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Research Article

Generalized Timelike Mannheim Curves in Minkowski Space-Time E_1^4

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We give the definition of generalized timelike Mannheim curve in Minkowski space-time E_1^4 . The necessary and sufficient conditions for the generalized timelike Mannheim curve are obtained. We show some characterizations of generalized Mannheim curve.

1. Introduction

The geometry of curves has long captivated the interests of mathematicians, from the ancient Greeks through to the era of Isaac Newton (1643–1727) and the invention of the calculus. It is a branch of geometry that deals with smooth curves in the plane and in the space by methods of differential and integral calculus. The theory of curves is the simpler and narrower in scope because a regular curve in a Euclidean space has no intrinsic geometry. One of the most important tools used to analyze curve is the Frenet frame, a moving frame that provides a coordinate system at each point of curve that is ”best adopted” to the curve near that point. Every person of classical differential geometry meets early in his course the subject of Bertrand curves, discovered in 1850 by J. Bertrand. A Bertrand curve is a curve such that its principal normals are the principal normals of a second curve. There are many works related with Bertrand curves in the Euclidean space and Minkowski space, [1–3].

Another kind of associated curve is called Mannheim curve and Mannheim partner curve. The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows:

a space curve is a Mannheim curve if and only if its curvature k_1 and torsion k_2 satisfy the relation

$$k_1 = \beta\left(k_1^2 + k_2^2\right) \tag{1.1}$$

for some constant β. The articles concerning Mannheim curves are rather few. In [4], a remarkable class of Mannheim curves is studied. General Mannheim curves in the Euclidean 3-space are obtained in [5–7]. Recently, Mannheim curves are generalized and some characterizations and examples of generalized Mannheim curves are given in Euclidean 4-space E^4 by [8].

In this paper, we study the generalized timelike Mannheim partner curves in 4-dimensional Minkowski space-time. We will give the necessary and sufficient conditions for the generalized timelike Mannheim partner curves.

2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves in Minkowski space-time E_1^4 are briefly presented in this section. A more complete elementary treatment can be found in [9].

Minkowski space-time E_1^4 is a usual vector space provided with the standard flat metric given by

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{2.1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since $\langle\ ,\ \rangle$ is an indefinite metric, recall that a $\mathbf{v} \in E_1^4$ can have one of the three causal characters; it can be spacelike if $\langle \mathbf{v}, \mathbf{v}\rangle > 0$ or $\mathbf{v} = 0$, timelike if $\langle \mathbf{v}, \mathbf{v}\rangle < 0$, and null(ligthlike) if $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ and $\mathbf{v} \neq 0$. Similarly, an arbitrary curve $\mathbf{c} = \mathbf{c}(t)$ in E_1^4 can locally be spacelike, timelike, or null (lightlike) if all of its velocity vectors $\mathbf{c}'(t)$ are, respectively, spacelike, timelike, or null. The norm of $\mathbf{v} \in E_1^4$ is given by $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v}\rangle|}$. If $\|\mathbf{c}'(t)\| = \sqrt{|\langle \mathbf{c}'(t), \mathbf{c}'(t)\rangle|} \neq 0$ for all $t \in I$, then C is a regular curve in E_1^4. A timelike (spacelike) regular curve C is parameterized by arc-length parameter t which is given by $\mathbf{c} : I \to E_1^4$, then the tangent vector $\mathbf{c}'(t)$ along C has unit length, that is, $\langle \mathbf{c}'(t), \mathbf{c}'(t)\rangle = -1$, $(\langle \mathbf{c}'(t), \mathbf{c}'(t)\rangle = 1)$ for all $t \in I$.

Hereafter, curves considered are timelike and regular C^∞ curves in E_1^4. Let $\mathbf{T}(t) = \mathbf{c}'(t)$ for all $t \in I$; then the vector field $\mathbf{T}(t)$ is timelike and it is called timelike unit tangent vector field on C.

The timelike curve C is called special timelike Frenet curve if there exist three smooth functions k_1, k_2, k_3 on C and smooth nonnull frame field $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ along the curve C. Also, the functions k_1, k_2, and k_3 are called the first, the second, and the third curvature function on C, respectively. For the C^∞ special timelike Frenet curve C, the following Frenet formula is

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \tag{2.2}$$

see [9].

Here, due to characters of Frenet vectors of the timelike curve, $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are mutually orthogonal vector fields satisfying equations

$$\langle \mathbf{T}, \mathbf{T} \rangle = -1, \qquad \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}_1, \mathbf{B}_1 \rangle = \langle \mathbf{B}_2, \mathbf{B}_2 \rangle = 1. \tag{2.3}$$

For $t \in I$, the nonnull frame field $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ and curvature functions k_1, k_2, and k_3 are determined as follows:

$$\begin{aligned}
\text{1st step} \quad & \mathbf{T}(t) = \mathbf{c}'(t), \\
\text{2nd step} \quad & k_1(t) = \left\| \mathbf{T}'(t) \right\| > 0, \\
& \mathbf{N}(t) = \frac{1}{k_1(t)} \mathbf{T}'(t), \\
\text{3rd step} \quad & k_2(t) = \left\| \mathbf{N}'(t) - k_1(t)\mathbf{T}(t) \right\| > 0, \\
& \mathbf{B}_1(t) = \frac{1}{k_2(t)} \left(\mathbf{N}'(t) - k_1(t)\mathbf{T}(t) \right), \\
\text{4th step} \quad & \mathbf{B}_2(t) = \delta \frac{1}{\left\| \mathbf{B}_1'(t) + k_2(t)\mathbf{N}(t) \right\|} \left(\mathbf{B}_1'(t) + k_2(t)\mathbf{N}(t) \right),
\end{aligned} \tag{2.4}$$

where δ is determined by the fact that orthonormal frame field $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}_1(t), \mathbf{B}_2(t)\}$ is of positive orientation. The function k_3 is determined by

$$k_3(t) = \left\langle \mathbf{B}_1'(t), \mathbf{B}_2(t) \right\rangle \neq 0. \tag{2.5}$$

So the function k_3 never vanishes.

