A new partial regularity result for non-autonomous convex integrals with non standard growth conditions

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ABSTRACT. We establish $C^{1,\gamma}$-partial regularity of minimizers of non autonomous convex integral functionals of the type: $\mathcal{F}(u;\Omega) := \int_\Omega f(x,Du)\ dx$, with non standard growth conditions into the gradient variable

$$\frac{1}{L}|\xi|^p \le f(x,\xi) \le L(1+|\xi|^q)$$

for a couple of exponents p,q such that

$$1 < p \le q < \min\left\{p\frac{n}{n-1}, p+1\right\},$$

and α- Hölder continuous dependence with respect to the x variable. The significant point here is that the distance between the exponents p and q is independent of α. Moreover this bound on the gap between the growth and the coercitivity exponents improves previous results in this setting.

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1 Introduction

In the last few years there has been an increasing interest in variational integrals exhibiting a gap between the growth and the coercitivity exponents of the form

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x,Du)\ dx \tag{1.1}$$

with

$$\frac{1}{L}|\xi|^p \le f(x,\xi) \le L(1+|\xi|^q) \qquad \text{for some} \qquad L \ge 1, \tag{F1}$$

where $1 < p \leq q < +\infty$, $u : \Omega \to \mathbb{R}^N$ and Ω is a bounded open set in $\mathbb{R}^n$.

Here we shall assume that there exist constants $C, \nu > 0$ and an exponent $\alpha \in (0,1)$ such that $f(x,\xi)$ is a $C^2(\Omega, \mathbb{R}^{n\times N})$ function fulfilling (F1) and whose derivatives satisfy the following assumptions:

$$|D_\xi f(x_1,\xi) - D_\xi f(x_2,\xi)| \leq C|x_1 - x_2|^\alpha \ (1 + |\xi|^{q-1}); \tag{F2}$$

$$\nu(1 + |\xi|^2)^{\frac{p-2}{2}} |\zeta|^2 \leq \langle D_{\xi\xi} f(x,\xi)\zeta, \zeta\rangle; \tag{F3}$$

for any $\xi \in \mathbb{R}^{nN}$ and for any $x, x_1, x_2 \in \Omega$.

By assumption (F1), we are dealing with functionals satisfying the so-called non standard growth conditions. Moreover it is well known that condition (F3), which is a strict uniform ellipticity condition on $D^2 f$, is equivalent to the strict uniform convexity of f. As in our previous paper [11], no control on the growth of the second derivatives of f from above will be assumed.

The theory of functionals with non standard growth conditions started with a series of well known papers by Marcellini ([26, 27, 28]) and after has been developed in many different aspects. The main topics treated in this setting are related to the lower semicontinuity, the relaxation and the regularity of the minimizers of such functionals (see for example [3, 5, 6, 16, 17, 18, 24, 31] and the references in [29] for a complete list).
From the very beginning it has been clear that, even in the scalar case, no regularity can be expected if the exponents p and q are too far apart.

In fact, Marcellini himself produced an example of functional with non standard growth conditions having unbounded minimizers (see [21] and [26]).

On the other hand if the ratio

$$\frac{q}{p} \leq c(n) \to 1 \tag{1.2}$$

as $n \to +\infty$, many regularity results are available both in the scalar and in the vectorial setting. The starting issue in the analysis of the regularity is just to improve the integrability of the gradient of a minimizer from L^p to L^q. In this direction we quote for example [13, 14, 20]. We stress that this kind of regularity has revealed to be crucial when one try to argue approximating the integrand with a sequence of functions having standard growth conditions. In fact, the useful apriori estimates depend on the L^q norm of the gradient of minimizer because of the right hand side of (F2) (for a self contained treatment we refer to [4] and the references therein).

On the other hand, $C^{1,\gamma}$ partial regularity results have been established by means of a linearization argument that avoids the approximation procedure based on suitable apriori estimates. The first result in this direction has been obtained in [3], under special structure assumptions on the integrand f and afterwards in [30], without any structure assumption on the integrand.

It is worth pointing out that all the quoted results concern autonomous functionals, i.e. $f \equiv f(Du)$.

The study of the regularity in the non autonomous case $f \equiv f(x, Du)$, started with the paper [15] by Esposito, Leonetti and Mingione. The result of [15] states that if f is convex with respect to the gradient variable, it satisfies assumption (F1) and (F2) with p, q such that

$$1 < p \leq q < p\frac{n+\alpha}{n} \tag{1.3}$$

and if there is no Lavrentiev Phenomenon for the functional, then a $W^{1,p}$ local minimizer of $\mathcal{F}$ actually belongs to $W^{1,q}$.

Note that the combination of the facts that f both depends on x and exhibits a gap could determine the occurrence of the Lavrentiev Phenomenon, that translates into the impossibility of approximate in energy a $W^{1,p}$ function with $W^{1,q}$ functions.

In this paper shall prove $C^{1,\gamma}$ partial regularity of minimizers of $\mathcal{F}$ with the following gap between growth and coercivity exponent:

$$1 < p \leq q < \min\left\{p\frac{n}{n-1}, p+1\right\}. \tag{1.4}$$

This is somehow surprising, since the condition (1.4) is independent of the exponent α, which is produced by the α-Hölder continuity dependence of Df with respect to the x variable. Moreover the new range in (1.4) is wider than the one given by (1.3).

In our previous paper [11], we proved a $C^{1,\gamma}$ partial regularity result for minimizers under the same set of assumptions (F1), (F2) and (F3), and provided that no Lavrentiev Phenomenon occured. But in that paper we were forced to assume that

$$2 \leq p \leq q < p\frac{n+\alpha}{n}, \tag{1.5}$$

that is condition (1.3) with $p \geq 2$, because we first established an higher integrability property of the minimizers following [15], and afterwards we performed a blow-up procedure. Moreover, we also confined ourselves to the case $p \geq 2$, because the usual finite difference quotient method used to prove higher integrability, led us to heavy technical difficulties in the case $1 < p < 2$. Indeed, even if the result of Esposito, Leonetti and Mingione [15] is proved for every $p > 1$, in [11] we needed an higher integrability result which had to be uniform with respect the rescaling procedure necessary for the blow-up method. However, in [11] we sensibly improved the outcome of Bildhauer and Fuchs' work [6], where Df was assumed to be Lipschitz continuous with respect the x variable and D^2f had controlled growth from above. We also would like to stress that even in the case $\alpha = 1$, which is the situation considered by Bildhauer and Fuchs in [6], our new range (1.4) is still better than (1.3) .

In the current context, we present a completely new proof which allows us to improve the quoted results on partial regularity and directly treat the case $p > 1$. The higher integrability step, which entailed the bound (1.3), is replaced by the proof of a Caccioppoli type inequality for the minimizers of a suitable perturbation of the rescaled functionals. The Caccioppoli type estimate will present some extra terms that won't effect the blow-up procedure. The main difficulty in studying the regularity properties of minimizers of integrals with non-standard growth is that the usual test functions, whose gradient is essentially proportional to the gradient of the minimizers, don't have the right degree of integrability. A gluing Lemma due to Fonseca and Maly ([17]), used to connect in an annulus two $W^{1,p}$ functions with a $W^{1,q}$ function, will play a key role to overcome this difficulty and partly provide the bound (1.4). In fact the gluing Lemma holds if

$$q < p\frac{n}{n-1}.$$

To be more precise we could allow $q \leq p+1$ if

$$p+1 < p\frac{n}{n-1},$$

that is when $p > n-1$. This restriction on q is explained in the following remark, taken from [30].

