10. Linear controllers for tracking chained-form systems

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Summary.

In this paper we study the tracking problem for the class of nonholonomic systems in chained-form. In particular, with the first and the last state component of the chained-form as measurable output signals, we suggest a solution for the tracking problem using output feedback by combining a time-varying state feedback controller with an observer for the chained-form system. For the stability analysis of the "certainty equivalence type" of controller we use a cascaded systems approach. The resulting closed loop system is globally $\mathcal{K}$-exponentially stable.

10.1 Introduction

In recent years a lot of interest has been devoted to (mainly) stabilization and tracking of nonholonomic dynamic systems, see e.g. [1, 6, 8, 15, 17]. One of the reasons for the attention is the lack of a continuous static state feedback control since Brockett's necessary condition for smooth stabilization is not met, see [3]. The proposed solutions to this problem follow mainly two routes, namely discontinuous and/or time-varying control. For a good overview, see the survey paper [12] and the references therein.

It is well known that the kinematic model of several nonholonomic systems can be transformed into a *chained-form system*. The global tracking problem for chained-form systems has recently been addressed in [4, 6, 7, 8, 17, 20]. In this paper we consider the tracking problem for chained form systems by means of output feedback, where we consider as output the first and last state component of the chained-form. To our knowledge, this problem has only been addressed in [9] where a backstepping approach is used. Our results are based on the construction of a linear time varying state feedback controller in combination with an observer. However, the stability analysis and design are based on results for (time-varying) cascaded systems [18]. In the design we divide the chained-form into a cascade of two sub-systems which we can stabilize independently of each other, and furthermore a similar partition into cascaded systems can be done for the controller-observer combination, where the same stability results apply. Regarding the latter part, similar ideas were recently presented for the combination of high-gain controllers and high-gain observer for a class of triangular nonlinear systems [2], see also [13].

The organization of the paper is as follows. Section 10.2 contains some definitions, preliminary results and the problem formulation. Section 10.3 addresses the tracking problem based on time-varying state feedback and in section 10.4 we design an exponentially convergent observer for the chained-form system. In section 10.5 we combine the control law from section 10.3 with the observer from section 10.4 in a "certainty equivalence" sense. This yields a globally $\mathcal{K}$-exponentially stable closed loop system under the condition of persistently exciting reference trajectories. Finally, section 10.6 concludes the paper.

10.2 Preliminaries and problem formulation

In this section we introduce the definitions and theorems used in the remainder of this paper and formulate the problem under consideration. We start with some basic stability concepts in 10.2.1, present a result for cascaded systems in 10.2.2 and recall some basic results from linear systems theory in 10.2.3. We conclude this section with the problem formulation in 10.2.4.

10.2.1 Stability

To start with, we recall some basic concepts (see e.g. [11, 23]).

Definition 10.2.1. *A continuous function* $\alpha : [0,a) \to [0,\infty)$ *is said to belong to* class $\mathcal{K}$ *if it is strictly increasing and* $\alpha(0) = 0$.

Definition 10.2.2. *A continuous function* $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ *is said to belong to* class $\mathcal{KL}$ *if, for each fixed* s*, the mapping* $\beta(r,s)$ *belongs to class* $\mathcal{K}$ *with respect to* r *and, for each fixed* r*, the mapping* $\beta(r,s)$ *is decreasing with respect to* s *and* $\beta(r,s) \to 0$ *as* $s \to \infty$.

Consider the system

$$\dot{x} = f(t,x), \quad f(t,0) = 0 \quad \forall t \geq 0 \tag{10.1}$$

with $x \in I\!R^n$ and $f(t,x)$ piecewise continuous in t and locally Lipschitz in x.

Definition 10.2.3. *The system (10.1) is* uniformly stable *if for each* $\epsilon > 0$ *there is* $\delta = \delta(\epsilon) > 0$*, independent of* t_0*, such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0.$$

Definition 10.2.4. *The system (10.1) is* globally uniformly asymptotically stable
(GUAS) *if it is uniformly stable and globally attractive, that is, there exists a class* $\mathcal{KL}$ *function* $\beta(\cdot,\cdot)$ *such that for every initial state* $x(t_0)$*:*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0$$