In order to make sure that the curve C is a special timelike Frenet curve, above steps must be checked, from 1st step to 4th step, for $t \in I$.

Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ be the moving Frenet frame along a unit speed timelike curve C in E_1^4, consisting of the tangent, the principal normal, the first binormal, and the second binormal vector field, respectively. Since C is a timelike curve, its Frenet frame contains only nonnull vector fields.

3. Generalized Timelike Mannheim Curves in E_1^4

Mannheim curves are generalized by Matsuda and Yorozu in [8]. In this paper, we have investigated the generalization of timelike Mannheim curves in Minkowski space E_1^4.

Definition 3.1. A special timelike curve C in E_1^4 is a generalized timelike Mannheim curve if there exists a special timelike Frenet curve C^* in E_1^4 such that the first normal line at each point of C is included in the plane generated by the second normal line and the third normal line of C^* at the corresponding point under ϕ. Here, ϕ is a bijection from C to C^*. The curve C^* is called the generalized timelike Mannheim mate curve of C.

By the definition, a generalized Mannheim mate curve C^* is given by the map $\mathbf{c}^* : I^* \to E_1^4$ such that

$$\mathbf{c}^*(t) = \mathbf{c}(t) + \beta(t)\mathbf{N}(t), \quad t \in I. \tag{3.1}$$

Here β is a smooth function on I. Generally, the parameter t is not an arc-length of C^*. Let t^* be the arc-length of C^* defined by

$$t^* = \int_0^t \left\| \frac{d\mathbf{c}^*(t)}{dt} \right\| dt. \tag{3.2}$$

If a smooth function $f : I \to I^*$ is given by $f(t) = t^*$, then for all $t \in I$, we have

$$f'(t) = \frac{dt^*}{dt} = \left\| \frac{d\mathbf{c}^*(t)}{dt} \right\| = \sqrt{\left| -\left(1 + \beta(t)k_1(t)\right)^2 + \left(\beta'(t)\right)^2 + \left(\beta(t)k_2(t)\right)^2 \right|}. \tag{3.3}$$

The representation of timelike curve C^* with arc-length parameter t^* is

$$\begin{aligned} \mathbf{c}^* :\ I^* &\longrightarrow E_1^4, \\ t^* &\longrightarrow \mathbf{c}^*(t^*). \end{aligned} \tag{3.4}$$

For a bijection $\phi : C \to C^*$ defined by $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$, the reparameterization of C^* is

$$\mathbf{c}^*\left(f(t)\right) = \mathbf{c}(t) + \beta(t)\mathbf{N}(t), \tag{3.5}$$

where β is a smooth function on I. Thus, we have

$$\frac{d\mathbf{c}^*\left(f(t)\right)}{dt} = \left.\frac{d\mathbf{c}^*(t^*)}{dt}\right|_{t^*=f(t)} f'(t) = f'(t)\mathbf{T}^*\left(f(t)\right), \quad t \in I. \tag{3.6}$$

Theorem 3.2. *If a special timelike Frenet curve C in E_1^4 is a generalized timelike Mannheim curve, then the following relation between the first curvature function k_1 and the second curvature function k_2 holds:*

$$k_1(t) = -\beta\left(k_1^2(t) - k_2^2(t)\right), \quad t \in I, \tag{3.7}$$

where β is a constant number.

Proof. Let C be a generalized timelike Mannheim curve and C^* the generalized timelike Mannheim mate curve of C, as the following diagram:

$$\begin{array}{rccc} f: & I & \longrightarrow & I^* \\ & \Big\downarrow \mathbf{c} & & \Big\downarrow \overset{*}{\mathbf{c}} \\ \phi: & E_1^4 & \longrightarrow & E_1^4 \end{array} \tag{3.8}$$

A smooth function f is defined by $f(t) = \int \|d\mathbf{c}^*(t)/dt\| dt = t^*$ and t^* is the arc-length parameter of C^*. Also, ϕ is a bijection defined by $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$. Thus, the timelike curve C^* is reparametrized as follows

$$\mathbf{c}^*(f(t)) = \mathbf{c}(t) + \beta(t)\mathbf{N}(t), \tag{3.9}$$

where $\beta : I \subset \mathbb{R} \to \mathbb{R}$ is a smooth function. By differentiating both sides of (3.9) with respect to t, we have

$$f'(t)\mathbf{T}^*(f(t)) = \left(1 + \beta(t)k_1(t)\right)\mathbf{T} + \beta'(t)\mathbf{N}(t) + \beta(t)k_2(t)\mathbf{B}_1(t). \tag{3.10}$$

On the other hand, since the first normal line at each point of C is lying in the plane generated by the second normal line and the third normal line of C^* at the corresponding points under bijection ϕ, the vector field $\mathbf{N}(t)$ is given by

$$\mathbf{N}(t) = g(t)\mathbf{B}_1{}^*(f(t)) + h(t)\mathbf{B}_2{}^*(f(t)), \tag{3.11}$$

where g and h are some smooth functions on $I \subset \mathbb{R}$. If we take into consideration

$$\left\langle \mathbf{T}^*(f(t)), g(t)\mathbf{B}_1{}^*(f(t)) + h(t)\mathbf{B}_2{}^*(f(t)) \right\rangle = 0 \tag{3.12}$$

and (3.10), then we have $\beta'(t) = 0$. So we rewrite (3.10) as

$$f'(t)\mathbf{T}^*(f(t)) = \left(1 + \beta k_1(t)\right)\mathbf{T}(t) + \beta k_2(t)\mathbf{B}_1(t), \tag{3.13}$$

that is,

$$\mathbf{T}^*(f(t)) = \frac{(1+\beta k_1(t))}{f'(t)}\mathbf{T}(t) + \frac{\beta k_2(t)}{f'(t)}\mathbf{B}_1(t), \tag{3.14}$$

where

$$f'(t) = \sqrt{\left|-(1+\beta k_1(t))^2 + (\beta k_2(t))^2\right|}. \tag{3.15}$$