Remark 1.1. [The Euler-Lagrange system for $q \leq p+1$.] If u is a local minimizer of the functional $\mathcal{F}$ and $\phi \in C_c^1(\Omega, \mathbb{R}^N)$ we get by the minimality condition that for any $\varepsilon > 0$:

$$0 \leq \int_\Omega [F(Du + \varepsilon D\phi) - F(Du)]\,\mathrm{d}x = \varepsilon \int_\Omega \int_0^1 \frac{\partial F}{\partial \xi_\alpha^i}(Du + \varepsilon t D\phi) D_\alpha \phi^i \,\mathrm{d}t\,\mathrm{d}x\,,$$

where the usual summation convention is in force. Dividing this inequality by ε, and letting $\varepsilon \searrow 0$, we infer from the growth assumptions and since $q \leq p+1$, that

$$\int_\Omega \frac{\partial F}{\partial \xi_\alpha^i}(Du) D_\alpha \phi^i \,\mathrm{d}x \geq 0.$$

Consequently, u is a weak solution to the Euler-Lagrange system for $\mathcal{I}$:

$$\int_\Omega \frac{\partial F}{\partial \xi_\alpha^i}(Du) D_\alpha \phi^i \,\mathrm{d}x = 0 \qquad \forall \phi \in C_c^1(\Omega, \mathbb{R}^N).$$

After having established the Caccioppoli type estimate, the blow-up argument, aimed to establish a decay estimate for the excess function of a minimizer, can be started up. The excess function, roughly speaking, measures how the gradient of the minimizer is far from being constant on small balls.

Moreover, by skipping the higher integrability step, it is not necessary to assume the non occurrence of the Lavrentiev Phenomenon (see [15]).
We also point out that regularity for minimizers of non autonomous functionals with standard growth conditions is usually achieved via the Ekeland principle after a comparison between the minimizer of the original functional and the minimizer of a suitable "frozen" one (see [2, 19]).
However, owing to the anisotropic growth of the functional, it seems that the comparison method cannot work in our context.

The main result of this paper is the following.

Theorem 1.2. *Let f be a $C^2(\Omega, \mathbb{R}^{n\times N})$ integrand satisfying the assumptions (F1), (F2) and (F3) with growth exponents p, q such that*

$$1 < p \leq q < \min\left\{p\frac{n}{n-1}, p+1\right\}. \tag{1.6}$$

If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$, then there exists an open subset Ω_0 of Ω such that

$$meas(\Omega \setminus \Omega_0)$$

and

$$u \in C^{1,\gamma}_{loc}(\Omega_0, \mathbb{R}^N) \qquad \text{for every} \qquad \gamma < \frac{\alpha}{2},$$

where α is the exponent appearing in (F2).

Since our regularity result is only partial, we are not in contradiction with the counterexample of [15], which shows that (1.3) is unavoidable to boost the integrability of the $W^{1,p}$-minimizers up to $W^{1,q}$.

Partial regularity results are a common feature when treating vectorial minimizers, because everywhere regularity cannot be proved in this case (see the counterexample due to De Giorgi and those due to Sverak and Yan [9, 32, 33]). Hence, the next issue is trying to estimate the Hausdorff dimension of the singular set. In the case of functionals with standard growth conditions, these estimates have been established in [25] (see also [10]). But in our setting, this kind of result cannot be achieved. In fact, an example constructed in [18] shows that if p and q are far enough, depending on the dimension n and the regularity of $x \mapsto f(x, Du)$, then the set of non-Lebesgue points of a minimizer can be nearly as bad as that of any other $W^{1,p}$ function.

2 Preliminaries

In this section we recall some standard definitions and collect several Lemmas that we shall need to establish our main result.

We begin with the definition of local minimizer for a functional with nonstandard growth conditions.

Definition 2.1. A function $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if $f(x, Du(x)) \in L^1_{loc}(\Omega)$ and

$$\int_{supp\,\varphi} f(x, Du)\,dx \leq \int_{supp\,\varphi} f(x, Du + D\varphi)\,dx,$$

for any $\varphi \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ with $supp\,\varphi \subset \Omega$.

In order to deal with the case $1 < p \leq 2$, we shall use the following auxiliary function defined for $\xi \in \mathbb{R}^k$

$$V_\beta(\xi) = (1 + |\xi|^2)^{\frac{\beta-2}{4}}\xi,$$

for any exponent $\beta > 1$. Recall that

$$|V_\beta(\xi)| \text{ is a non-decreasing function of } |\xi|; \tag{2.1}$$

$$|V_\beta(\xi+\eta)| \le c(\beta)(|V_\beta(\xi)| + |V_\beta(\eta)|); \tag{2.2}$$

$$\min\{t^2, t^\beta\}|V_\beta(\xi)|^2 \le |V_\beta(t\xi)|^2 \le \max\{t^2, t^\beta\}|V_\beta(\xi)|^2; \tag{2.3}$$

$$(1+|\xi|^2+|\eta|^2)^{\frac{\beta}{2}} \le 1 + (1+|\xi|^2+|\eta|^2)^{\frac{\beta-2}{2}}(|\xi|^2+|\eta|^2) \qquad \text{if} \quad \beta \le 2; \tag{2.4}$$

$$c(\beta)(|\xi|^2+|\xi|^\beta) \le |V_\beta(\xi)|^2 \le C(\beta)(|\xi|^2+|\xi|^\beta) \qquad \text{if} \quad \beta \ge 2; \tag{2.5}$$

$$|V_\beta(\xi)|^2 \qquad \text{is convex if} \qquad 1 < \beta < 2. \tag{2.6}$$

Many of the previous properties of the function V_β can be easily checked and they have been successfully employed in the study of the regularity of minimizers of convex and quasiconvex integrals under subquadratic growth conditions ([1, 7, 8, 31]).

In the linearization procedure we shall use the translated functional of $\mathcal{F}$ on the unit ball $B \equiv B_1(0)$

$$\mathcal{I}(v) := \int_B g(y, Dv)\, dy$$

defined by setting

$$g(y,\xi) = f(x_0 + r_0 y, A+\xi) - f(x_0 + r_0 y, A) - D_\xi f(x_0 + r_0 y, A)\xi, \tag{2.7}$$

where A is a matrix such that $|A|$ is uniformly bounded by a positive constant M. Next Lemma, whose proof is given in [11], contains the growth conditions on g.

Lemma 2.2. *Let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ be a function satisfying the assumptions (F1), (F2) and (F3) and let $g(y,\xi)$ be the function defined by (2.7). Then we have*

$$c_1|V_p(\xi)|^2 \le g(y,\xi) \le c_2|V_q(\xi)|^2; \tag{I1}$$

$$|D_\xi g(y,\xi)| \le c(1+|\xi|^2)^{\frac{q-2}{2}}|\xi|; \tag{I2}$$

$$|D_\xi g(y_1,\xi) - D_\xi g(y_2,\xi)| \le c r_0^\alpha \; |y_1-y_2|^\alpha \; (1+|\xi|^{q-1}); \tag{I3}$$

$$c(1+|\xi|^2)^{\frac{p-2}{2}}|\zeta|^2 \le \langle D_{\xi\xi} g(y,\xi)\zeta, \zeta\rangle \tag{I4}$$

where the constant c, c_1 and c_2 depend on M, p and q.

The following result is standard if $p \ge 2$ and can be inferred from [1] (Lemma 2.2) in the case $1 < p < 2$.

