tag{3.1}$$

We first determine a necessary and sufficient condition for a function $f \in \Sigma$ of the form (1.13) to be in the class $f \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$ of meromorphically univalent functions with positive coefficients.

Theorem 3.1. *Let $f \in \Sigma$ be given by* (1.13). *Then $f \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$ if and only if*

$$\sum_{n=1}^{\infty} n^k [n(1-B) + (1-A)] \frac{|(\alpha)_{n+1}|}{\left|(\beta)_{n+1}\right|} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} |a_n| \leq A - B, \tag{3.2}$$

where, for convenience, the result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(A-B)(n+1+\lambda)^{\mu}\left|(\beta)_{n+1}\right|}{n^k \lambda^{\mu}[n(1-B)+(1-A)]|(\alpha)_{n+1}|} z^n, \tag{3.3}$$

for all $z \neq 0$.

Proof. Suppose that the function $f \in \Sigma$ is given by (1.13) and is in the class $\Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$. Then, from (1.13) and (1.12), we find that

$$\begin{aligned} &\left| \frac{z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)' + I_{\alpha,\beta,\lambda}^{\mu,k} f(z)}{Bz\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)' + A I_{\alpha,\beta,\lambda}^{\mu,k} f(z)} \right| \\ &\quad = \left| \frac{\sum_{n=1}^{\infty} n^k (n+1)\left(|(\alpha)_{n+1}|/\left|(\beta)_{n+1}\right|\right)(\lambda/(n+1+))^{\mu}|a_n| z^n}{(A-B) + \sum_{n=1}^{\infty} n^k (A+nB)(\lambda/(n+1+))^{\mu}\left(|(\alpha)_{n+1}|/\left|(\beta)_{n+1}\right|\right)|a_n| z^n} \right| \leq 1 \quad (z \in U). \end{aligned} \tag{3.4}$$

Since $|\Re(z)| \leq |z|$ for any z, therefore, we have

$$\Re\left(\frac{\sum_{n=1}^{\infty} n^k (n+1)\left(|(\alpha)_{n+1}|/\left|(\beta)_{n+1}\right|\right)(\lambda/(n+1+\lambda))^{\mu}|a_n| z^n}{(A-B) + \sum_{n=1}^{\infty} n^k (A+nB)(\lambda/(n+1+\lambda))^{\mu}\left(|(\alpha)_{n+1}|/\left|(\beta)_{n+1}\right|\right)|a_n| z^n} \right) \leq 1 \qquad (z \in U). \tag{3.5}$$

Choosing z to be real and letting $z \to 1$ through real values, (3.5) yields

$$\begin{aligned}\sum_{n=1}^{\infty} n^k (n+1) \frac{|(\alpha)_{n+1}|}{|(\beta)_{n+1}|} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} |a_n| \\ \leq (A-B) + \sum_{n=1}^{\infty} n^k (A+nB) \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} \frac{|(\alpha)_{n+1}|}{|(\beta)_{n+1}|} |a_n|,\end{aligned} \tag{3.6}$$

which leads us to the desired inequality (3.2).

Conversely, by applying hypothesis (3.2), we get

$$\begin{aligned}&\left| \frac{z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)' + I_{\alpha,\beta,\lambda}^{\mu,k} f(z)}{Bz\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)' + AI_{\alpha,\beta,\lambda}^{\mu,k} f(z)} \right| \\ &\quad \leq \frac{\sum_{n=1}^{\infty} n^k (n+1) \left(|(\alpha)_{n+1}| / |(\beta)_{n+1}|\right) (\lambda/(n+1+\lambda))^{\mu} |a_n|}{(A-B) + \sum_{n=1}^{\infty} n^k (A+nB) \left(|(\alpha)_{n+1}| / |(\beta)_{n+1}|\right) (\lambda/(n+1+\lambda))^{\mu} |a_n|} \leq 1 \quad (z \in U).\end{aligned} \tag{3.7}$$

Hence, we have $f(z) \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$. By observing that the function $f(z)$, given by (3.3), is indeed an extremal function for the assertion (3.2), we complete the proof of Theorem 3.1. □

By applying Theorem 3.1, we obtain the following sharp coefficient estimates.