Definition 10.2.5. *The system (10.1) is* globally exponentially stable (GES) *if there exist* $k > 0$ *and* $\gamma > 0$ *such that for any initial state*

$$\|x(t)\| \leq \|x(t_0)\| k \exp[-\gamma(t - t_0)].$$

A slightly weaker notion of exponential stability is the following (cf. [21])

Definition 10.2.6. *We call the system (10.1)* globally $\mathcal{K}$-exponentially stable *if there exist* $\gamma > 0$ *and a class* $\mathcal{K}$ *function* $\kappa(\cdot)$ *such that*

$$\|x(t)\| \leq \kappa(\|x(t_0)\|) \exp[-\gamma(t - t_0)] \tag{10.2}$$

Definition 10.2.7. *We call the (locally integrable) vector-valued function*

$$w(t) = [w_1(t), \ldots, w_n(t)]^T$$

persistently exciting *if there exist* $\delta, \varepsilon_1, \varepsilon_2 > 0$ *such that for all* $t > 0$*:*

$$\varepsilon_1 I \leq \int_t^{t+\delta} w(\tau) w(\tau)^T d\tau \leq \varepsilon_2 I$$

10.2.2 Cascaded systems

Consider the system

$$\begin{cases} \dot{z}_1 = f_1(t, z_1) + g(t, z_1, z_2)z_2 \\ \dot{z}_2 = f_2(t, z_2) \end{cases} \tag{10.3}$$

where $z_1 \in I\!R^n$, $z_2 \in I\!R^m$, $f_1(t, z_1)$ is continuously differentiable in (t, z_1) and $f_2(t, z_2)$, $g(t, z_1, z_2)$ are continuous in their arguments, and locally Lipschitz in z_2 and (z_1, z_2) respectively.

We can view the system (10.3) as the system

$$\Sigma_1 : \dot{z}_1 = f_1(t, z_1)$$

that is perturbed by the state of the system

$$\Sigma_2 : \dot{z}_2 = f_2(t, z_2).$$

When Σ_2 is asymptotically stable, we have that z_2 tends to zero, which means that the z_1 dynamics in (10.3) asymptotically reduces to Σ_1. Therefore, we can hope that asymptotic stability of both Σ_1 and Σ_2 implies asymptotic stability of (10.3).

Unfortunately, this is not true in general. However, from the proof presented in [18] it can be concluded that:

Theorem 10.2.1 (based on [18]). *The cascaded system (10.3) is GUAS if the following three assumptions hold:*

- assumption on Σ_1: *the system $\dot{z}_1 = f_1(t, z_1)$ is GUAS and there exists a continuously differentiable function $V(t, z_1) : I\!R_+ \times I\!R^n \to I\!R$ that satisfies*

$$W(z_1) \leq V(t, z_1), \tag{10.4}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1} \cdot f_1(t, z_1) \leq 0, \quad \forall \|z_1\| \geq \eta, \tag{10.5}$$

$$\left\| \frac{\partial V}{\partial z_1} \right\| \|z_1\| \leq cV(t, z_1), \quad \forall \|z_1\| \geq \eta, \tag{10.6}$$

  where $W(z_1)$ is a positive definite proper function and $c > 0$ and $\eta > 0$ are constants,

- assumption on the interconnection: *the function $g(t, z_1, z_2)$ satisfies for all $t \geq t_0$:*

$$\|g(t, z_1, z_2)\| \leq \theta_1(\|z_2\|) + \theta_2(\|z_2\|)\|z_1\|,$$

  where $\theta_1, \theta_2 : I\!R_+ \to I\!R_+$ are continuous functions,

- assumption on Σ_2: *the system* $\dot{z}_2 = f_2(t, z_2)$ *is GUAS and for all* $t_0 \geq 0$*:*

$$\int_{t_0}^{\infty} \|z_2(t_0, t, z_2(t_0))\| dt \leq \kappa(\|z_2(t_0)\|),$$

where the function $\kappa(\cdot)$ *is a class* $\mathcal{K}$ *function,*

Remark 10.2.1. Notice that the assumption on Σ_1 is slightly weaker than the one presented in [18]. However, under the assumption mentioned above the result can still be shown to be true by (almost) exactly copying the proof presented in [18].