Lemma 2.3. *For $\beta > 1$ and $\eta, \xi \in \mathbb{R}^{N\times n}$ there holds*

$$C_1(1+|\eta|^2+|\xi|^2)^{\frac{\beta-2}{2}} \le \int_0^1 (1+|\eta+t\xi|^2)^{\frac{\beta-2}{2}}\, dt \le C_2(1+|\eta|^2+|\xi|^2)^{\frac{\beta-2}{2}}$$

with some positive constants C_1, C_2 depending only on β.

Next Lemma can be found in a slightly different form in [17] (Lemma 2.2), see also [30] and [31], and it will be crucial in our proofs. In fact it will allow us to construct admissible test functions needed to establish the Caccioppoli inequality.

Lemma 2.4. *Let $0 < r < s < 1$ and let $v \in W^{1,p}(B_1(0);\mathbb{R}^N)$. If $1 < p \le q < \frac{pn}{n-1}$ there exist a function $w \in W^{1,p}(B_1(0);\mathbb{R}^N)$ and two radii $0 < r < r' < s' < s < 1$ depending on v such that*

$$w = \begin{cases} v & in \quad B_{r'} \\ \\ v & in \quad B_1 \setminus B_{s'} \end{cases} \tag{2.8}$$

$$\frac{s-r}{3} \le s' - r' \le s - r$$

and

$$\int_{B_s \setminus B_r} |w|^p \, dx \le c(n,p) \int_{B_s \setminus B_r} |v|^p \, dx; \tag{2.9}$$

$$\int_{B_s \setminus B_r} |Dw|^p \, dx \le c(n,p) \int_{B_s \setminus B_r} |Dv|^p \, dx. \tag{2.10}$$

Moreover if $p \ge 2$ we have

$$\int_{B_{s'} \setminus B_{r'}} |w|^q \, dx \le c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |v|^p \, dx \right)^{\frac{q}{p}}; \tag{2.11}$$

$$\int_{B_{s'} \setminus B_{r'}} |Dw|^q \, dx \le c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |Dv|^p \, dx \right)^{\frac{q}{p}}. \tag{2.12}$$

While, in case $1 < p < 2$, we have that

$$\int_{B_s \setminus B_r} |V_p(w)|^2 \, dx \le c(n,p) \int_{B_s \setminus B_r} |V_p(v)|^2 \, dx. \tag{2.13}$$

$$\int_{B_s \setminus B_r} |V_p(Dw)|^2 \, dx \le c(n,p) \int_{B_s \setminus B_r} |V_p(Dv)|^2 \, dx. \tag{2.14}$$

$$\int_{B_{s'} \setminus B_{r'}} |V_p(w)|^{\frac{2q}{p}} \, dx \le c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |V_p(v)|^2 \, dx \right)^{\frac{q}{p}}; \tag{2.15}$$

$$\int_{B_{s'} \setminus B_{r'}} |V_p(Dw)|^{\frac{2q}{p}} \, dx \le c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |V_p(Dv)|^2 \, dx \right)^{\frac{q}{p}}. \tag{2.16}$$

Next Lemma finds an important application in the so called hole-filling method. Its proof can be found in [23] (See Lemma 6.1).

Lemma 2.5. *Let $h : [\rho, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \theta < 1$, $0 \le A$, $0 \le B$ and $0 < \beta$. Assume that*

$$h(r) \le \frac{A}{(d-r)^\beta} + B + \theta h(d)$$

for $\rho \le r < d \le R_0$. Then

$$h(\rho) \le \frac{cA}{(R_0-\rho)^\beta} + B,$$

where $c = c(\theta, \beta) > 0$.

In order to deal with the case $1 < p < 2$, we shall need the following Poincaré-Sobolev inequality, whose proof can be found in [12] (for other versions of this inequality we refer to [7, 8]).

Lemma 2.6. *Assume $1 < p < 2$ and let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Then there exists a positive constant $c \equiv c(n, N, p)$ such that*

$$\left(\fint_{B_\rho(x_0)} \left| V_p \left(\frac{u-(u)_\rho}{\rho} \right) \right|^{\frac{2n}{n-p}} dx \right)^{\frac{n-p}{2n}} \le c \left(\fint_{B_\rho(x_0)} |V(Du)|^2 \, dx \right)^{\frac{1}{2}}.$$

Next result is a simple consequence of the a priori estimates for solutions to linear elliptic systems with constant coefficients.

Proposition 2.7. *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, $p \ge 1$ be such that*

$$\int_\Omega A^{ij}_{\alpha\beta} D_\alpha u^i D_\beta \varphi^j \, dx = 0$$

for every $\varphi \in C^\infty_0(\Omega; \mathbb{R}^N)$, where $A^{ij}_{\alpha\beta}$ is a constant matrix satisfying the strong Legendre Hadamard condition

$$A^{ij}_{\alpha\beta} \lambda^i \lambda^j \mu_\alpha \mu_\beta \ge \nu |\lambda|^2 |\mu|^2 \qquad \forall \lambda \in \mathbb{R}^N, \, \mu \in \mathbb{R}^n.$$

Then $u \in C^\infty$ and for any ball $B_R(x_0) \Subset \Omega$ we have

$$\sup_{B_{\frac{R}{2}}(x_0)} |Du| \le \frac{c}{R^n} \int_{B_R} |Du| \, dx$$

For the proof see [22], [23] in case $p \ge 2$ and see [7], [8] in case $1 \le p < 2$.

3 A Caccioppoli type inequality

In order to perform the blow up procedure, it will be convenient to introduce suitable translations of minimizers of the functional $\mathcal{F}$. More precisely, if u is a local minimizer of $\mathcal{F}$ we shall consider the function

$$v(y) = \frac{u(x_0 + r_0 y) - r_0 A y - (u)_{B_1(0)}}{r_0}.$$

The minimality of u implies that

$$\int_{B_1(0)} f(x_0+r_0y, Du(x_0+r_0y))\,dy \le \int_{B_1(0)} f(x_0+r_0y, Du(x_0+r_0y)+D\varphi(x_0+r_0y))\,dy$$

that is

$$\int_{B_1(0)} f(x_0+r_0y, Dv(y)+A)\,dy \le \int_{B_1(0)} f(x_0+r_0y, Dv(y)+A+D\varphi(x_0+r_0y))\,dy$$

and hence

$$\int_{B_1(0)} g(y,Dv)\,dy \le \int_{B_1(0)} g(y,Dv+D\varphi)\,dy + cr_0^\alpha \int_{B_1(0)} |D\varphi|\,dy, \tag{3.1}$$

for every $\varphi \in W^{1,1}(B_1(0);\mathbb{R}^N)$ with compact support, where g is the function defined at (2.7).

Therefore, the first step in the proof of Theorem 1.2 is to obtain a Caccioppoli type inequality for every function $v \in W^{1,p}(B_1(0);\mathbb{R}^N)$ which satisfies the minimality inequality (3.1).