Corollary 3.2. *Let $f \in \Sigma$ be given by* (1.13). *If $f \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,\ B)$, then*

$$|a_n| \leq \frac{(A-B)(n+1+\lambda)^{\mu} |(\beta)_{n+1}|}{n^k \lambda^{\mu} [n(1-B) + (1-A)] |(\alpha)_{n+1}|} \quad (n \geq 1,\ z \in U), \tag{3.8}$$

where the equality holds true for the function $f(z)$ given by (3.3).

Next, we prove the following growth and distortion properties for the class $\Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}$.

Theorem 3.3. *If the function f defined by* (1.13) *is in the class $\Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,\ B)$, then, for $0 < |z| = r < 1$, we have*

$$\begin{aligned}\frac{1}{r} - \frac{(2+\lambda)^{\mu}(A-B)|(\beta)_2|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|} r \leq |f(z)| \leq \frac{1}{r} + \frac{(2+\lambda)^{\mu}(A-B)|(\beta)_2|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|} r, \\ \frac{1}{r^2} - \frac{(2+\lambda)^{\mu}(A-B)|(\beta)_2|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(2+\lambda)^{\mu}(A-B)|(\beta)_2|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|}.\end{aligned} \tag{3.9}$$

Each of these results is sharp with the extremal function $f(z)$ given by (3.3).

Proof. Since $f \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$. Theorem 3.1 readily yields the inequality

$$\sum_{n=1}^{\infty} a_n \le \frac{(2+\lambda)^{\mu}(A-B)\left|(\beta)_2\right|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|}. \tag{3.10}$$

Thus, for $0 < |z| = r < 1$ and utilizing (3.10), we have

$$\begin{aligned} \left|f(z)\right| &\le \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \le \frac{1}{r} + r\sum_{n=1}^{\infty} a_n \le \frac{1}{r} + r\frac{(2+\lambda)^{\mu}(A-B)\left|(\beta)_2\right|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|}, \\ \left|f(z)\right| &\ge \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n \ge \frac{1}{r} - r\sum_{n=1}^{\infty} a_n \ge \frac{1}{r} - r\frac{(2+\lambda)^{\mu}(A-B)\left|(\beta)_2\right|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|}. \end{aligned} \tag{3.11}$$

Also from Theorem 3.1, we get

$$\sum_{n=1}^{\infty} n a_n \le \frac{(2+\lambda)^{\mu}(A-B)\left|(\beta)_2\right|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|}. \tag{3.12}$$

Hence

$$\begin{aligned} \left|f'(z)\right| &\le \frac{1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1} \le \frac{1}{r^2} + \sum_{n=1}^{\infty} n a_n \le \frac{1}{r^2} + \frac{(2+\lambda)^{\mu}(A-B)\left|(\beta)_2\right|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|}, \\ \left|f'(z)\right| &\ge \frac{1}{z^2} - \sum_{n=1}^{\infty} n a_n z^{n-1} \ge \frac{1}{r^2} - \sum_{n=1}^{\infty} n a_n \ge \frac{1}{r^2} - \frac{(2+\lambda)^{\mu}(A-B)\left|(\beta)_2\right|}{\lambda^{\mu}[2-(A+B)]|(\alpha)_2|}. \end{aligned} \tag{3.13}$$

This completes the proof of Theorem 3.3. □

We conclude this section by determining the radii of meromorphically univalent starlikeness and meromorphically univalent convexity of the class $\Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$. We state our results as in the following theorems.

Theorem 3.4. *Let $f \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$. Then, f is meromorphically univalent starlike of order γ $(0 \le \gamma < 1)$ in $|z| < r_1$, where*

$$r_1 = r_1(A,B,\gamma) = \inf_{n\ge 0}\left\{\frac{(1-\gamma)[n(1-B)+(1-A)]}{(n+2-\gamma)(A-B)}\right\}^{n+1}. \tag{3.14}$$

The equality is attained for the function $f(z)$ given by (3.3).