Lemma 10.2.1 (see [17]). *If in addition to the assumptions in Theorem 10.2.1 both* $\dot{z}_1 = f_1(t, z_1)$ *and* $\dot{z}_2 = f_2(t, z_2)$ *are globally* $\mathcal{K}$*-exponentially stable, then the cascaded system (10.3) is globally* $\mathcal{K}$*-exponentially stable.*

10.2.3 Linear time-varying systems

Consider the linear time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{10.7}$$

and let $\Phi(t, t_0)$ denote the state-transition matrix for the system $\dot{x} = A(t)x$. We recall some results from linear control theory (cf. [10, 19]).

Definition 10.2.8. *The pair* $(A(t), B)$ *is* uniformly controllable *if there exist* $\delta, \varepsilon_1, \varepsilon_2 > 0$ *such that for all* $t > 0$*:*

$$\varepsilon_1 I \leq \int_t^{t+\delta} \Phi(t, \tau)BB^T\Phi^T(t, \tau)d\tau \leq \varepsilon_2 I$$

Definition 10.2.9. *The pair* $(A(t), C)$ *is* uniformly observable *if there exist* $\delta, \epsilon_1, \epsilon_2 > 0$ *such that for all* $t > 0$*:*

$$\epsilon_1 I \leq \int_{t-\delta}^{t} \Phi^T(\tau, t-\delta)C^T C\Phi(\tau, t-\delta)d\tau \leq \epsilon_2 I$$

From linear systems theory several methods are available to exponentially stabilize the linear time-varying system (10.7) via state or output feedback, in case the pairs $(A(t), B)$ and $(A(t), C)$ are uniformly controllable and observable respectively (cf. [19]):

Theorem 10.2.2. *Suppose that the system (10.7) is uniformly controllable and define for $\alpha > 0$*

$$W_\alpha(t, t+\delta) = \int_t^{t+\delta} 2e^{4\alpha(t-\tau)}\Phi(t,\tau)BB^T\Phi^T(t,\tau)d\tau \tag{10.8}$$

Then given any constant α the state feedback $u(t) = K_\alpha(t)x(t)$ where

$$K_\alpha(t) = -B^T W_\alpha^{-1}(t, t+\delta) \tag{10.9}$$

is such that the resulting closed-loop state equation is uniformly exponentially stable with rate α.

Theorem 10.2.3. *Suppose that the system (10.7) is uniformly controllable and uniformly observable and define for $\alpha > 0$*

$$M_\alpha(t-\delta, t) = \int_{t-\delta}^{t} 2e^{4\alpha(\tau-t)}\Phi^T(\tau, t-\delta)C^T C\Phi(\tau, t-\delta)d\tau$$

Then given $\alpha > 0$, for any $\eta > 0$ the linear dynamic output feedback

$$\begin{aligned}
u(t) &= K_{\alpha+\eta}(t)\hat{x}(t) \\
\dot{\hat{x}}(t) &= A(t)\hat{x}(t) + Bu(t) + H_{\alpha+\eta}(t)[y(t) - \hat{y}(t)], \qquad \hat{x}(t_0) = \hat{x}_0 \\
\hat{y}(t) &= C\hat{x}(t)
\end{aligned}$$

with feedback and observer gains

$$\begin{aligned}
K_{\alpha+\eta}(t) &= -B^T W_{\alpha+\eta}^{-1}(t, t+\delta) \\
H_{\alpha+\eta}(t) &= \left[\Phi^T(t-\delta, t)M_{\alpha+\eta}(t-\delta, t)\Phi(t-\delta, t)\right]^{-1} C^T
\end{aligned} \tag{10.10}$$

is such that the closed-loop state equation is uniformly exponentially stable with rate α.

10.2.4 Problem formulation

The class of chained-form nonholonomic systems we study in this paper is given by the following equations

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned} \tag{10.11}$$

where $x = (x_1, \ldots, x_n)$ is the state, u_1 and u_2 are control inputs.

Consider the problem of tracking a reference trajectory (x_r, u_r) generated by the chained-form system:

$$\begin{aligned}
\dot{x}_{1,r} &= u_{1,r} \\
\dot{x}_{2,r} &= u_{2,r} \\
\dot{x}_{3,r} &= x_{2,r}u_{1,r} \\
&\vdots \\
\dot{x}_{n,r} &= x_{n-1,r}u_{1,r}
\end{aligned} \tag{10.12}$$

where we assume $u_{1,r}(t)$ to $u_{2,r}(t)$ be continuous functions of time. This reference trajectory can be generated by any of the motion planning techniques available from the literature.