By taking the differentiations both sides of (3.13) with respect to $t \in I$, we get

$$\begin{aligned} f'(t)k_1^*(f(t))\mathbf{N}^*(f(t)) &= \left(\frac{1+\beta k_1(t)}{f'(t)}\right)'\mathbf{T}(t) \\ &+ \left(\frac{(1+\beta k_1(t))k_1(t) - \beta(k_2(t))^2}{f'(t)}\right)\mathbf{N}(t) \\ &+ \left(\frac{\beta k_2(t)}{f'(t)}\right)'\mathbf{B}_1(t) + \left(\frac{\beta k_2(t)k_3(t)}{f'(t)}\right)\mathbf{B}_2(t). \end{aligned} \tag{3.16}$$

Since

$$\langle \mathbf{N}^*(f(t)), g(t)\mathbf{B}_1{}^*(f(t)) + h(t)\mathbf{B}_2{}^*(f(t))\rangle = 0, \tag{3.17}$$

the coefficient of $\mathbf{N}(t)$ in (3.16) vanishes, that is,

$$(1+\beta k_1(t))k_1(t) - \beta(k_2(t))^2 = 0. \tag{3.18}$$

Thus, this completes the proof. □

Theorem 3.3. *In E_1^4, let C be a special timelike Frenet curve such that its nonconstant first and second curvature functions satisfy the equality $k_1(s) = -\beta(k_1^2(t) - k_2^2(t))$ for all $t \in I \subset \mathbb{R}$. If the timelike curve C^* given by*

$$\mathbf{c}^*(t) = \mathbf{c}(t) + \beta\mathbf{N}(t) \tag{3.19}$$

is a special timelike Frenet curve, then C^* is a generalized timelike Mannheim mate curve of C.

Proof. The arc-length parameter of C^* is given by

$$t^* = \int_0^t \left\|\frac{d\mathbf{c}^*(t)}{dt}\right\| dt, \quad t \in I. \tag{3.20}$$

Under the assumption of

$$k_1(t) = -\beta\left(k_1^2(t) - k_2^2(t)\right), \tag{3.21}$$

we obtain $f'(t) = \sqrt{|1+\beta k_1(t)|}$, $t \in I$.

Differentiating the equation $\mathbf{c}^*(f(t)) = \mathbf{c}(t) + \beta \mathbf{N}(t)$ with respect to t, we reach

$$f'(t)\mathbf{T}^*(f(t)) = (1+\beta k_1(t))\mathbf{T}(t) + \beta k_2(t)\mathbf{B}_1(t). \tag{3.22}$$

Thus, it is seen that

$$\mathbf{T}^*(f(t)) = \left(\frac{1+\beta k_1(t)}{\sqrt{|1+\beta k_1(t)|}}\mathbf{T}(t) + \frac{\beta k_2(t)}{\sqrt{|1+\beta k_1(t)|}}\mathbf{B}_1(t)\right), \quad t \in I. \tag{3.23}$$

The differentiation of the last equation with respect to t is

$$\begin{aligned} f'(t)k_1^*(f(t))\mathbf{N}^*(f(t)) &= \left(\sqrt{|1+\beta k_1(t)|}\right)'\mathbf{T}(t) \\ &\quad + \left(\frac{(1+\beta k_1(t))k_1(t) - \beta k_2^2(t)}{\sqrt{|1+\beta k_1(t)|}}\right)\mathbf{N}(t) \\ &\quad + \left(\frac{\beta k_2(t)}{\sqrt{|1+\beta k_1(t)|}}\right)'\mathbf{B}_1(t) + \left(\frac{\beta k_2(t)k_3(t)}{\sqrt{|1+\beta k_1(t)|}}\right)\mathbf{B}_2(t). \end{aligned} \tag{3.24}$$

From our assumption, we have

$$\frac{k_1(t) + \beta k_1^2(t) - \beta k_2^2(t)}{\sqrt{|1+\beta k_1(t)|}} = 0. \tag{3.25}$$

Thus, the coefficient of $\mathbf{N}(t)$ in (3.24) is zero. It is seen from (3.23) that $\mathbf{T}^*(f(t))$ is a linear combination of $\mathbf{T}(t)$ and $\mathbf{B}_1(t)$. Additionally, from (3.24), $\mathbf{N}^*(f(t))$ is given by linear combination of $\mathbf{T}(t)$, $\mathbf{B}_1(t)$, and $\mathbf{B}_2(t)$. On the other hand, C^* is a special timelike Frenet curve that the vector $\mathbf{N}(t)$ is given by linear combination of $\mathbf{B}_1^*(f(t))$ and $\mathbf{B}_2^*(f(t))$.

Therefore, the first normal line of C lies in the plane generated by the second normal line and third normal line of C^* at the corresponding points under a bijection ϕ defined by $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$.

This completes the proof. □

Remark 3.4. In 4-dimensional Minkowski space E_1^4, a special timelike Frenet curve C with curvature functions k_1 and k_2 satisfying $k_1(t) = -\beta(k_1^2(t) - k_2^2(t))$, it is not clear that a smooth timelike curve C^* given by (3.1) is a special Frenet curve. Thus, it is unknown whether the reverse of Theorem 3.2 is true or not.

Theorem 3.5. *Let C be a special timelike curve in E_1^4 with nonzero third curvature function k_3. There exists a timelike special Frenet curve C^* in E_1^4 such that the first normal line of C is linearly dependent with the third normal line of C^* at the corresponding points $\mathbf{c}(t)$ and $\mathbf{c}^*(t)$, respectively, under a bijection $\phi : C \to C^*$, if and only if the curvatures k_1 and k_2 of C are constant functions.*

Proof. Let C be a timelike Frenet curve in E_1^4 with the Frenet frame field $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ and curvature functions k_1, k_2, and k_3. Also, we assume that C^* is a timelike special Frenet curve in E_1^4 with the Frenet frame field $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}_1{}^*, \mathbf{B}_2{}^*\}$ and curvature functions k_1^*, k_2^*, and k_3^*. Let the first normal line of C be linearly dependent with the third normal line of C^* at the corresponding points C and C^*, respectively. Then the parameterization of C^* is

$$\mathbf{c}^*(f(t)) = \mathbf{c}(t) + \beta(t)\mathbf{N}(t), \quad t \in I. \tag{3.26}$$

If the arc-length parameter of C^* is given t^*, then

$$t^* = \int_0^t \sqrt{\left|-(1 + \beta(t)k_1(t))^2 + (\beta'(t))^2 + (\beta(t)k_2(t))^2\right|}dt, \tag{3.27}$$

$$\begin{aligned} f : I &\longrightarrow I^*, \\ t &\longrightarrow f(t) = t^*. \end{aligned} \tag{3.28}$$

Moreover, $\phi : C \to C^*$ is a bijection given by $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$.

