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# Modes of Ideal Continuity of $(\ell)$-Group-Valued Measures

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**Abstract**

In this paper we deal with (ideal) continuity of lattice group-valued finitely additive measures, and prove some basic properties and comparison results. We investigate the relations between different modes of ideal continuity, and give some characterization. Finally we pose some open problems.

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**Keywords:** lattice group, (admissible) ideal, measurable set, symmetric difference of sets, (pseudo)-distance between measurable sets, finitely additive measure, continuous measure, ideal continuous measure, modes of continuity, ideal exhaustive measure sequence

# 1 Introduction

Ideal convergence was introduced in [42], though a primitive version of [41] is also included in the references of [42], and independently in [49] under the name of "cofilter convergence", and was recently developed and investigated in several papers in the context of normed and/or metric spaces. Among the related literature, we quote [4, 5, 8, 9, 22, 30, 31, 35, 38, 41, 50, 53, 54, 55]. This concept has been studied also in topological spaces ([28, 29, 43, 44]) and $(\ell)$-groups ([7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24]). A particular case of ideal convergence is the *statistical convergence,* introduced in [34] and [58] (see also [27, 36, 39, 42, 47, 48, 52]).

In [42, Proposition 3.3] a characterization of the classical continuity is given with ideal convergent sequences. In the literature there are several studies about abstract methods of convergences and related modes of continuity, in particular associated with summability matrices and almost convergence. Some relations between them are established, for instance, in [6, 25, 26, 27, 40, 45, 51, 56]. In [21] some modes of continuity between almost and ideal convergence are investigated, for normed space-valued functions, while in [3, 46, 57] and [24] there are some comparison results between different types of ideal continuity for topological space- and Riesz space-valued functions, respectively.

In this paper we consider modes of ideal continuity for lattice group-valued measures, dealing with two fixed admissible ideals of $\mathbf{N}$. We give some characterization, relating them with continuity of finitely additive measures with respect to a positive real-valued measure (see also [10, 12, 13]). In this setting, we give some characterizations of this property in terms of ideal convergence. Similar results have been proved in [24] for modes of continuity of functions. Our concept of continuity of measures is similar to absolute continuity of measures and substantially different from that of continuity of functions, and together with ideal exhaustiveness plays a fundamental role in Measure Theory, for example in several types of Brooks-Jewett, Vitali-Hahn-Saks and Nikodým convergence theorems for $(\ell)$-group-valued measures, when it is dealt with ideal pointwise convergent of the measures involved (see also [12, 13, 14, 23]). Finally, we pose some open problems.

## 2 The main results

For the concepts of (admissible) ideal, filter, $(O)$-sequence, $(O)$-convergence and ideal $(O)$-convergence ($(O\mathcal{I})$-convergence) in lattice groups, we refer to [19, 20, 33, 42].

Let $\Sigma$ be a $\sigma$-algebra of parts of an abstract infinite set $G$, $\lambda : \Sigma \to \mathbf{R}$ be a positive finitely additive measure, $d_\lambda(A, B) := \lambda(A\Delta B)$, $A$, $B \in \Sigma$ be the *(pseudo)-$\lambda$-distance*, where $\Delta$ denotes the symmetric difference (see also [32]). A finitely additive measure $\mu : \Sigma \to R$ is said to be *$\lambda$-continuous* at $A \in \Sigma$, iff

$$(O) \lim_p \Big( \bigvee_{B \in \Sigma, d_\lambda(A,B) \le 1/p} |\mu(B) - \mu(A)| \Big) = 0,$$

namely there is an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that for any $p \in \mathbf{N}$ there is $q = q(p) \in \mathbf{N}$ with

H1) $|\mu(B) - \mu(A)| \le \sigma_p$ whenever $d_\lambda(A, B) \le 1/q$.

We say that $\mu : \Sigma \to R$ is *$\lambda$-continuous* on $\Sigma$ iff it is $\lambda$-continuous at every $A \in \Sigma$.

