# Pancyclicity of 4-connected \{Claw, Generalized Bull\} - free Graphs 

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November 7, 2012


#### Abstract

A graph $G$ is pancyclic if it contains cycles of each length $\ell, 3 \leq \ell \leq|V(G)|$. The generalized bull $B(i, j)$ is obtained by associating one endpoint of each of the paths $P_{i+1}$ and $P_{j+1}$ with distinct vertices of a triangle. Gould, Luczak and Pfender [4] showed that if $G$ is a 3 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph with $i+j=4$ then $G$ is pancyclic. In this paper, we prove that every 4 -connected, claw-free, $B(i, j)$-free graph with $i+j=6$ is pancyclic. As the line graph of the Petersen graph is $B(i, j)$-free for any $i+j=7$ and is not pancyclic, this result is best possible.


Keywords: Pancyclic, Claw-free, Generalized Bull

## 1 Introduction

All graphs in this paper are simple. A graph $G$ is hamiltonian if it contains a spanning cycle, and is pancyclic if it contains cycles of each length $\ell, 3 \leq \ell \leq|V(G)|$. We consider all cycles to have an implicit clockwise orientation. With this in mind, given a cycle $C$ and a vertex $x$ on $C$, we let $x^{+}$denote the successor of $x$ under this orientation and let $x^{-}$denote the predecessor. We define $x^{+i}$ recursively with $x^{+1}=x^{+}$and $x^{+(i+1)}=\left(x^{+i}\right)^{+}$for $i>1$ and define $x^{-i}$ analogously. For any other vertex $y$ on $C$, we let $x C y$ denote the path from $x$ to $y$ on $C$ in the clockwise direction of the orientation and $x C^{-} y$ denote the path from $x$ to $y$ on $C$ in the counterclockwise direction. When convenient, we will also let $C(x, y)$ denote $V\left(x^{+} C y^{-}\right)$, that is, the set of vertices lying between $x$ and $y$ on $C$ when traversed in the clockwise direction. We will use the term arc to describe these paths on a cycle. Given a subgraph $H$ of $G$ and a vertex $v \in G-H$, by a $v-H$ path we mean a path $P$ with endpoints $v$ and $w \in H$ such that $P \cap H=\{w\}$. For a set of vertices $A$ in $G$ and a subgraph $H$ of $G$, we let $N_{G}(A)$ denote the neighborhood of $A$ in $G$ and $N_{H}(A)$ denote the neighborhood of $A$ in $H$. When $A=\{x\}$, we write $N_{G}(x)$ and $N_{H}(x)$, respectively. Furthermore let $d_{G}(x)=\left|N_{G}(x)\right|$ and $d_{H}(x)=\left|N_{H}(x)\right|$.

Given a family $\mathcal{F}$ of graphs, a graph $G$ is said to be $\mathcal{F}$-free if $G$ contains no member of $\mathcal{F}$ as an induced subgraph. If $\mathcal{F}=\left\{K_{1,3}\right\}$, then $G$ is said to be claw-free. The net, $N$, is the graph

[^0]obtained by attaching a pendant vertex to each vertex in a triangle. The generalized net $N(i, j, k)$ is obtained by associating one endpoint of each of the paths $P_{i+1}, P_{j+1}$ and $P_{k+1}$ with distinct vertices of a triangle. We refer to the generalized net $N(i, j, 0)$ as the generalized bull, and denote this by $B(i, j)$.

The following well-known conjecture of Matthews and Sumner [9] has provided the impetus for a great deal of research into the hamiltonicity of claw-free graphs.

Conjecture 1.1 (The Matthews-Sumner Conjecture). If $G$ is a 4-connected claw-free graph, then $G$ is hamiltonian.

In [11] Ryjáček demonstrated that this was equivalent to a conjecture of Thomassen [15] that every 4 -connected line graph is hamiltonian. Also in [11], Ryjáček showed that every 7 -connected, claw-free graph is hamiltonian. More recently, in [6], Kaiser and Vrána showed that every 5connected claw-free graph $G$ with minimum degree at least six is hamiltonian, which currently represents the best general progress towards affirming Conjecture 1.1. As the general conjecture has proven difficult, a number of authors have considered the hamiltonicity of $\left\{K_{1,3}, G^{\prime}\right\}$-free graphs for various choices of $G^{\prime}$. These include proofs that every 4 -connected $\left\{K_{1,3}, H\right\}$-free graph is hamiltonian when $H$ is the hourglass [1] or a chain of three triangles [10], as well as results that any 3 -connected $\left\{K_{1,3}, P_{11}\right\}$-free [8] graph is hamiltonian.