Differentiating (3.26) with respect to t and using the Frenet formulas, we get

$$f'(t)\mathbf{T}^*(f(t)) = (1 + \beta(t)k_1(t))\mathbf{T}(t) + \beta'(t)\mathbf{N}(t) + \beta(t)k_2(t)\mathbf{B}_1(t). \tag{3.29}$$

Since $\mathbf{B}_2{}^*(f(t)) = \mp\mathbf{N}(t)$, then

$$\langle f'(t)\mathbf{T}^*(f(t)), \mathbf{B}_2{}^*(f(t))\rangle = \langle (1 + \beta(t)k_1(t))\mathbf{T}(t) + \beta'(t)\mathbf{N}(t) + \beta(t)k_2(t)\mathbf{B}_1(t), \mp\mathbf{N}(t)\rangle \tag{3.30}$$

that is,

$$0 = \mp\beta'(t). \tag{3.31}$$

From the last equation, it is easily seen that β is a constant number. Hereafter, we can denote $\beta(t) = \beta$, for all $t \in I$.

From (3.27), we have

$$f'(t) = \sqrt{\left|-\left(1+\beta k_1(t)\right)^2 + \left(\beta k_2(t)\right)^2\right|} > 0. \tag{3.32}$$

Thus, we rewrite (3.29) as follows:

$$\mathbf{T}^*\left(f(t)\right) = \left(\frac{1+\beta k_1(t)}{f'(t)}\right)\mathbf{T}(t) + \left(\frac{\beta k_2(t)}{f'(t)}\right)\mathbf{B}_1(t). \tag{3.33}$$

The differentiation of the last equation with respect to t is

$$\begin{aligned} f'(t)k_1^*\left(f(t)\right)\mathbf{N}^*\left(f(t)\right) = {} & \left(\frac{1+\beta k_1(t)}{f'(t)}\right)'\mathbf{T}(t) \\ & + \left(\frac{\left(1+\beta k_1(t)\right)k_1(t) - \beta k_2^2(t)}{f'(t)}\right)\mathbf{N}(t) \\ & + \left(\frac{\beta k_2(t)}{f'(t)}\right)'\mathbf{B}_1(t) + \left(\frac{\beta k_2(t)k_3(t)}{f'(t)}\right)\mathbf{B}_2(t). \end{aligned} \tag{3.34}$$

Since $\langle f'(t)k_1^*(f(t))\mathbf{N}^*(f(t)), \mathbf{B}_2{}^*(f(t))\rangle = 0$ and $\mathbf{B}_2{}^*(f(t)) = \mp\mathbf{N}(t)$ for all $t \in I$,

$$k_1(t) + \beta k_1^2(t) - \beta k_2^2(t) = 0 \tag{3.35}$$

is satisfied. Then

$$\beta = -\frac{k_1(t)}{k_1^2(t) - k_2^2(t)} \tag{3.36}$$

is a nonzero constant number. Thus, from (3.34), we reach

$$\begin{aligned} \mathbf{N}^*\left(f(t)\right) = {} & \frac{1}{f'(t)K(t)}\left(\frac{1+\beta k_1(t)}{f'(t)}\right)'\mathbf{T}(t) + \frac{1}{f'(t)K(t)}\left(\frac{\beta k_2(t)}{f'(t)}\right)'\mathbf{B}_1(t) \\ & + \frac{1}{f'(t)K(t)}\left(\frac{\beta k_2(t)k_3(t)}{f'(t)}\right)\mathbf{B}_2(t), \end{aligned} \tag{3.37}$$

where $K(t) = k_1^*(f(t))$ for all $t \in I$. Differentiating the last equation with respect to t, then we have

$$\begin{aligned}
&f'(t)\left[k_1^*(f(t))\mathbf{T}^*(f(t)) + k_2^*(f(t))\mathbf{B}_1^*(f(t))\right] \\
&\quad = \left(\frac{1}{f'(t)K(t)}\left(\frac{1+\beta k_1(t)}{f'(t)}\right)'\right)'\mathbf{T}(t) \\
&\quad + \left(\frac{k_1(t)}{f'(t)K(t)}\left(\frac{1+\beta k_1(t)}{f'(t)}\right)' - \frac{k_2(t)}{f'(t)K(t)}\left(\frac{\beta k_2(t)}{f'(t)}\right)'\right)\mathbf{N}(t) \\
&\quad + \left(\left(\frac{1}{f'(t)K(t)}\left(\frac{\beta k_2(t)}{f'(t)}\right)'\right)' - \frac{k_3(t)}{f'(t)K(t)}\left(\frac{\beta k_2(t)k_3(t)}{f'(t)}\right)\right)\mathbf{B}_1(t) \\
&\quad + \left(\left(\frac{1}{f'(t)K(t)}\left(\frac{\beta k_2(t)k_3(t)}{f'(t)}\right)\right)' + \frac{k_3(t)}{f'(t)K(t)}\left(\frac{\beta k_2(t)}{f'(t)}\right)'\right)\mathbf{B}_2(t)
\end{aligned} \tag{3.38}$$

for all $t \in I$. Considering

$$\begin{gathered}
\left\langle f'(t)\left(k_1^*(f(t))\mathbf{T}^*(f(t)) + k_2^*(f(t))\mathbf{B}_1^*(f(t))\right), \mathbf{B}_2^*(f(t))\right\rangle = 0, \\
\mathbf{B}_2^*(f(t)) = \mp\mathbf{N}(t),
\end{gathered} \tag{3.39}$$

then we get

$$k_1(t)\left(\frac{1+\beta k_1(t)}{f'(t)}\right)' - k_2(t)\left(\frac{\beta k_2(t)}{f'(t)}\right)' = 0. \tag{3.40}$$