Proposition 3.1. *Let us suppose that $g(y,\xi) \in C^2(B_1(0);\mathbb{R}^{nN})$ satisfies the assumptions $(I1)$, $(I2)$, $(I3)$ with*

$$1 < p \le q < p\left(\frac{n}{n-1}\right) \tag{3.2}$$

and set $t = \min\{2,p\}$. If the function $v \in W^{1,p}(B_1(0);\mathbb{R}^N)$ satisfies the inequality (3.1) then, for every $\rho < 1$, we have

$$\begin{aligned}\fint_{B_{\frac{\rho}{2}}} |V_p(Dv)|^2\,dy \le{}& c\fint_{B_\rho} \left|V_p\left(\frac{v}{\rho}\right)\right|^2\,dy + c\left(\fint_{B_\rho} |V_p(Dv)|^2 + \left|V_p\left(\frac{v}{\rho}\right)\right|^2\,dy\right)^{\frac{q}{p}} \\ &+ cr_0^\alpha\left(\fint_{B_\rho} |Dv|^t\,dy\right)^{\frac{1}{t}} + cr_0^\alpha\left(\fint_{B_\rho} \frac{|v|^t}{\rho^t}\,dy\right)^{\frac{1}{t}},\end{aligned} \tag{3.3}$$

for a positive constant c independent of the parameter r_0 and of the point x_0 appearing in the definition of $g(y,\xi)$.

Proof. Let us fix two radii $\frac{\rho}{2} < r < s < \rho$. Lemma 2.4 implies that there exist $\psi \in W^{1,p}(B_1(0))$ and $r < r' < s' < s$ such that

$$\psi = v \quad \text{on} \quad B_{r'} \qquad\qquad \psi = v \quad \text{on} \quad B_1 \setminus B_{s'},$$

$$\frac{s-r}{3} \le s'-r' \le s-r. \tag{3.4}$$

Thanks to the assumption (3.2), the function ψ satisfies the estimates (2.9)–(2.12) in case $p \ge 2$ and (2.13)–(2.16) in case $1 < p < 2$.

Fix now a cut-off function $\eta \in C_0^\infty(B_{s'})$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{r'}$ and $|D\eta| \le \frac{c}{s'-r'}$ and set

$$\varphi = (1-\eta)\psi \qquad\qquad \tilde{\varphi} = \eta\psi.$$

By the left hand inequality in assumption $(I1)$, we get

$$
\begin{aligned}
&\int_{B_{r'}} (1+|Dv|^2)^{\frac{p-2}{2}}|Dv|^2\,dy \le c\int_{B_{s'}} g(y,D\tilde{\varphi})\,dy \\
=&\int_{B_{s'}\setminus B_{r'}} [g(y,D\tilde{\varphi})-g(y,Dv)]\,dy + \int_{B_{s'}} [g(y,Dv)-g(y,D\varphi)]\,dy + \int_{B_{s'}\setminus B_{r'}} [g(y,D\varphi)]\,dy \\
=&I+II+III, \qquad (3.5)
\end{aligned}
$$

where we used that in $B_{r'}$ one has $\tilde{\varphi}=v$ and $\varphi=0$. By the minimality inequality (3.1) for v we have that

$$
II \le cr_0^{\alpha}\left(\int_{B_{s'}} |Dv-D\varphi|\,dy\right), \qquad (3.6)
$$

since $v-\varphi\in W_0^{1,p}(B_{s'})$ Moreover, since $g(y,\xi)\ge 0$ for all $y\in B_1$ and all $\xi\in\mathbb{R}^{n\times N}$, we have that

$$
I \le \int_{B_{s'}\setminus B_{r'}} [g(y,D\tilde{\varphi})]\,dy. \qquad (3.7)
$$

Hence inserting (3.6) and (3.7) in (3.5) we get

$$
\begin{aligned}
&\int_{B_{r'}} (1+|Dv|^2)^{\frac{p-2}{2}}|Dv|^2\,dy \\
\le\ & c\int_{B_{s'}\setminus B_{r'}} [g(y,D\tilde{\varphi})]\,dy + \int_{B_{s'}\setminus B_{r'}} [g(y,D\varphi)]\,dy + cr_0^{\alpha}\left(\int_{B_{s'}} |Dv-D\varphi|\,dy\right) \\
=\ & J+JJ+JJJ. \qquad (3.8)
\end{aligned}
$$

Now we treat the cases $1<p\le 2$ and $p>2$ separately.

- *The case* $1<p\le 2$.

In order to estimate J, we use the right inequality in assumption $(I1)$ thus getting

$$
\begin{aligned}
J &\le c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{q-2}{2}}|D\tilde{\varphi}|^2\,dy = c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}+\frac{q-p}{2}}|D\tilde{\varphi}|^2\,dy \\
&= c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}(1+|D\tilde{\varphi}|^2)^{\frac{p}{2}\frac{q-p}{p}}|D\tilde{\varphi}|^2\,dy \\
&\le c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}|D\tilde{\varphi}|^2\Big[1+|D\tilde{\varphi}|^2(1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}\Big]^{\frac{q-p}{p}}\,dy. \qquad (3.9)
\end{aligned}
$$

where we used (2.4) in the last line. Hence

$$
\begin{aligned}
J &\le c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}|D\tilde{\varphi}|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} \Big(|D\tilde{\varphi}|^2(1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}\Big)^{\frac{q}{p}}\,dy \\
&\le c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^{\frac{2q}{p}}\,dy. \qquad (3.10)
\end{aligned}
$$

Arguing exactly in the same way we have

$$JJ \leq c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^{\frac{2q}{p}}\,dy. \tag{3.11}$$

From (3.10) and (3.11), using the properties of the function V_p and the definition of $\tilde{\varphi}$ and φ we obtain

$$\begin{aligned}
J + JJ \;&\leq\; c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^{\frac{2q}{p}}\,dy \\
&+\; c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^{\frac{2q}{p}}\,dy \\
&=\; c\int_{B_{s'}\setminus B_{r'}} |V_p(D(1-\eta)\psi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D(1-\eta)\psi)|^{\frac{2q}{p}}\,dy \\
&+\; c\int_{B_{s'}\setminus B_{r'}} |V_p(D(\eta\psi))|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D(\eta\psi))|^{\frac{2q}{p}}\,dy \\
&\leq\; c\int_{B_{s'}\setminus B_{r'}} |V_p(D\psi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} \left|V_p\left(\frac{\psi}{s'-r'}\right)\right|^2\,dy \\
&+\; c\int_{B_{s'}\setminus B_{r'}} |V_p(D\psi)|^{\frac{2q}{p}}\,dy + c\int_{B_{s'}\setminus B_{r'}} \left|V_p\left(\frac{\psi}{s'-r'}\right)\right|^{\frac{2q}{p}}\,dy,
\end{aligned} \tag{3.12}$$

where we also used the properties of η. Therefore, using (2.13)–(2.16) and (3.4), we get

$$\begin{aligned}
J + JJ \;&\leq\; c\int_{B_s\setminus B_r} |V_p(Dv)|^2\,dy + c\int_{B_s\setminus B_r} \left|V_p\left(\frac{v}{s-r}\right)\right|^2\,dy \\
&+\; c(s-r)^n\left(\frac{1}{(s-r)^n}\int_{B_s\setminus B_r} |V_p(Dv)|^2 + \left|V_p\left(\frac{v}{s-r}\right)\right|^2\,dy\right)^{\frac{q}{p}}.
\end{aligned} \tag{3.13}$$