In this paper, we are not only interested in the hamiltonicity of highly connected claw-free graphs, but also in their pancyclicity. Significantly fewer results of this type can be found in the literature, in part because it has been shown in many cases $[12,13]$ that closure techniques such as those in [11] do not apply to pancyclicity.

In [14], Shepherd showed the following, which extended a well-known result of Duffus, Gould and Jacobson [2].

Theorem 1.2. Every 3-connected, $\left\{K_{1,3}, N\right\}$-free graph is pancyclic.
Gould, Luczak and Pfender [4] obtained the following characterization of forbidden pairs of subgraphs that imply pancyclicity in 3 -connected graphs. Here L denotes the graph obtained by connecting two disjoint triangles with a single edge.

Theorem 1.3. Let $X$ and $Y$ be connected graphs on at least three vertices such that neither $X$ nor $Y$ are $P_{3}$ and $Y$ is not $K_{1,3}$. Then the following statements are equivalent:

1. Every 3-connected $\{X, Y\}$-free graph $G$ is pancyclic.
2. $X=K_{1,3}$ and $Y$ is a subgraph of one of the graphs from the family

$$
\mathcal{F}=\left\{P_{7}, E, B(4,0), B(3,1), B(2,2), N(2,1,1)\right\} .
$$

The Matthews-Sumner conjecture and Theorem 1.3 together inspire the following general question.

Problem 1. Characterize those pairs of graphs $(X, Y)$ such that every 4 -connected, $(X, Y)$-free graph is pancyclic.

In [3], the following was shown.
Theorem 1.4. Every 4-connected, claw-free, $P_{10}$-free graph is either pancyclic or is the line graph of the Petersen graph. Consequently, every 4 -connected, claw-free, $P_{9}$-free graph is pancyclic.

The line graph of the Petersen graph is 4 -connected and contains no cycle of length 4 (see Figure 1).


Figure 1: The line graph of the Petersen graph.
The main result of this paper is the following, which represents new progress towards Problem 1.

Theorem 1.5. Every 4 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$, is pancyclic.
As the line graph of the Petersen graph is $B(i, j)$-free for all $i+j=7$, this result is best possible in the sense that the condition on $i+j$ could not be increased.

## 2 Proof of Theorem 1.5

Before we proceed, we introduce some additional notation. For the remainder of the paper, we will let $\langle w+x y z\rangle$ denote a $K_{1,3}$ in $G$, induced or otherwise, with center vertex $w$ and pendant vertices $x, y$ and $z$. Also, we let $N\left(x y z ; x_{1} \ldots x_{i}, y_{1} \ldots y_{j}, z_{1} \ldots z_{k}\right)$ denote a copy of $N(i, j, k)$ with central triangle $x y z$ and appended paths $x x_{1} \ldots x_{i}, y y_{1} \ldots y_{j}$, and $z z_{1} \ldots z_{k}$. A copy of the bull $B(i, j)$ is denoted $B\left(x y z ; x_{1} \ldots x_{i}, y_{1} \ldots y_{j}\right)$ where $x y z$ is the central triangle with appended paths $x x_{1} \ldots x_{i}$ and $y y_{1} \ldots y_{j}$.

The following two results allow us to establish the hamiltonicity of the graphs under consideration.

Theorem 2.1 (Hu and Lin [5]). If $G$ is a 3 -connected, $\left\{K_{1,3}, N(5,2,2)\right\}$ - or $\left\{K_{1,3}, N(4,3,2)\right\}$-free graph, then $G$ is hamiltonian.

Theorem 2.2 (Lai et al. [7]). If $G$ is a 3 -connected, $\left\{K_{1,3}, N(8,0,0)\right.$ )\}-free graph, then $G$ is hamiltonian.

By these results, we immediately get the following corollary which provides hamiltonicity of all graphs considered in this paper.

Corollary 2.3. If $G$ is a 3 -connected, $\left\{K_{1,3}, B(6,0)\right\}$-, $\left\{K_{1,3}, B(5,1)\right\}$-, $\left\{K_{1,3}, B(4,2)\right\}$-or $\left\{K_{1,3}, B(3,3)\right\}$-free graph, then $G$ is hamiltonian.

Our strategy for the proof of Theorem 1.5 is to show that for $t \geq 4$ the presence of a $t$-cycle in our graph implies the existence of a $(t-1)$-cycle. In the absence of such a cycle, we show that the graph contains either an induced $K_{1,3}$ or each of $B(6,0), B(5,1), B(4,2)$ and $B(3,3)$. Given a cycle $C$, an edge $x y \notin C$ with $x, y \in V(C)$ is called a chord of $C$, and $x$ and $y$ are called chordal vertices of $C$. A hop is a chord $x y$ of $C$ where there is exactly one vertex between $x$ and $y$ on $C$.