Arranging the last equation, we find

$$\beta\left[k_1(t)k_1'(t) - k_2(t)k_2'(t)\right]f'(t) - \left[k_1(t) + \beta k_1^2(t) - \beta k_2^2(t)\right]f''(t) = 0. \tag{3.41}$$

Moreover, the differentiation of (3.36) with respect to t is

$$k_1'(t) + 2\beta\left(k_1(t)k_1'(t) - k_2(t)k_2'(t)\right) = 0. \tag{3.42}$$

From the above equation, it is seen that

$$-\frac{k_1'(t)}{2} = \beta\left(k_1(t)k_1'(t) - k_2(t)k_2'(t)\right). \tag{3.43}$$

Substituting (3.36) and (3.43) into (3.41), we obtain

$$-\frac{k_1'(t)}{2} = 0. \tag{3.44}$$

This means that the first curvature function is constant (that is, positive constant). Additionally, from (3.43) it is seen that the second curvature function k_2 is positive constant, too.

Conversely, suppose that C is a timelike Frenet curve in E_1^4 with the Frenet frame field $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ and curvature functions k_1, k_2, and k_3. The first curvature function k_1 and the second curvature function k_2 of C are of positive constant. Thus, $k_1/(k_2^2 - k_1^2)$ is a positive constant number, say β.

The representation of timelike curve C^* with arc-length parameter t is

$$\begin{aligned} \mathbf{c}^* : \ I &\longrightarrow E_1^4, \\ t &\longrightarrow \mathbf{c}^*(t) = \mathbf{c}(t) + \beta(t)\mathbf{N}(t). \end{aligned} \tag{3.45}$$

Let t^* denote the arc-length parameter of C^*; we have

$$\begin{aligned} f : \ I &\longrightarrow I^*, \\ t &\longrightarrow t^* = f(t) = \sqrt{|1 + \beta k_1|}\,t. \end{aligned} \tag{3.46}$$

Then, we obtain $f'(t) = \sqrt{|1 + \beta k_1|}$ and

$$\begin{aligned} f'(t)\mathbf{T}^*(f(t)) &= \mathbf{T}(t) + \beta\mathbf{N}'(t) \\ &= (1 + \beta k_1)\mathbf{T}(t) + \beta k_2 \mathbf{B}_1(t), \end{aligned} \tag{3.47}$$

that is,

$$\mathbf{T}^*(f(t)) = \sqrt{|1 + \beta k_1|}\,\mathbf{T}(t) + \frac{\beta k_2}{\sqrt{|1 + \beta k_1|}}\mathbf{B}_1(t). \tag{3.48}$$

By differentiating the both sides of the above equality with respect to t, we find

$$\begin{aligned} f'(t)\frac{d\mathbf{T}^*(t^*)}{dt^*}\bigg|_{t^*=f(t)} &= \sqrt{|1 + \beta k_1|}\,\mathbf{T}'(t) + \frac{\beta k_2}{\sqrt{|1 + \beta k_1|}}\mathbf{B}_1'(t) \\ &= \left[\frac{k_1(1 + \beta k_1) - \beta k_2^2}{\sqrt{|1 + \beta k_1|}}\right]\mathbf{N}(t) + \left[\frac{\beta k_2 k_3(t)}{\sqrt{|1 + \beta k_1|}}\right]\mathbf{B}_2(t) \\ &= \left[\frac{\beta k_2 k_3(t)}{\sqrt{|1 + \beta k_1|}}\right]\mathbf{B}_2(t). \end{aligned} \tag{3.49}$$

Hence, since k_3 does not vanish, we get

$$k_1^*(f(t)) = \left\| \left. \frac{d\mathbf{T}^*(t^*)}{dt^*} \right|_{t^*=f(t)} \right\| = \varepsilon \frac{\beta k_2 k_3(t)}{|1+\beta k_1|} > 0, \tag{3.50}$$

where $\varepsilon = \operatorname{sign}(k_3)$ denotes the sign of function k_3. That is, ε is -1 or $+1$.

We can put

$$\mathbf{N}^*(t^*) = \frac{1}{k_1^*(t^*)} \frac{d\mathbf{T}^*(t^*)}{dt^*}, \quad t \in I. \tag{3.51}$$

Then, we get

$$\mathbf{N}^*(f(t)) = \varepsilon \mathbf{B}_2(t). \tag{3.52}$$

Differentiating of the last equation with respect to t, we reach

$$f'(t) \left. \frac{d\mathbf{N}^*(t^*)}{dt^*} \right|_{t^*=f(t)} = -\varepsilon k_3 \mathbf{B}_1(t), \tag{3.53}$$

and we have

$$\left. \frac{d\mathbf{N}^*(t^*)}{dt^*} \right|_{t^*=f(t)} - k_1^*(f(t))\mathbf{T}^*(f(t)) = -\varepsilon \frac{\beta k_2 k_3(t)}{\sqrt{|1+\beta k_1|}} \mathbf{T}(t) - \varepsilon \sqrt{|1+\beta k_1|}\, k_3(t) \mathbf{B}_1(t). \tag{3.54}$$

Since $\varepsilon k_3(t)$ is positive for $t \in I$, we have

$$\begin{aligned} k_2^*(f(t)) &= \left\| \left. \frac{d\mathbf{N}^*(t^*)}{dt^*} \right|_{t^*=f(t)} - k_1^*(f(t))\mathbf{T}^*(f(t)) \right\| \\ &= \sqrt{\left| -\frac{\beta^2 k_2^2 (k_3(t))^2}{|1+\beta k_1|} + (|1+\beta k_1|)(k_3(t))^2 \right|} \\ &= \sqrt{(k_3(t))^2} = \varepsilon k_3(t) > 0. \end{aligned} \tag{3.55}$$