Let $\mathcal{I}_1$, $\mathcal{I}_2$ be two admissible ideals of $\mathbf{N}$, and $A \in \Sigma$. A finitely additive measure $\mu : \Sigma \to R$ is *$(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuous at $A$* iff there is an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that for each sequence $(A_n)_n$ in $\Sigma$ with $(\mathcal{I}_1) \lim_n d_\lambda(A_n, A) = 0$ we have $(O\mathcal{I}_2) \lim_n \mu(A_n) = \mu(A)$ with respect to $(\sigma_p)_p$.

We say that $\mu$ is *$(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuous on* $\Sigma$, iff it is $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuous at every $A \in \Sigma$.

Note that $\mu$ is $\lambda$-absolutely continuous if and only if it is $\lambda$-continuous at $\emptyset$ (see also [12]).

We will use the definitions and notations above, unless differently stated.

We now give our main result, which extends [3, Theorem 3] to the $(\ell)$-group setting (concerning similar results for functions, see also [24]).

**Theorem 2.1** *Let $\lambda : \Sigma \to [0, +\infty[$, $\mu : \Sigma \to R$ be two finitely additive measures, and fix $A \in \Sigma$. Then the following properties hold:*

*(a) If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $\mu$ is $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuous at $A$ if and only if $\mu$ is $\lambda$-continuous at $A$.*

*(b) If $\mathcal{I}_1 \setminus \mathcal{I}_2 \ne \emptyset$, then $\mu$ is $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuous at $A$ if and only if $\mu(B) = 0$ for every $B \in \Sigma$.*

**Proof:** We begin with (a). We first prove the "if" part. Let $\mathcal{I}_1 \subset \mathcal{I}_2$, and $\mathcal{F}(\mathcal{I}_1)$, $\mathcal{F}(\mathcal{I}_2)$ be the dual filters associated with $\mathcal{I}_1$, $\mathcal{I}_2$ respectively, $(\sigma_p)_p$ be an $(O)$-sequence in $R$ related with $\lambda$-continuity of $\mu$ and $(A_n)_n$ be a sequence in $\Sigma$, with $(\mathcal{I}_1) \lim_n d_\lambda(A_n, A) = 0$. Fix arbitrarily $p \in \mathbf{N}$. By H1), we get $\{n \in$

$\mathbf{N} : d_\lambda(A_n, A) \leq 1/p\} \subset \{n \in \mathbf{N} : |\mu(A_n) - \mu(A)| \leq \sigma_p\} \in \mathcal{F}(\mathcal{I}_1)) \subset \mathcal{F}(\mathcal{I}_2)$, that is $(O\mathcal{I}_2) \lim_n \mu(A_n) = \mu(A)$.

We now turn to the "only if" part. Suppose that $\mu$ is $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuous at $A$. We claim that $\mu$ is $\lambda$-continuous at $A$. By $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuity of $\mu$, there is an $(O)$-sequence $(\sigma_p)_p$ in $R$ with $(O\mathcal{I}_2) \lim_n \mu(A_n) = \mu(A)$ w.r.t. $(\sigma_p)_p$ whenever $(\mathcal{I}_1) \lim\limits_n d_\lambda(A_n, A) = 0$. We now prove that $(\sigma_p)_p$ satifies the condition H1). If not, then there are $\overline{p} \in \mathbf{N}$ and a sequence $(A_q)_q$ in $\Sigma$, with $d_\lambda(A_q, A) \leq 1/q$ and $|\mu(A_q) - \mu(A)| \not\leq \sigma_{\overline{p}}$ for each $q \in \mathbf{N}$. Since $(O) \lim\limits_q d_\lambda(A_q, A) = 0$, then $(\mathcal{I}_1) \lim\limits_q d_\lambda(A_q, A) = 0$ and so, by $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuity of $\mu$, $(O\mathcal{I}_2) \lim\limits_q \mu(A_q) = \mu(A)$ with respect to $(\sigma_p)_p$. Hence, $\mathbf{N} = \{q \in \mathbf{N} : |\mu(A_q) - \mu(A)| \not\leq \sigma_{\overline{p}}\} \in \mathcal{I}_2$, which is impossible, since $\mathcal{I}_2$ is non-trivial. This concludes the proof of (a).