Concerning JJJ, recalling that $\varphi = 0$ on $B_{r'}$, using Hölder's inequality and Lemma 2.4 we have

$$\begin{aligned}
JJJ =& cr_0^\alpha\left[\int_{B_{s'}} |Dv|\,dy + \int_{B_{s'}\setminus B_{r'}} |D\psi|\,dy\right] \\
\leq& cr_0^\alpha\left[\int_{B_\rho} |Dv|\,dy + \int_{B_{s'}\setminus B_{r'}} |D\psi|\,dy + \int_{B_{s'}\setminus B_{r'}} \frac{|\psi|}{(s'-r')}\,dy\right] \\
\leq& cr_0^\alpha \rho^{\frac{n}{p'}}\left[\int_{B_\rho} |Dv|^p\,dy\right]^{\frac{1}{p}} + cr_0^\alpha \rho^{\frac{n}{p'}}\left[\int_{B_{s'}\setminus B_{r'}} |D\psi|^p\,dy + \int_{B_{s'}\setminus B_{r'}} \frac{|\psi|^p}{(s'-r')^p}\,dy\right]^{\frac{1}{p}} \\
\leq& cr_0^\alpha \rho^{\frac{n}{p'}}\left[\int_{B_\rho} |Dv|^p\,dy\right]^{\frac{1}{p}} + cr_0^\alpha \rho^{\frac{n}{p'}}\left[\int_{B_\rho} \frac{|v|^p}{(s-r)^p}\,dy\right]^{\frac{1}{p}},
\end{aligned} \tag{3.14}$$

where p' is the Hölder conjugate of p and we used again (2.9), (2.10) and (3.4).

- *The case* $p \geq 2$.

In this case we use the right inequality in assumption $(I1)$, property (2.5) and the definition of φ and $\tilde{\varphi}$ as follows

$$\begin{aligned} J + JJ &\leq c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{q-2}{2}}|D\tilde{\varphi}|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} (1+|D\varphi|^2)^{\frac{q-2}{2}}|D\varphi|^2\,dy \\ &\leq c\int_{B_{s'}\setminus B_{r'}} |D\tilde{\varphi}|^2 + |D\tilde{\varphi}|^q\,dy + c\int_{B_{s'}\setminus B_{r'}} |D\varphi|^2 + |D\varphi|^q\,dy \\ &\leq c\int_{B_{s'}\setminus B_{r'}} |D\psi|^2 + |D\psi|^q\,dy + c\int_{B_{s'}\setminus B_{r'}} \left|\frac{\psi}{s'-r'}\right|^2 + \left|\frac{\psi}{s'-r'}\right|^q\,dy \qquad (3.15) \end{aligned}$$

Hence, by Lemma 2.4, we get

$$\begin{aligned} J + JJ \leq c\int_{B_s\setminus B_r} |Dv|^2 &+ c(s-r)^{n\left(1-\frac{q}{p}\right)}\left(\int_{B_s\setminus B_r} |Dv|^p\,dy\right)^{\frac{q}{p}} \\ + \;\; c\int_{B_s\setminus B_r}\left|\frac{v}{s-r}\right|^2 &+ c(s-r)^{n\left(1-\frac{q}{p}\right)}\left(\int_{B_s\setminus B_r}\left|\frac{v}{s-r}\right|^p dy\right)^{\frac{q}{p}} \\ \leq \;\; c\int_{B_s\setminus B_r} |V_p(Dv)|^2\,dy &+ c\int_{B_s\setminus B_r}\left|V_p\left(\frac{v}{s-r}\right)\right|^2 dy \\ + \;\; c(s-r)^n &\left(\frac{1}{(s-r)^n}\int_{B_s\setminus B_r} |V_p(Dv)|^2 + \left|V_p\left(\frac{v}{s-r}\right)\right|^2 dy\right)^{\frac{q}{p}}, \qquad (3.16) \end{aligned}$$

where we used again (3.4).

Now we argue exactly as in (3.14) and obtain that

$$\begin{aligned} JJJ =& cr_0^\alpha\left[\int_{B_{s'}} |Dv|\,dy + \int_{B_{s'}\setminus B_{r'}} |D\psi|\,dy\right] \\ \leq& cr_0^\alpha\left[\int_{B_\rho} |Dv|\,dy + \int_{B_{s'}\setminus B_{r'}} |D\psi|\,dy + \int_{B_{s'}\setminus B_{r'}} \frac{|\psi|}{(s'-r')}\,dy\right] \\ \leq& cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_\rho} |Dv|^2\,dy\right]^{\frac12} + cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_{s'}\setminus B_{r'}} |D\psi|^2\,dy + \int_{B_{s'}\setminus B_{r'}} \frac{|\psi|^2}{(s'-r')^2}\,dy\right]^{\frac12} \\ \leq& cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_\rho} |Dv|^2\,dy\right]^{\frac12} + cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_\rho} \frac{|v|^2}{(s-r)^2}\,dy\right]^{\frac12}. \qquad (3.17) \end{aligned}$$

Hence we can write a final estimate for JJJ as follows:

$$JJJ \leq cr_0^\alpha\rho^{\frac{n}{t'}}\left(\int_{B_\rho} |Dv|^t\,dy\right)^{\frac1t} + cr_0^\alpha\rho^{\frac{n}{t'}}\left(\int_{B_\rho} \frac{|v|^t}{\rho^t}\,dy\right)^{\frac1t}. \qquad (3.18)$$

where $t = \min\{2, p\}$ and t' is the Hölder conjugate of t.

Inserting (3.13) and (3.18) or (3.16) and (3.18) in (3.8) in case $1 < p \leq 2$ and $p \geq 2$ respectively, we obtain

$$\int_{B_r} |V_p(Dv)|^2\,dy \leq c\int_{B_s\setminus B_r} |V_p(Dv)|^2\,dy + c\int_{B_s\setminus B_r}\left|V_p\left(\frac{v}{s-r}\right)\right|^2 dy$$

$$
\begin{aligned}
&+\quad c(s-r)^n\left(\frac{1}{(s-r)^n}\int_{B_s\setminus B_r}|V_p(Dv)|^2+\left|V_p\left(\frac{v}{s-r}\right)\right|^2 dy\right)^{\frac{q}{p}}\\
&+\quad cr_0^{\alpha}\rho^{\frac{n}{t'}}\left(\int_{B_\rho}|Dv|^t\,dy\right)^{\frac{1}{t}}+cr_0^{\alpha}\rho^{\frac{n}{t'}}\left(\int_{B_\rho}\frac{|v|^t}{\rho^t}\,dy\right)^{\frac{1}{t}}, \qquad (3.19)
\end{aligned}
$$

where $t=\min\{2,p\}$.

Now, we fill the hole by adding the quantity

$$
c\int_{B_r}|V_p(Dv)|^2\,dy
$$

to both sides of (3.19) and use the iteration Lemma 2.5 to obtain that

$$
\begin{aligned}
\int_{B_{\frac{\rho}{2}}}|V_p(Dv)|^2\,dy \quad &\le\quad c\int_{B_\rho}\left|V_p\left(\frac{v}{\rho}\right)\right|^2\,dy+c\rho^n\left(\frac{1}{\rho^n}\int_{B_\rho}|V_p(Dv)|^2+\left|V_p\left(\frac{v}{\rho}\right)\right|^2\,dy\right)^{\frac{q}{p}}\\
&+\quad cr_0^{\alpha}\rho^{\frac{n}{t'}}\left(\int_{B_\rho}|Dv|^t\,dy\right)^{\frac{1}{t}}+cr_0^{\alpha}\rho^{\frac{n}{t'}}\left(\int_{B_\rho}\frac{|v|^t}{\rho^t}\,dy\right)^{\frac{1}{t}}. \qquad (3.20)
\end{aligned}
$$