When we define the tracking error $x_e = x - x_r$ we obtain as tracking error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} &&= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} &&= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_2u_1 - x_{2,r}u_{1,r} &&= x_{2,e}u_{1,r} + x_2(u_1 - u_{1,r}) \\
&\vdots &&\vdots \\
\dot{x}_{n,e} &= x_{n-1}u_1 - x_{n-1,r}u_{1,r} &&= x_{n-1,e}u_{1,r} + x_{n-1}(u_1 - u_{1,r})
\end{aligned} \tag{10.13}$$

The state feedback tracking control problem then can be formulated as

Problem 10.2.1 (State feedback tracking control problem). Find appropriate state feedback laws u_1 and u_2 of the form

$$u_1 = u_1(t, x, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, x, x_r, u_r) \tag{10.14}$$

such that the closed-loop trajectories of (10.13,10.14) are globally uniformly asymtotically stable.

Consider the system (10.11) with output

$$y = \begin{bmatrix} x_1 \\ x_n \end{bmatrix} \tag{10.15}$$

then it is easy to show (see e. g. [1]) that the system (10.11) with output (10.15) is locally observable at any $x \in I\!R^n$. Clearly, this is the minimal number of state components we need to know for solving the output-feedbacl tracking problem.

Now we can formulate the output feedback tracking problem as

Problem 10.2.2 (Output feedback tracking control problem). Find appropriate control laws u_1 and u_2 of the form

$$u_1 = u_1(t, \hat{x}, y, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, \hat{x}, y, x_r, u_r) \tag{10.16}$$

where $\hat{x}$ is generated from an observer

$$\dot{\hat{x}} = f(t, \hat{x}, y, x_r, u_r) \tag{10.17}$$

such that the closed-loop trajectories of (10.13,10.16,10.17) are globally uniformly asymptotically stable.

10.3 The state feedback problem

The approach we use to solve our problem is based on the recently developed studies on cascaded systems [5, 14, 16, 18, 22], and that of Theorem 10.2.1 in particular, since it deals with time-varying systems.

We search for a subsystem which, with a stabilizing control law, can be written in the form $\dot{z}_2 = f_2(t, z_2)$ that is asymptotically stable. In the remaining dynamics we can then replace the appearance of z_2 by 0, leading to the system $\dot{z}_1 = f_1(t, z_1)$. If this system is asymptotically stable we might be able to conclude asymptotic stability of the overall system using Theorem 10.2.1.

Consider the tracking error dynamics (10.13). We can stabilize the $x_{1,e}$ dynamics by using the linear controller

$$u_1 = u_{1,r} - c_1 x_{1,e} \tag{10.18}$$

which yields GES for $x_{1,e}$, provided $c_1 > 0$.

If we now set $x_{1,e}$ equal to 0 in (10.13) we obtain

$$\begin{aligned} \dot{x}_{2,e} &= u_2 - u_{2,r} \\ \dot{x}_{3,e} &= x_{2,e} u_{1,r} \\ &\vdots \\ \dot{x}_{n,e} &= x_{n-1,e} u_{1,r} \end{aligned} \tag{10.19}$$

where we used (10.18).

Notice that the system (10.19) is a linear time-varying system:

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & \ddots & & & \vdots \\ 0 & u_{1,r}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix}}_{A(t)} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} (u_2 - u_{2,r}) \tag{10.20}$$

that can be made exponentially stable by means of the controller $u(t) = K(t)x(t)$ provided the system (10.20) is uniformly controllable (cf. Theorem 10.2.2).

This observation leads to the following

Proposition 10.3.1. *Assume that the reference trajectory (x_r, u_r) satisfying (10.12) to be tracked by our chained form system is given. Define*

$$w_r(t, t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^{t} u_{1,r}(\tau)d\tau \\ \left(\int_{t_0}^{t} u_{1,r}(\tau)d\tau\right)^2 \\ \vdots \\ \left(\int_{t_0}^{t} u_{1,r}(\tau)d\tau\right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_{1,r}(t) - x_{1,r}(t_0) \\ (x_{1,r}(t) - x_{1,r}(t_0))^2 \\ \vdots \\ (x_{1,r}(t) - x_{1,r}(t_0))^{n-2} \end{bmatrix}$$

and assume that there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_t^{t+\delta} w_r(t,\tau) w_r(t,\tau)^T d\tau \leq \varepsilon_2 I. \tag{10.21}$$