We now turn to (b), and we prove only the "only if" part, since the "if" part is straightforward. Choose arbitrarily $B \in \Sigma$, and let us show that $\mu(B) = \mu(A)$. From this and arbitrariness of $B \in \Sigma$ it will follow that $\mu(B) = \mu(\emptyset) = 0$ for all $B \in \Sigma$. If $\mu(B) \neq \mu(A)$, then for every $(O)$-sequence $(\sigma_p)_p$ there is $\overline{p} \in \mathbf{N}$ with $|\mu(B) - \mu(A)| \not\leq \sigma_{\overline{p}}$. By hypothesis, there is a set $C \in \mathcal{I}_1 \setminus \mathcal{I}_2$. Note that $\mathbf{N} \setminus C$ is infinite: otherwise, since $\mathcal{I}_1$ is admissible, we should have $\mathbf{N} \setminus C \in \mathcal{I}_1$ and hence also $\mathbf{N} \in \mathcal{I}_1$, which is impossible, because $\mathcal{I}_1$ is non-trivial. Set $\mathbf{N} \setminus C := \{n_k : k \in \mathbf{N}\}$. Let $(Z_k)_k$ be a sequence in $\Sigma$, with $\lim_k d_\lambda(Z_k, A) = 0$. For each $n \in \mathbf{N}$, let

$$A_n := \begin{cases} Z_k, & \text{if } n = n_k; \\ B, & \text{if } n \in C. \end{cases}$$

For each $p \in \mathbf{N}$ there is $k_0 \in \mathbf{N}$ with $d_\lambda(A_{n_k}, A) = d_\lambda(Z_k, A) \leq 1/p$ whenever $k \geq k_0$, and so the set $\{n \in \mathbf{N} : d_\lambda(A_n, A) \not\leq 1/p\}$ is contained in $C \cup \{n_1, \ldots, n_{k_0 - 1}\} \in \mathcal{I}_1$. Thus $(\mathcal{I}_1) \lim\limits_n d_\lambda(A_n, A) = 0$, and hence, by $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-continuity of $\mu$, there is an $(O)$-sequence $(\sigma_p)_p$ with $(O\mathcal{I}_2) \lim\limits_n \mu(A_n) = \mu(A)$ with respect to $(\sigma_p)_p$, namely

$$D := \{n \in \mathbf{N} : |\mu(A_n) - \mu(A)| \not\leq \sigma_p\} \in \mathcal{I}_2 \quad \text{for each}\, p \in \mathbf{N}.$$

As $\mu(A_n) = \mu(B)$ for every $n \in C$, it follows that $C \subset D$, and thus $C \in \mathcal{I}_2$, obtaining a contradiction. This proves the "only if" part.

**Open problems:** (a) Prove similar results by using $(D)$-convergence.

(b) Examine similar results by requiring $\lambda$ to take values in the positive cone on an arbitrary $(\ell)$-group.

(c) Investigate the $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-equicontinuity of a family $\mathcal{M}$ of measures (finitely and/or $\sigma$-additive), defined on a $\sigma$-algebra $\Sigma$ and with values in an $(\ell)$-group $R$, $\lambda$ being a positive finitely additive measure defined on $\Sigma$.

(d) A sequence of measures $\mu_n : \Sigma \to R$, $n \in \mathbf{N}$, is said to be $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-*exhaustive at* $A \in \Sigma$ iff there is an $(O)$-sequence $(\sigma_p)_p$ (depending on $A$) such that for every $p \in \mathbf{N}$ and for each sequence $(A_k)_k$ in $\Sigma$ with

$$(O\mathcal{I}_1) \lim_k d_\lambda(A_k, A) = 0$$

there are $C$, $D \in \mathcal{I}_2$ with $|\mu_n(A_k) - \mu_n(A)| \leq \sigma_p$ for all $k \in \mathbf{N} \setminus D$ and $n \in \mathbf{N} \setminus C$.

The sequence $(\mu_n)_n$ is $(\mathcal{I}_1, \mathcal{I}_2)$-$\lambda$-*exhaustive on* $\Sigma$ iff it is $(\mathcal{I}_1, \mathcal{I}_2)$-exhaustive at every $A \in \Sigma$.

Establish some relations between $(\mathcal{I}_1, \mathcal{I}_2)$-exhaustiveness, (ideal) continuity and ideal limit theorems for $(\ell)$-group-valued measures (see also [2, 10, 13, 14, 23, 24]).

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