The conclusion follows dividing both sides by ρ^n. □

4 Decay estimate

As usual the proof of Theorem 1.2 relies on a blow up argument aimed to establish a decay estimate for the excess function of the minimizer, which is defined as

$$
E(x,r)=\fint_{B_r(x)}|V_p(Du-(Du)_r)|^2+r^\beta \qquad (4.1)
$$

with $\beta<\alpha$. The blow up argument for a local minimizer $u\in W^{1,p}_{\mathrm{loc}}$ of $\mathcal{F}$ with an integrand function $f(x,\xi)\in C^2(\Omega,\mathbb{R}^{n\times N})$ fulfilling assumptions (F1), (F2) and (F3) for a couple of exponents satisfying (1.4), is contained in the following

Proposition 4.1. *Fix $M>0$. There exists a constant $C(M)>0$ such that, for every $0<\tau<\frac{1}{4}$, there exists $\varepsilon=\varepsilon(\tau,M)$ such that, if*

$$
|(Du)_{x_0,r}|\le M \qquad \text{and} \qquad E(x_0,r)\le\varepsilon,
$$

then

$$
E(x_0,\tau r)\le C(M)\,\tau^\beta\,E(x_0,r).
$$

Proof. Step 1. Blow up

Fix $M>0$. Assume by contradiction that there exists a sequence of balls $B_{r_j}(x_j)\subset\subset\Omega$ such that

$$
|(Du)_{x_j,r_j}|\le M \qquad \text{and} \qquad \lambda_j^2=E(x_j,r_j)\to 0 \qquad (4.2)
$$

but

$$\frac{E(x_j, \tau r_j)}{\lambda_j^2} > \tilde{C}(M)\tau^\beta \tag{4.3}$$

where $\tilde{C}(M)$ will be determined later. Setting $A_j = (Du)_{x_j,r_j}$, $a_j = (u)_{x_j,r_j}$ and

$$v_j(y) = \frac{u(x_j + r_j y) - a_j - r_j A_j y}{\lambda_j r_j} \tag{4.4}$$

for all $y \in B_1(0)$, one can easily check that $(Dv_j)_{0,1} = 0$ and $(v_j)_{0,1} = 0$. By the definition of λ_j at (4.2), we get

$$\fint_{B_1(0)} \frac{|V(\lambda_j Dv_j)|^2}{\lambda_j^2}\, dy + \frac{r_j^\beta}{\lambda_j^2} = 1, \tag{4.5}$$

and hence

$$\fint_{B_1(0)} |Dv_j|^p \, dy \le C \qquad\qquad 1 < p < 2 \tag{4.6}$$

$$\fint_{B_1(0)} |Dv_j|^2 + \lambda_j^{p-2}|Dv_j|^p \, dy \le C \qquad\qquad p \ge 2. \tag{4.7}$$

Therefore passing possibly to not relabeled sequences

$$\begin{aligned}
&v_j \rightharpoonup v && \text{weakly in } W^{1,p}(B_1(0);\mathbb{R}^N) && 1 < p < 2;\\
&v_j \rightharpoonup v && \text{weakly in } W^{1,2}(B_1(0);\mathbb{R}^N) && p \ge 2;\\
&A_j \longrightarrow A && && \\
&r_j \longrightarrow 0; && \frac{r_j^\gamma}{\lambda_h^2} \longrightarrow 0, && \forall\, \gamma > \beta.
\end{aligned} \tag{4.8}$$

Step 2. Minimality of v_j

We normalize f around A_j as follows

$$f_j(y,\xi) = \frac{f(x_j + r_j y, A_j + \lambda_j \xi) - f(x_j + r_j y, A_j) - D_\xi f(x_j + r_j y, A_j)\lambda_j \xi}{\lambda_j^2} \tag{4.9}$$

and we consider the corresponding rescaled functionals

$$\mathcal{I}_j(w) = \int_{B_1(0)} [f_j(y, Dw)] dy. \tag{4.10}$$

The minimality of u yields that

$$\int_{B_1(0)} f(x_j + r_j y, Du(x_j + r_j y))\, dy \le \int_{B_1(0)} f(x_j + r_j y, Du(x_j + r_j y) + D\varphi(x_j + r_j y))\, dy$$

for every $\varphi \in W_0^{1,1}(B_{r_j}(x_j);\mathbb{R}^N)$, that is

$$\int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y))\, dy \le \int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y) + D\varphi(x_j + r_j y))\, dy,$$

for every $\varphi \in W_0^{1,1}(B_{r_j}(x_j);\mathbb{R}^N)$. Thus, by the definition of the rescaled functionals, we have

$$\mathcal{I}_j(v_j) \le \mathcal{I}_j(v_j + \varphi) + \int_{B_1(0)} \frac{D_\xi f(x_j + r_j y, A_j) D\varphi}{\lambda_j}\, dy. \tag{4.11}$$

Using (F2) we conclude that

$$\begin{aligned} \mathcal{I}_j(v_j) &\le \mathcal{I}_j(v_j + \varphi) + \int_{B_1(0)} \frac{[D_\xi f(x_j + r_j y, A_j) - D_\xi f(x_j, A_j)] D\varphi}{\lambda_j}\, dy \\ &\le \mathcal{I}_j(v_j + \varphi) + c(M)\frac{r_j^\alpha}{\lambda_j} \int_{B_1(0)} |D\varphi|\, dy. \end{aligned} \tag{4.12}$$

Step 3. v solves a linear system

Since v_j satisfies inequality (4.12) we have that

$$0 \le \mathcal{I}_j(v_j + s\varphi) - \mathcal{I}_j(v_j) + c(M)\frac{r_j^\alpha}{\lambda_j} \int_{B_1(0)} |sD\varphi|\, dy, \tag{4.13}$$

for every $\varphi \in C_0^1(B)$ and for every $s \in (0,1)$. Now, by the definition of the rescaled functionals we get

$$\begin{aligned} \mathcal{I}_j(v_j + s\varphi) - \mathcal{I}_j(v_j) &= \int_{B_1(0)} \int_0^1 [D_\xi f_j(x_j + r_j y, A_j + \lambda_j(Dv_j + tsD\varphi))] sD\varphi\, dt\, dy \\ &= \frac{c}{\lambda_j} \int_{B_1(0)} [D_\xi f(x_j + r_j y, A_j + \lambda_j(Dv_j + sD\varphi)) - D_\xi f(x_j + r_j y, A_j)] sD\varphi\, dy. \end{aligned} \tag{4.14}$$

Inserting (4.14) in (4.13), dividing by s and taking the limit as $s \to 0$, we conclude that

$$\begin{aligned} 0 &\le \frac{c}{\lambda_j} \int_{B_1(0)} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi\, dy \\ &+ \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_1(0)} |D\varphi|\, dy. \end{aligned} \tag{4.15}$$

Let us split

$$B_1(0) = E_j^+ \cup E_j^- = \{y \in B_1 :\ \lambda_j |Dv_j| > 1\} \cup \{y \in B_1 :\ \lambda_j |Dv_j| \le 1\}.$$

By (4.6), in case $1 < p < 2$, we get

$$|E_j^+| \le \int_{E_j^+} \lambda_j^p |Dv_j|^p\, dy \le \lambda_j^p \int_{E_j^+} |Dv_j|^p\, dy \le c\lambda_j^p. \tag{4.16}$$