Thus, we can put

$$\begin{aligned} \mathbf{B}_1{}^*(f(t)) &= \frac{1}{k_2^*(f(t))} \left(\left. \frac{d\mathbf{N}^*(t^*)}{dt^*} \right|_{t^*=f(t)} - k_1^*(f(t))\mathbf{T}^*(f(t)) \right) \\ &= -\frac{\beta k_2}{\sqrt{|1+\beta k_1|}} \mathbf{T}(t) - \sqrt{|1+\beta k_1|}\, \mathbf{B}_1(t), \quad t \in I. \end{aligned} \tag{3.56}$$

By differentiation of the above equation with respect to t, we get

$$f'(t)\frac{d\mathbf{B}_1{}^*(t^*)}{dt^*}\bigg|_{t^*=f(t)} = \frac{k_2}{\sqrt{|1+\beta k_1|}}\mathbf{N}(t) - k_3(t)\sqrt{|1+\beta k_1|}\mathbf{B}_2(t). \tag{3.57}$$

Since $f'(t) = \sqrt{|1+\beta k_1|}$ and $k_2^*(f(t))\mathbf{N}^*(f(t)) = k_3(t)\mathbf{B}_2(t)$, we have

$$\frac{d\mathbf{B}_1{}^*(t^*)}{dt^*}\bigg|_{t^*=f(t)} + k_2^*(f(t))\mathbf{N}^*(f(t)) = \frac{k_2}{|1+\beta k_1|}\mathbf{N}(t). \tag{3.58}$$

Thus, we obtain $\mathbf{B}_2{}^*(f(t)) = \delta\mathbf{N}(t)$ for $t \in I$, where $\delta = \mp 1$. We must determine whether δ is -1 or $+1$ under the condition that the frame field $\{\mathbf{T}^*(t), \mathbf{N}^*(t), \mathbf{B}_1^*(t), \mathbf{B}_2^*(t)\}$ is of positive orientation.

We have, by $\det[\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}_1(t), \mathbf{B}_2(t)] = 1$ for $t \in I$,

$$\begin{aligned}
&\det\left[\mathbf{T}^*(t), \mathbf{N}^*(t), \mathbf{B}_1^*(t), \mathbf{B}_2^*(t)\right] \\
&\quad = \det\begin{bmatrix} \sqrt{|1+\beta k_1|}\mathbf{T}(t) + \dfrac{\beta k_2}{\sqrt{|1+\beta k_1|}}\mathbf{B}_1(t), \\ \varepsilon\mathbf{B}_2(t), -\dfrac{\beta k_2}{\sqrt{|1+\beta k_1|}}\mathbf{T}(t) - \sqrt{|1+\beta k_1|}\mathbf{B}_1(t), \delta\mathbf{N}(t) \end{bmatrix} \\
&\quad = \varepsilon\delta\left((|1+\beta k_1|) - \frac{\beta^2 k_2^2}{|1+\beta k_1|} \right) = \varepsilon\delta
\end{aligned} \tag{3.59}$$

and $\det[\mathbf{T}^*(t), \mathbf{N}^*(t), \mathbf{B}_1^*(t), \mathbf{B}_2^*(t)] = 1$ for any $t \in I$. Therefore, we get $\varepsilon = \delta$. Thus, we get

$$\begin{aligned}
\mathbf{B}_2^*(f(t)) &= \varepsilon\mathbf{N}(t), \\
k_3^*(f(t)) &= \left\langle \frac{d\mathbf{B}_1^*(t^*)}{dt^*}\bigg|_{t^*=f(t)}, \mathbf{B}_2^*(f(t)) \right\rangle \\
&= \varepsilon\frac{k_2}{|1+\beta k_1|}, \quad t \in I.
\end{aligned} \tag{3.60}$$

By the above facts, C^* is a special Frenet curve in E_1^4 and the first normal line at each point of C is the third normal line of C^* at corresponding each point under the bijection $\phi : \mathbf{c}(t) \to \phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t)) \in C^*$.

Thus, the proof is completed. □

The following theorem gives a parametric representation of a generalized timelike Mannheim curves E_1^4.

Theorem 3.6. *Let C be a timelike special curve defined by*

$$\mathbf{c}(s) = \begin{bmatrix} \beta \int f(s)\cosh s\, ds \\ \beta \int f(s)\sinh s\, ds \\ \beta \int f(s) g(s) ds \\ \beta \int f(s) h(s) ds \end{bmatrix}, \quad s \in U \subset \mathbb{R}. \tag{3.61}$$

Here, β is a nonzero constant number, $g : U \to \mathbb{R}$ and $h : U \to \mathbb{R}$ are any smooth functions, and the positive-valued smooth function $f : U \to \mathbb{R}$ is given by

$$\begin{aligned} f = {} & \left(1 - g^2(s) - h^2(s)\right)^{-3/2} \left(1 - g^2(s) - h^2(s) + \dot{g}^2(s) + \dot{h}^2(s) - \left(\dot{g}(s)h(s) - g(s)\dot{h}(s)\right)^2\right)^{-5/2} \\ & \times \Bigg[-\left(1 - g^2(s) - h^2(s) + \dot{g}^2(s) + \dot{h}^2(s) - \left(\dot{g}(s)h(s) - g(s)\dot{h}(s)\right)^2\right)^3 \\ & + \left(1 - g^2(s) - h^2(s)\right)^3 \begin{pmatrix} -\left(g(s) - \ddot{g}(s)\right)^2 - \left(h(s) - \ddot{h}(s)\right)^2 \\ -\left(\left(g(s)\dot{h}(s) - \dot{g}(s)h(s)\right) - \left(\dot{g}(s)\ddot{h}(s) - \ddot{g}(s)\dot{h}(s)\right)\right)^2 \\ +\left(g(s)\ddot{h}(s) - \ddot{g}(s)h(s)\right)^2 \end{pmatrix} \Bigg], \end{aligned} \tag{3.62}$$

for $s \in U$. Then, the curvature functions k_1 and k_2 of C satisfy

$$k_1 = -\beta\left(k_1^2 - k_2^2\right) \tag{3.63}$$

at each point $\mathbf{c}(s)$ of C.