Consider the system (10.13) in closed-loop with the linear controller

$$\begin{aligned} u_1 &= u_{1,r} - c_1 x_{1,e} \\ u_2 &= u_{2,r} + K(t) \begin{bmatrix} x_{2,e} \\ \vdots \\ x_{n,e} \end{bmatrix} \end{aligned} \tag{10.22}$$

where $c_1 > 0$ and $K(t)$ is given by

$$K(t) = -[1\ 0\ 0\ \dots\ 0] \left[\int_t^{t+\delta} 2e^{4\alpha(t-\tau)} w_r(t,\tau) w_r(t,\tau)^T d\tau \right]^{-1} \tag{10.23}$$

with $\alpha > 0$. If $x_{2,r}(t), \dots, x_{n-1,r}(t)$ are bounded then the closed-loop system (10.13,10.22) is globally $\mathcal{K}$-exponentially stable.

Proof. We can see the closed-loop system (10.13,10.22) as a system of the form (10.3) where

$$
\begin{aligned}
z_1 &= [x_{2,e}, \dots, x_{n,e}]^T && (10.24)\\
z_2 &= x_{1,e} && (10.25)\\
f_1(t, z_1) &= (A(t) - BK(t))z_1 && (10.26)\\
f_2(t, z_2) &= -c_1 z_2 && (10.27)\\
g(t, z_1, z_2) &= -c_1[0, x_2, x_3, \dots, x_{n-1}]^T && (10.28)
\end{aligned}
$$

with

$$
A(t) = \begin{bmatrix} 0 & \dots\dots & \dots & 0 \\ u_{1,r}(t) & \ddots & & \vdots \\ 0 & \ddots\ \ddots & & \vdots \\ \vdots & \ddots\ \ddots & \ddots & \vdots \\ 0 & \dots\ 0 & u_{1,r}(t) & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}
$$

To be able to apply Theorem 10.2.1 we need to verify the three assumptions:

- assumption on Σ_1: Due to the assumption (10.21) on $u_{1,r}(t)$ we have that the system (10.20) is uniformly controllable (cf. Remark 10.3.2). Therefore, from Theorem 10.2.2 we know that $\dot{z}_1 = f_1(t, z_1)$ is GES and therefore GUAS. From converse Lyapunov theory (see e. g. [11]) the existence of a suitable V is guaranteed.
- assumption on connecting term: Since $x_{2,r}, \dots, x_{n-1,r}$ are bounded, we have

$$
\begin{aligned}
\|g(t, z_1, z_2)\| &\leq c_1 \left(\| \begin{bmatrix} 0 \\ x_{2,r} \\ \vdots \\ x_{n-1,r} \end{bmatrix} \| + \| \begin{bmatrix} 0 \\ x_{2,e} \\ \vdots \\ x_{n-1,e} \end{bmatrix} \| \right) && (10.29)\\
&\leq c_1 M + c_1 \|x\| && (10.30)
\end{aligned}
$$

- assumption on Σ_2: Follows from GES of $\dot{x}_2 = -c_1 x_2$.

Therefore, we can conclude GUAS from Theorem 10.2.1. Since both Σ_1 and Σ_2 are GES, Lemma 10.2.1 gives the desired result.

Remark 10.3.1. Notice that since

$$
u_1(t) = u_{1,r}(t) - c_1 x_{1,e}(t_0) \exp(-c_1(t - t_0))
$$

the condition (10.21) on $u_{1,r}(t)$ is satisfied if and only if a similar condition on $u_1(t)$ is satisfied (i.e. in which the r is omitted).

Therefore, we can also see the closed-loop system (10.13,10.22) as a system of the form (10.3) where

$$z_1 = [x_{2,e}, \ldots, x_{n,e}]^T \tag{10.31}$$
$$z_2 = x_{1,e} \tag{10.32}$$
$$f_1(t, z_1) = (A(t) - BK(t))z_1 \tag{10.33}$$
$$f_2(t, z_2) = -c_1 z_2 \tag{10.34}$$
$$g(t, z_1, z_2) = -c_1[0, x_{2,r}, x_{3,r}, \ldots, x_{n-1,r}]^T \tag{10.35}$$