By assumption (F1) and the convexity of f we have that

$$|D_\xi f(x,\xi)| \le c(1+|\xi|^{q-1})$$

Since $q < p+1$, we can apply Hölder's inequality thus obtaining

$$\begin{aligned}
& \frac{1}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j+r_jy, A_j+\lambda_j Dv_j) - D_\xi f(x_j+r_jy, A_j)]D\varphi\,dy\right| \\
\le\ & \frac{c}{\lambda_j}|E_j^+| + c\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^{q-1}\,dy \\
\le\ & c\lambda_j^{p-1} + c\lambda_j^{q-2}\left(\int_{E_j^+}|Dv_j|^p\,dy\right)^{\frac{q-1}{p}}|E_j^+|^{\frac{p-q+1}{p}} \\
\le\ & c\lambda_j^{p-1}. \qquad (4.17)
\end{aligned}$$

In case $p \ge 2$, by (4.7) we get

$$|E_j^+| \le \int_{E_j^+}\lambda_j^2|Dv_j|^2\,dy \le \lambda_j^2\int_{E_j^+}|Dv_j|^2\,dy \le c\lambda_j^2. \qquad (4.18)$$

Arguing as before, we have

$$\begin{aligned}
& \frac{1}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j+r_jy, A_j+\lambda_j Dv_j) - D_\xi f(x_j+r_jy, A_j)]D\varphi\,dy\right| \\
\le\ & \frac{c}{\lambda_j}|E_j^+| + c\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^{q-1}\,dy \\
\le\ & c\lambda_j + c\lambda_j^{\frac{2q-p-2}{p}}\left(\int_{E_j^+}\lambda_j^{p-2}|Dv_j|^p\,dy\right)^{\frac{q-1}{p}}|E_j^+|^{\frac{p-q+1}{p}} \\
\le\ & c\lambda_j. \qquad (4.19)
\end{aligned}$$

Hence, for every $p > 1$, we infer that

$$\lim_{j\to\infty}\frac{c}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j+r_jy, A_j+\lambda_j Dv_j) - D_\xi f(x_j+r_jy, A_j)]D\varphi\,dy\right| = 0. \qquad (4.20)$$

On E_j^- we have

$$\begin{aligned}
& \frac{1}{\lambda_j}\int_{E_j^-}[D_\xi f(x_j+r_jy, A_j+\lambda_j Dv_j) - D_\xi f(x_j+r_jy, A_j)]D\varphi\,dy \\
=\ & \int_{E_j^-}\int_0^1 D_{\xi\xi}f(x_j+r_jy, A_j+t\lambda_j Dv_j)\,dt Dv_j D\varphi\,dy. \qquad (4.21)
\end{aligned}$$

Note that (4.16) yields that $\chi_{E_j^-} \to \chi_{B_1}$ in L^r, for every $r < \infty$. Moreover by (4.8) we have, at least for subsequences, that

$$\lambda_j Dv_j \to 0 \qquad \text{a.e. in } B_1$$

$$r_j \to 0$$

and

$$x_j \to x_0.$$

Hence the uniform continuity of $D_{\xi\xi}f$ on bounded sets implies

$$\begin{aligned} &\lim_j \frac{1}{\lambda_j} \int_{E_j^-} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \\ = &\int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy. \end{aligned} \tag{4.22}$$

Since $\beta < \alpha$, by (4.8) we deduce that

$$\lim_j \frac{r_j^\alpha}{\lambda_j} = 0. \tag{4.23}$$

By estimates (4.20), (4.22) and (4.23), passing to the limit as $j \to \infty$ in (4.15) yields

$$0 \leq \int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy$$

Changing φ in $-\varphi$ we finally get

$$\int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy = 0$$

i.e. v solves a linear system which is uniformly elliptic thanks to the uniform convexity of f. The regularity result stated in Proposition 2.7 implies that $v \in C^\infty(B_1)$ and for any $0 < \tau < 1$

$$⨍_{B_\tau} |Dv - (Dv)_\tau|^2 \, dy \leq c\tau^2 ⨍_{B_1} |Dv - (Dv)_1|^2 \, dy \leq c\tau^2, \tag{4.24}$$

for a constant c depending on M.

Step 4. Conclusion

Fix $\tau \in (0, \frac{1}{4})$, set $b_j = (v_j)_{B_{2\tau}}$, $B_j = (Dv_j)_{B_\tau}$ and define

$$w_j(y) = v_j(y) - b_j - B_j y.$$

After rescaling, we note that $\lambda_j w_j$ satisfies the following integral inequality

$$\int_{B_1(0)} g_j(y, \lambda_j Dw_j) \, dy \leq \int_{B_1(0)} g_j(y, \lambda_j Dw_j + D\varphi) \, dy + c r_j^\alpha \int_{B_1(0)} |D\varphi| \, dy,$$

for every $\varphi \in W_0^{1,1}(B_1(0))$ where

$$g_j(y, \xi) = f(x_j + r_j y, A_j + \lambda_j B_j + \xi) - f(x_j + r_j y, A_j + \lambda_j B_j) - D_\xi f(x_j + r_j y, A_j + \lambda_j B_j)\xi.$$

It is easy to check that Lemma 2.2 applies to each g_j, for some constants that could depend on τ through $|\lambda_j B_j|$. But, given τ, we may always choose j large enough to have $|\lambda_j B_j| <$

$\frac{\lambda_j}{\tau^{\frac{n}{t}}} < 1$, where $t = \min\{2, p\}$. Hence we can apply Proposition 3.1 to each $\lambda_j w_j$. In case $1 < p < 2$ we have that

$$
\begin{aligned}
\lim_j \frac{E(x_j, \tau r_j)}{\lambda_j^2} &= \lim_j \frac{1}{\lambda_j^2} \fint_{B_{\tau r_j}(x)} |V_p(Du - (Du)_{\tau r_j})|^2 \, dy + \lim_j \frac{\tau^\beta r_j^\beta}{\lambda_j^2} \\
&\leq \lim_j \frac{1}{\lambda_j^2} \fint_{B_\tau} |V_p(\lambda_j Dw_j)|^2 \, dy + \tau^\beta \\
&\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j w_j}{\tau}\right)\right|^2 \, dy \\
&+ c \lim_j \lambda_j^{\frac{2(q-p)}{p}} \left(\fint_{B_{2\tau}} \frac{|V_p(\lambda_j Dw_j)|^2}{\lambda_j^2} + \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j w_j}{\tau}\right)\right|^2 \, dy \right)^{\frac{q}{p}} \\
&+ c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\fint_{B_\tau} \lambda_j^p |Dw_j|^p \, dy \right)^{\frac{1}{p}} + c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\fint_{B_\tau} \lambda_j^p \frac{|w_j|^p}{\tau^p} \, dy \right)^{\frac{1}{p}} + \tau^\beta \\
&\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j w_j}{\tau}\right)\right|^2 \, dy + \tau^\beta
\end{aligned}
$$

since

$$
\lim_j \lambda_j^{\frac{2(q-p)}{p}} = 0, \qquad \lim_j \frac{r_j^\alpha}{\lambda_j^2} = 0
$$

and the integrals appearing as their factors are bounded as $j \to \infty$. Now, since $v_j \to v$ strongly in $L^p(B_1(0))$, using the Sobolev-Poincaré inequality stated in Lemma 2.6, one can easily check that

$$
\lim_{j \to +\infty} \int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j(v_j - v))|^2}{\lambda_j^2} \, dy = 0. \tag{4.25}
$$