Proof. Let C be a timelike special curve defined by

$$\mathbf{c}(s) = \begin{bmatrix} \beta \int f(s)\cosh s\, ds \\ \beta \int f(s)\sinh s\, ds \\ \beta \int f(s) g(s)\, ds \\ \beta \int f(s) h(s)\, ds \end{bmatrix}, \quad s \in U \subset \mathbb{R}, \tag{3.64}$$

where β is a nonzero constant number g and h are any smooth functions. f is a positive-valued smooth function. Thus, we obtain

$$\dot{\mathbf{c}}(s) = \begin{bmatrix} \beta f(s)\cosh s \\ \beta f(s)\sinh s \\ \beta f(s) g(s) \\ \beta f(s) h(s) \end{bmatrix}, \quad s \in U \subset \mathbb{R}, \tag{3.65}$$

where the subscript prime $(\cdot)$ denotes the differentiation with respect to s.

The arc-length parameter t of C is given by

$$t = \psi(s) = \int_{s_0}^{s} \|\dot{\mathbf{c}}(s)\| ds, \tag{3.66}$$

where $\|\dot{\mathbf{c}}(s)\| = \beta f(s)\sqrt{-1 + g^2(s) + h^2(s)}$.

If φ denotes the inverse function of $\psi : U \to I \subset \mathbb{R}$, then $s = \varphi(t)$ and we get

$$\varphi'(t) = \left\| \left. \frac{d\mathbf{c}(s)}{ds} \right|_{s=\varphi(t)} \right\|^{-1}, \quad t \in I, \tag{3.67}$$

where the prime $(')$ denotes the differentiation with respect to t.

The unit tangent vector $\mathbf{T}(t)$ of the curve C at the each point $\mathbf{c}(\varphi(t))$ is given by

$$\mathbf{T}(t) = \left(-1 + g^2(\varphi(t)) + h^2(\varphi(t))\right)^{-1/2} \begin{bmatrix} \cosh(\varphi(t)) \\ \sinh(\varphi(t)) \\ g(\varphi(t)) \\ h(\varphi(t)) \end{bmatrix}, \quad t \in I. \tag{3.68}$$

Some simplifying assumptions are made for the sake of brevity as follows:

$$\begin{gathered} \sinh := \sinh(\varphi(t)), \qquad \cosh := \cosh(\varphi(t)), \\ f := f(\varphi(t)), \qquad g := g(\varphi(t)), \qquad h := h(\varphi(t)), \\ \dot{g} := \dot{g}(\varphi(t)) = \left.\frac{dg(s)}{ds}\right|_{s=\varphi(t)}, \qquad \dot{h} := \dot{h}(\varphi(t)) = \left.\frac{dh(s)}{ds}\right|_{s=\varphi(t)}, \\ \ddot{g} := \ddot{g}(\varphi(t)) = \left.\frac{d^2 g(s)}{ds^2}\right|_{s=\varphi(t)}, \qquad \ddot{h} := \ddot{h}(\varphi(t)) = \left.\frac{d^2 h(s)}{ds^2}\right|_{s=\varphi(t)}, \\ \varphi' := \varphi'(t) = \left.\frac{d\varphi}{dt}\right|_{t}, \\ A := 1 - g^2 - h^2, \qquad B := -g\dot{g} - h\dot{h}, \qquad C := -\dot{g}^2 - \dot{h}^2, \\ D := -g\ddot{g} - h\ddot{h}, \qquad E := -\dot{g}\ddot{g} - \dot{h}\ddot{h}, \qquad F := \ddot{g}^2 + \ddot{h}^2. \end{gathered} \tag{3.69}$$

Thus, we get

$$\dot{A} = 2B, \qquad \dot{B} = C + D, \qquad \dot{C} = 2E, \qquad \varphi' = \beta^{-1} f^{-1} A^{-1/2}. \tag{3.70}$$

So, we rewrite (3.68) as

$$\mathbf{T} := \mathbf{T}(t) = A^{-1/2} \begin{bmatrix} \cosh \\ \sinh \\ g \\ h \end{bmatrix}. \tag{3.71}$$

Differentiating the last equation with respect to t, we find

$$\mathbf{T}' = \varphi' \begin{bmatrix} -\frac{1}{2} A^{-3/2} \dot{A} \cosh + A^{-1/2} \sinh \\ -\frac{1}{2} A^{-3/2} \dot{A} \sinh + A^{-1/2} \cosh \\ -\frac{1}{2} A^{-3/2} \dot{A} g + A^{-1/2} \dot{g} \\ -\frac{1}{2} A^{-3/2} \dot{A} h + A^{-1/2} \dot{h} \end{bmatrix}, \tag{3.72}$$

that is,

$$\mathbf{T}' = -\varphi' A^{-1/2} \begin{bmatrix} A^{-1} B \cosh - \sinh \\ A^{-1} B \sinh - \cosh \\ A^{-1} B g - \dot{g} \\ A^{-1} B h - \dot{h} \end{bmatrix}. \tag{3.73}$$

From the last equation, we find

$$k_1 := k_1(t) = \left\| \mathbf{T}'(t) \right\| = \varphi' A^{-1} \left(A - AC + B^2 \right)^{1/2}. \tag{3.74}$$

By the fact that $\mathbf{N}(t) = (k_1(t))^{-1} \mathbf{T}'(t)$, we get

$$\mathbf{N} := \mathbf{N}(t) = -A^{1/2} \left(A - AC + B^2 \right)^{-1/2} \begin{bmatrix} A^{-1} B \cosh - \sinh \\ A^{-1} B \sinh - \cosh \\ A^{-1} B g - \dot{g} \\ A^{-1} B h - \dot{h} \end{bmatrix}. \tag{3.75}$$

In order to get second curvature function k_2, we need to calculate $k_2(t) = \|\mathbf{N}'(t) - k_1(t)\mathbf{T}(t)\|$. After a long process of calculations and using abbreviations, we obtain

$$\mathbf{N}' - k_1\mathbf{T} = \varphi' A^{-3/2}\left(A - AC + B^2\right)^{-3/2} \begin{bmatrix} (P+Q)\cosh - R\sinh \\ (P+Q)\sinh - R\cosh \\ Pg - R\dot{g} + Q\ddot{g} \\ Ph - R\dot{h} + Q\ddot{h} \end{bmatrix}, \tag{3.76}$$

where

$$\begin{gathered} P = \left(A - AC + B^2\right)\left(B^2 - AC - AD\right) - \left(A - AC + B^2\right)^2 + AB(B - AE + BD), \\ Q = A^2\left(A - AC + B^2\right), \\ R = A^2(B - AE + BD). \end{gathered} \tag{3.77}$$