with

$$A(t) = \begin{bmatrix} 0 & \ldots\ldots & \ldots & 0 \\ u_1(t) & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & u_1(t) & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Notice that we redefined $A(t)$ and that correspondingly the connecting term $g(t, z_1, z_2)$ changed. When we modify our controller accordingly, i.e. redefine $K(t)$ in (10.22) as

$$K(t) = -[1\ 0\ 0\ \ldots\ 0] \left[\int_t^{t+\delta} 2e^{4\alpha(t-\tau)} w(t,\tau) w(t,\tau)^T d\tau \right]^{-1} \tag{10.36}$$

with $\alpha > 0$, where

$$w(t, t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^{t} u_1(\tau) d\tau \\ \left(\int_{t_0}^{t} u_1(\tau) d\tau \right)^2 \\ \vdots \\ \left(\int_{t_0}^{t} u_1(\tau) d\tau \right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_1(t) - x_1(t_0) \\ (x_1(t) - x_1(t_0))^2 \\ \vdots \\ (x_1(t) - x_1(t_0))^{n-2} \end{bmatrix}$$

we can copy the proof.

Moreover, since the connecting term $g(t, z_1, z_2)$ now can be bounded by a constant, we can claim not only global $\mathcal{K}$-exponential stability, but even GES. However, the disadvantage of (10.36) in comparison to (10.23) is that it depends on the state and therefore can not be determined a priori for a known reference trajectory in contrast to (10.23).

Remark 10.3.2. Notice that in general it is not easy to compute $\Phi(t, t_0)$. However, for the system (10.20) this turns out not to be too difficult, due to the nice and simple structure of the matrix $A(t)$. We find:

$$\Phi(t, t_0) = \begin{bmatrix} f_0(t, t_0) & 0 & \dots & 0 \\ f_1(t, t_0) & f_0(t, t_0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{n-2}(t, t_0) & \dots & f_1(t, t_0) & f_0(t, t_0) \end{bmatrix}$$

where

$$f_k(t, t_0) = \frac{1}{k!} \left[\int_{t_0}^{t} u_{1,r}(\sigma) d\sigma \right]^k = \frac{1}{k!} \left[x_{1,r}(t) - x_{1,r}(t_0) \right]^k$$

¿From this it is also straightforward to see that uniform controllability of the system (10.20) can also rephrased as persistency of excitation of the vector

$$\begin{bmatrix} f_0(t, t_0) \\ f_1(t, t_0) \\ \vdots \\ f_{n-2}(t, t_0) \end{bmatrix}$$

Remark 10.3.3. Notice that the persistency of excitation condition (10.21) is obviously met in case $\liminf_{t \to \infty} u_{1,r}(t) = \varepsilon > 0$, so that the results of [6, 7, 8, 17] are included in this result.

10.4 An observer

The observability property for chained-form systems was considered in [1], in which a (local) observer was proposed in case $u_1(t) = -c_1 x_1(t)$. In this section we propose a globally exponentially stable observer for the chained system under an observability condition which is related to the persistence of excitation with respect to the first component of the state.

Proposition 10.4.1. *Consider the chained-form system (10.11) with output (10.15). Define*

$$w(t, t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^{t} u_1(\tau) d\tau \\ \left(\int_{t_0}^{t} u_1(\tau) d\tau \right)^2 \\ \vdots \\ \left(\int_{t_0}^{t} u_1(\tau) d\tau \right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_1(t) - x_1(t_0) \\ (x_1(t) - x_1(t_0))^2 \\ \vdots \\ (x_1(t) - x_1(t_0))^{n-2} \end{bmatrix}$$

Assume that there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ *such that for all* $t > 0$*:*

$$\varepsilon_1 I \leq \int_t^{t+\delta} w(t,\tau)w(t,\tau)^T d\tau \leq \varepsilon_2 I.$$

Then the observer

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \\ \vdots \\ \dot{\hat{x}}_n \end{bmatrix} = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ u_1 & \ddots & & & \vdots \\ 0 & u_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & u_1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \vdots \\ \hat{x}_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 + H(t)\tilde{x}_n$$

where $\tilde{x}_n = x_n - \hat{x}_n$ *and*

$$H(t) = \left[\Phi^T(t-\delta,t)M_\alpha(t-\delta,t)\Phi(t-\delta,t)\right]^{-1} C^T \quad (\alpha > 0)$$

guarantees that the observation error $\tilde{x} = x - \hat{x}$ *converges to zero exponentially.*

Proof. Because of the assumption on $u_1(t)$ we have a uniformly observable linear time-varying system. The result follows readily from standard linear theory (see e. g. [19]).