In fact, for every $\vartheta \in (0, \frac{p}{2})$ we can use Hölder's inequality of exponents $\frac{p}{2\vartheta}$ and $\frac{p}{p-2\vartheta}$ as follows

$$
\begin{aligned}
&\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j(v_j - v))|^2}{\lambda_j^2} \, dy = \int_{B_{\frac{1}{2}}} |v_j - v|^2 (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p-2}{2}} \, dy \\
&\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)}{4}} dy \right)^{\frac{2\vartheta}{p}} \\
&\qquad \times \left(\int_{B_{\frac{1}{2}}} |v_j - v|^{\frac{2p(1-\vartheta)}{p-2\vartheta}} (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)(1-\vartheta)}{2(p-2\vartheta)}} dy \right)^{\frac{p-2\vartheta}{p}} \\
&\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p \, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \left(\frac{|V_p(\lambda_j(v_j - v))|^2}{\lambda_j^2} \right)^{\frac{p(1-\vartheta)}{p-2\vartheta}} dy \right)^{\frac{p-2\vartheta}{p}} \\
&\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p \, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j(Dv_j - Dv))|^2}{\lambda_j^2} \, dy \right)^{1-\vartheta}.
\end{aligned}
$$

Last inequality is obtained applying Lemma 2.6 to the second integral, choosing $\vartheta \in (0, \frac{p}{2})$ such that $\frac{p(1-\vartheta)}{p-2\vartheta} = \frac{n}{n-p}$. Hence (4.25) follows noticing that the first integral vanishes as j goes to infinity and second one stays bounded thanks to (4.5), since $v \in C_0^\infty(B_1(0))$.
Since $b_j \to (v)_{2\tau}$ and $B_j \to (Dv)_\tau$, using (4.25) and the definition of w_j we get

$$
\begin{aligned}
\lim_j \frac{E(x_j, \tau r_j)}{\lambda_j^2} &\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p \left(\frac{\lambda_j (w_j - v + v)}{\tau} \right) \right|^2 \, dy + \tau^\beta \\
&= c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p \left(\frac{\lambda_j (v_j - v + v - b_j - B_j y)}{\tau} \right) \right|^2 \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_\tau y|^2}{\tau^2} \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_{2\tau} y|^2}{\tau^2} \, dy + c \fint_{B_{2\tau}} \frac{|(Dv)_\tau y - (Dv)_{2\tau} y|^2}{\tau^2} \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \, dy + c |(Dv)_\tau - (Dv)_{2\tau}|^2 + \tau^\beta \\
&\leq c\tau^2 + c\tau^\beta \leq c_M^\star \tau^\beta .
\end{aligned}
$$

The contradiction follows, if $1 < p < 2$, by choosing $c_M^\star > \tilde{C}(M)$.

Now we face the case $p \geq 2$. Arguing as we did for the case $1 < p < 2$ and using property (2.5) we get

$$
\begin{aligned}
\lim_j \frac{E(x_j, \tau r_j)}{\lambda_j^2} &\leq c \lim_j \fint_{B_\tau} (|Dw_j|^2 + \lambda_j^{p-2} |Dw_j|^p) \, dy + \tau^\beta \\
&\leq c \lim_j \fint_{B_{2\tau}} \left(\frac{|w_j|^2}{\tau^2} + \lambda_j^{p-2} \frac{|w_j|^p}{\tau^p} \right) \, dy \\
&+ c \lim_j \lambda_j^{\frac{2(q-p)}{p}} \left(\fint_{B_{2\tau}} \left(|Dw_j|^2 + \lambda_j^{p-2} |Dw_j|^p \right) \, dy \right)^{\frac{q}{p}} \\
&+ c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\int_{B_{2\tau}} \lambda_j^2 |Dw_j|^2 \, dy \right)^{\frac{1}{2}} + c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\int_{B_{2\tau}} \lambda_j^2 \frac{|w_j|^2}{\tau^2} \, dy \right)^{\frac{1}{2}} + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_\tau y|^2}{\tau^2} \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_{2\tau} y|^2}{\tau^2} \, dy + c \fint_{B_{2\tau}} \frac{|(Dv)_\tau y - (Dv)_{2\tau} y|^2}{\tau^2} \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \, dy + c |(Dv)_\tau - (Dv)_{2\tau}|^2 + \tau^\beta \\
&\leq c\tau^2 + c\tau^\beta \leq c_M^\star \tau^\beta .
\end{aligned}
$$

The contradiction follows, if $p \geq 2$, by choosing $c_M^\star > \tilde{C}(M)$. □

5 Proof of Theorem 1.2

The proof of our regularity result follows from the decay estimate of Proposition 4.1 by a standard iteration argument. We sketch it here for the reader's convenience.

Proof of Theorem 1.2. Following the arguments used in Section 6 of [19], from Proposition 4.1 we deduce that for every $M > 0$ there exist $0 < \tau < \frac{1}{4}$ and $\eta > 0$ such that if

$$|(Du)_{x_0,R}| \leq M \qquad \text{and} \qquad E(x_0, R) < \eta \tag{5.1}$$

then

$$|(Du)_{x_0,\tau^k R}| \leq 2M \qquad \text{and} \qquad E(x_0, \tau^k R) < c(M)\tau^{\beta k} E(x_0, R) \tag{5.2}$$

for every $k \in \mathbb{N}$. Estimate (5.2) yields that if (5.1) holds for any $\rho \in (0, R)$ we have

$$|(Du)_{x_0,\rho}| \leq c(M) \qquad \text{and} \qquad E(x_0, \rho) < c(M) \left(\frac{\rho}{R}\right)^{\beta} E(x_0, R)$$

Therefore, in case $1 < p < 2$, using (2.3) we obtain

$$\begin{aligned}
\fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|\, dx &= \fint_{B_\rho(x_0)\cap\{x:\, |Du-(Du)_{x_0,\rho}|\leq 1\}} |Du - (Du)_{x_0,\rho}|\, dx \\
&\quad + \fint_{B_\rho(x_0)\cap\{x:\, |Du-(Du)_{x_0,\rho}|> 1\}} |Du - (Du)_{x_0,\rho}|\, dx \\
&\leq c \fint_{B_\rho(x_0)} |V_p(Du - (Du)_{x_0,\rho})|\, dx + \left(\fint_{B_\rho(x_0)} |V_p(Du - (Du)_{x_0,\rho})|^2\, dx\right)^{\frac{1}{p}} \\
&\leq cE^{\frac{1}{2}}(x_0, \rho) + cE^{\frac{1}{p}}(x_0, \rho) \leq c(M, R)\rho^{\frac{\beta}{2}}
\end{aligned} \tag{5.3}$$

while in case $p \geq 2$ we use (2.5) thus getting

$$\begin{aligned}
\fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|\, dx &\leq \left(\fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2\, dx\right)^{\frac{1}{2}} \\
&\leq \left(\fint_{B_\rho(x_0)} |V_p(Du - (Du)_{x_0,\rho})|^2\, dx\right)^{\frac{1}{2}} = cE^{\frac{1}{2}}(x_0, \rho) \leq c(M, R)\rho^{\frac{\beta}{2}}
\end{aligned} \tag{5.4}$$

From estimates (5.3) and (5.4) it is clear that, setting

$$\Omega_0 = \{x \in \Omega : \sup_{r>0} |(Du)_{x_0,r}| < \infty \text{ and } \lim_{r\to 0} E(x_0, r) = 0\},$$

Ω_0 is an open subset of Ω of full measure and $u \in C^{1,\gamma}(\Omega_0)$ for every $\gamma < \frac{\beta}{2}$, and the conclusion follows since β is any number less than α. $\square$

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