If we simplify P, then we have

$$P = A^2(C - BE - D + CD - 1). \tag{3.78}$$

Therefore, we rewrite (3.76) and (3.77) as

$$\mathbf{N}' - k_1\mathbf{T} = \varphi' A^{-1/2}\left(A - AC + B^2\right)^{-3/2} \begin{bmatrix} \left(\widetilde{P}+\widetilde{Q}\right)\cosh - \widetilde{R}\sinh \\ \left(\widetilde{P}+\widetilde{Q}\right)\sinh - \widetilde{R}\cosh \\ \widetilde{P}g - \widetilde{R}\dot{g} + \widetilde{Q}\ddot{g} \\ \widetilde{P}h - \widetilde{R}\dot{h} + \widetilde{Q}\ddot{h} \end{bmatrix}, \tag{3.79}$$

where

$$\begin{gathered} \widetilde{P} = C - D + CD - BE - 1, \\ \widetilde{Q} = A - AC + B^2, \\ \widetilde{R} = B - AE + BD. \end{gathered} \tag{3.80}$$

Consequently, from (3.79) and (3.80), we have

$$\left\|\mathbf{N}' - k_1\mathbf{T}\right\|^2 = \left(\varphi'\right)^2 A\left(A - AC + B^2\right)^{-3} \begin{bmatrix} -\left(\widetilde{P}+\widetilde{Q}\right)^2 + \widetilde{R}^2 + \widetilde{P}^2\left(g^2+h^2\right) + \widetilde{R}^2\left(\dot{g}^2+\dot{h}^2\right) \\ +\widetilde{Q}^2\left(\ddot{g}^2+\ddot{h}^2\right) - 2\widetilde{P}\widetilde{R}\left(g\dot{g}+h\dot{h}\right) \\ -2\widetilde{R}\widetilde{Q}\left(\dot{g}\ddot{g}+\dot{h}\ddot{h}\right) + 2\widetilde{P}\widetilde{Q}\left(g\ddot{g}+h\ddot{h}\right) \end{bmatrix}. \tag{3.81}$$

Substituting the abbreviations into the last equation, we have

$$\|\mathbf{N}' - k_1\mathbf{T}\|^2 = (\varphi')^2 A\left(A - AC + B^2\right)^{-3} \times \left[-\widetilde{P}^2 A - 2\widetilde{P}\widetilde{Q} - \widetilde{Q}^2 + \widetilde{R}^2 - \widetilde{R}^2 C + \widetilde{Q}^2 F + 2\widetilde{P}\widetilde{R}B + 2\widetilde{R}\widetilde{Q}E - 2\widetilde{P}\widetilde{Q}D\right]. \tag{3.82}$$

After substituting (3.80) into the last equation and simplifying it, we get

$$\begin{aligned} k_2^2 &= \|\mathbf{N}' - k_1\mathbf{T}\|^2 \\ &= (\varphi')^2 A\left(A - AC + B^2\right)^{-2}\left[\left(A - AC + B^2\right)(1 - F) + (C-1)(1+D)^2 - 2BE(1+D) + AE^2\right]. \end{aligned} \tag{3.83}$$

Moreover, from (3.74) it is seen that

$$k_1^2 = (\varphi')^2 A^{-2}\left(A - AC + B^2\right). \tag{3.84}$$

The last two equations show us that

$$\begin{aligned} k_2^2 - k_1^2 = (\varphi')^2 A^{-2}\left(A - AC + B^2\right)^{-2} &\times \left[-\left(A - AC + B^2\right)^3\right. \\ &\left. + A^3\left(\left(A - AC + B^2\right)(1 - F) + (C-1)(1+D)^2 - 2BE(1+D) + AE^2\right)\right]. \end{aligned} \tag{3.85}$$

By the fact $\varphi' = \beta^{-1} f^{-1} A^{-1/2}$, we obtain

$$\begin{aligned} k_2^2 - k_1^2 &= \beta^{-2} f^{-2} A^{-3}\left(A - AC + B^2\right)^{-2} \\ &\times\left[\left(A - AC + B^2\right)^3\right. \\ &\left. + A^3\left(\left(A - AC + B^2\right)(1 - F) + (C-1)(1+D)^2 - 2BE(1+D) + AE^2\right)\right], \end{aligned} \tag{3.86}$$

$$k_1 = \beta^{-1} f^{-1} A^{-3/2}\left(A - AC + B^2\right)^{1/2}. \tag{3.87}$$

According to our assumption

$$f = \left(1 - g^2 - h^2\right)^{-3/2}\left(1 - g^2 - h^2 + \dot{g}^2 + \dot{h}^2 - (\dot{g}h - g\dot{h})^2\right)^{-5/2}$$
$$\times \left[-\left(1 - g^2 - h^2 + \dot{g}^2 + \dot{h}^2 - (\dot{g}h - g\dot{h})^2\right)^3\right.$$
$$\left.+\left(1 - g^2 - h^2\right)^3\left(-(g - \ddot{g})^2 - (h - \ddot{h})^2 - ((g\dot{h} - \dot{g}h) - (\dot{g}\ddot{h} - \ddot{g}\dot{h}))^2 + (g\ddot{h} - \ddot{g}h)^2\right)\right], \tag{3.88}$$

we obtain

$$f = A^{-3/2}\left(A - AC + B^2\right)^{-5/2}\begin{bmatrix} \left(A - AC + B^2\right)^3 \\ +A^3\begin{pmatrix} (A - AC + B^2)(1 - F) + (C - 1)(1 + D)^2 \\ -2BE(1 + D) + AE^2 \end{pmatrix}\end{bmatrix}. \tag{3.89}$$

Substituting the above equation into (3.86) and (3.87), we obtain

$$k_1 = -\beta\left(k_1^2 - k_2^2\right). \tag{3.90}$$

The proof is completed. □

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