Proof. It suffices to prove that

$$\left|\frac{z\left(I_{\alpha,\beta,\lambda}^{\mu,k}\right)'}{I_{\alpha,\beta,\lambda}^{\mu,k}} + 1\right| \le 1-\gamma, \tag{3.15}$$

for $|z| < r_1$, we have

$$\left|\frac{z\left(I_{\alpha,\beta,\lambda}^{\mu,k}\right)'}{I_{\alpha,\beta,\lambda}^{\mu,k}}+1\right| = \left|\frac{\sum_{n=1}^{\infty} n^k(n+1)\left((\alpha)_{n+1}/(\beta)_{n+1}\right)(\lambda/(n+1+\lambda))^{\mu} a_n z^n}{(1/z)+\sum_{n=1}^{\infty} n^k\left((\alpha)_{n+1}/(\beta)_{n+1}\right)(\lambda/(n+1+\lambda))^{\mu} a_n z^n}\right|$$
$$\leq \frac{\sum_{n=1}^{\infty} n^k(n+1)\left(|(\alpha)_{n+1}|/\left|(\beta)_{n+1}\right|\right)(\lambda/(n+1+\lambda))^{\mu}|a_n|\left|z^{n+1}\right|}{1-\sum_{n=1}^{\infty} n^k\left(|(\alpha)_{n+1}|/\left|(\beta)_{n+1}\right|\right)(\lambda/(n+1+\lambda))^{\mu}|a_n||z^{n+1}|}. \tag{3.16}$$

Hence, (3.16) holds true if

$$\sum_{n=1}^{\infty} n^k(n+1)\frac{|(\alpha)_{n+1}|}{\left|(\beta)_{n+1}\right|}\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu}|a_n|\left|z^{n+1}\right|$$
$$\leq (1-\gamma)\left(1-\sum_{n=1}^{\infty} n^k\frac{|(\alpha)_{n+1}|}{\left|(\beta)_{n+1}\right|}\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu}|a_n|\left|z^{n+1}\right|\right) \tag{3.17}$$

or

$$\sum_{n=1}^{\infty} n^k\left(\frac{n+2-\gamma}{1-\gamma}\right)\frac{|(\alpha)_{n+1}|}{\left|(\beta)_{n+1}\right|}\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu}|a_n|\left|z^{n+1}\right| \leq 1, \tag{3.18}$$

with the aid of (3.18) and (3.2), it is true to have

$$\left(\frac{n^k(n+2-\gamma)}{1-\gamma}\right)\frac{|(\alpha)_{n+1}|}{\left|(\beta)_{n+1}\right|}\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu}\left|z^{n+1}\right|$$
$$\leq \frac{n^k\lambda^{\mu}[n(1-B)+(1-A)]|(\alpha)_{n+1}|}{(A-B)(n+1+\lambda)^{\mu}\left|(\beta)_{n+1}\right|} \quad (n \geq 1). \tag{3.19}$$

Solving (3.19) for $|z|$, we obtain

$$|z| \leq \left\{\frac{(1-\gamma)[n(1-B)+(1-A)]}{(n+2-\gamma)(A-B)}\right\}^{n+1} \quad (n \geq 1). \tag{3.20}$$

This completes the proof of Theorem 3.4. □

Theorem 3.5. *Let* $f \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k,+}(A,B)$. *Then,* f *is meromorphically univalent convex of order* γ $(0 \leq \gamma < 1)$ *in* $|z| < r_2$, *where*

$$r_2 = r_2(A,B,\gamma) = \inf_{n\geq 0}\left\{\frac{(1-\gamma)\left[n^{k-1}(1-B)+(1-A)\right]}{(n+2-\gamma)(A-B)}\right\}^{n+1}. \tag{3.21}$$

The equality is attained for the function $f(z)$ *given by* (3.3).

Proof. By using the technique employed in the proof of Theorem 3.4, we can show that

$$\left| \frac{z\left(I_{\alpha,\beta,\lambda}^{\mu,k,+}\right)''}{\left(I_{\alpha,\beta,\lambda}^{\mu,k,+}\right)'} + 2 \right| \leq (1-\gamma), \tag{3.22}$$

for $|z| < r_2$, with the aid of Theorem 3.1. Thus, we have the assertion of Theorem 3.5. □

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