10.5 The output feedback problem

In section 3 we derived a state feedback controller for tracking a desired trajectory, whereas in section 4 we derived an observer for a system in chained-form. We can also combine these two results in a "certainty equivalence" sense:

Proposition 10.5.1. *For the reference trajectory* $x_r, u_r)$ *satisfying (10.12) define*

$$w_r(t,t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^t u_{1,r}(\tau)d\tau \\ \left(\int_{t_0}^t u_{1,r}(\tau)d\tau\right)^2 \\ \vdots \\ \left(\int_{t_0}^t u_{1,r}(\tau)d\tau\right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_{1,r}(t) - x_{1,r}(t_0) \\ (x_{1,r}(t) - x_{1,r}(t_0))^2 \\ \vdots \\ (x_{1,r}(t) - x_{1,r}(t_0))^{n-2} \end{bmatrix}$$

and assume that there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ *such that for all* $t > 0$*:*

$$\varepsilon_1 I \leq \int_t^{t+\delta} w_r(t,\tau) w_r(t,\tau)^T d\tau \leq \varepsilon_2 I.$$

Consider the system (10.13) in closed-loop with the linear controller-observer-combination

$$\begin{aligned}
u_1 &= u_{1,r} - c_1 x_{1,e} \\
u_2 &= u_{2,r} + K(t) \begin{bmatrix} \hat{x}_{2,e} \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \dot{\hat{x}}_{4,e} \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \hat{x}_{4,e} \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 + H(t)\tilde{x}_n \tag{10.37}$$

where $\tilde{x}_n = x_n - \hat{x}_n$*,* $c_1 > 0$ *and* $K(t)$ *and* $H(t)$ *are given by*

$$\begin{aligned}
K(t) &= -[1\ 0\ 0\ \dots\ 0] \left[\int_t^{t+\delta} 2e^{4\alpha(t-\tau)} w_r(t,\tau) w_r(t,\tau)^T d\tau \right]^{-1} \\
H(t) &= \left[2e^{4\alpha(\tau-t)} w_r(\tau, t-\delta) w_r(\tau, t-\delta)^T d\tau\ \Phi(t-\delta, t) \right]^{-1} w_r(t, t-\delta)
\end{aligned}$$

with $\alpha > 0$*. If* $x_{2,r}, \dots, x_{n-1,r}$ *are bounded then the closed-loop system (10.13,10.37) is globally* $\mathcal{K}$*-exponentially stable.*

Proof. Similar to that of Proposition 10.3.1. Note that due to the assumption on $u_{1,r}$ we have both uniform controllability and uniform controllability. ¿From Theorem 10.2.3 we then know that the system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{\hat{z}}_1 \end{bmatrix} = \begin{bmatrix} A(t) & -BK(t) \\ A(t) + H(t)C & -BK(t) - H(t)C \end{bmatrix} \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix}$$

is globally exponentially stable.

Since we can write the closed-loop system (10.13,10.37) as

$$\begin{aligned}
\begin{bmatrix} \dot{z}_1 \\ \dot{\hat{z}}_1 \end{bmatrix} &= \begin{bmatrix} A(t) & -BK(t) \\ A(t) + H(t)C & -BK(t) - H(t)C \end{bmatrix} \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix} + \begin{bmatrix} g(t, \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix}, z_2) \\ 0 \end{bmatrix} z_2 \\
\dot{z}_2 &= -c_1 z_2
\end{aligned}$$

where

$$z_1 = [x_{2,e}, \ldots, x_{n,e}]^T$$
$$z_2 = x_{1,e}$$
$$g(t, \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix}, z_2) = -c_1[0, x_2, x_3, \ldots, x_{n-1}]^T$$

The proof can be completed similar to that of Proposition 10.3.1.

10.6 Conclusions

In this paper we considered the tracking problem for nonholonomic systems in chained-form by means of output feedback. We combined a time-varying state feedback controller with an observer for the chained-form in a "certainty equivalence" way. The stability of the closed loop system is shown using results from time-varying cascaded systems. Under a condition of persistence of excitation, we have shown globally $\mathcal{K}$-exponential stability of the closed loop system.

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