Lemma 2.4. Let $G$ be a 4-connected $K_{1,3}$-free graph containing a cycle $C$ of length $t \geq 4$. If $C$ has a chord or if there is a vertex $w \in G \backslash C$ with at least 4 neighbors on $C$, then $G$ contains another cycle $C^{\prime}$ of length $t-1$.

Proof. Given a cycle $C$, a path $P$ with endpoints $x$ and $y$ such that $V(P) \cap V(C)=\{x, y\}$ shortens $x C y$ if $|V(P)|<|x C y|$. In this case we say that $P$ is a shortening path that covers the arc $x C y$. Note that a chord of $C$ is certainly a shortening path, but other paths may be as well. Let $X$ denote the set of vertices on $C$ that are not incident to a chord of $C$, and call any vertex in $V(C)-X$ a chordal vertex of $C$.

Let $C$ be a cycle as given in the statement of the lemma and note that we may assume $C$ has no hops. We would now like to show that there exist a pair of (not necessarily disjoint) shortening paths of $C$, each of length at most two, that shorten disjoint arcs of $C$. Recall that either $C$ has a chord, or there is some vertex $w \in G-C$ such that $d_{C}(w) \geq 4$. Assume the latter, and note that since $G$ is claw-free and has no hops, each vertex with a neighbor $x$ on $C$ must also be adjacent to either $x^{+}$or $x^{-}$. The assumption that $d_{C}(w) \geq 4$ implies that there must be two pairs of vertices in $N_{C}(w)$ that are consecutive on $C$. Let $w_{1}, w_{1}^{+}, w_{2}$ and $w_{2}^{+}$denote these vertices, and note that $w_{2}$ is neither $w_{1}^{+2}$ nor $w_{1}^{+3}$ and similarly that $w_{1}$ is neither $w_{2}^{+2}$ nor $w_{2}^{+3}$ as any of these possibilities results in a cycle of length $t-1$ in $G$. Thus $w_{1} w w_{2}^{+}$and $w_{1}^{+} w w_{2}$ comprise the desired shortening paths.

If there is no vertex outside $C$ with four neighbors, then by the conditions of the lemma, $C$ must have at least one chord. Among all chords of $C$, choose the chord $x y$ so that $|x C y|$ is a minimum. We will show that we can either find a cycle of length $t-1$ or that there are in fact two vertex-disjoint, non-crossing chords. Now, to avoid the induced claw $\left\langle y+y^{-} x y^{+}\right\rangle$, we must have that $x y^{+} \in E(G)$ as the edge $x y^{-}$would create a chord with $\left|x C y^{-}\right|<|x C y|$. Similarly, to avoid the induced claw $\left\langle x+x^{-} y x^{+}\right\rangle$, we have $x^{-} y \in E(G)$. To avoid the induced claw $\left\langle y+y^{-} x^{-} y^{+}\right\rangle$, either $x^{-} y^{-} \in E(G)$ or $x^{-} y^{+} \in E(G)$ since $C$ has no hops. If $x^{-} y^{-} \in E(G)$, then the cycle $x^{-} y^{-} C^{-} x y^{+} C x^{-}$is the desired cycle of length $t-1$. If $x^{-} y^{-} \notin E(G)$ and $x^{-} y^{+} \in E(G)$, then the chords $x y$ and $x^{-} y^{+}$are the desired vertex-disjoint, non-crossing chords. Note that we can consider these chords as shortening paths that cover disjoint arcs of $C$.

We now select two shortening paths $P_{L}$ and $P_{R}$ of length at most two which cover disjoint arcs of $C$. Let $x_{L}$ and $y_{L}$ (respectively $x_{R}$ and $y_{R}$ ) denote the endpoints of $P_{L}$ (resp. $P_{R}$ ). In particular, assume that $x_{R}, y_{R}, x_{L}$ and $y_{L}$ appear in that order when $C$ is traversed in the clockwise direction where $x_{R} y_{R} \notin E(G)$ and $x_{L} y_{L} \notin E(G)$. We select $P_{L}$ and $P_{R}$ such that $\left|x_{L} C y_{L} \cap X\right|+\left|x_{R} C y_{R} \cap X\right|$
is minimum and, subject to this, such that $\left|x_{L} C y_{L}\right|+\left|x_{R} C y_{R}\right|$ is minimum. As each chord of $C$ is a shortening path, this implies that there is no chord of $C$ with both endpoints in $x_{R} C y_{R}$, with the possible exception of $x_{R} y_{R}$, and we may draw a similar conclusion about $x_{L} C y_{L}$. Finally, without loss of generality suppose that $x_{R} C y_{R}$ contains at least as many vertices of $X$ as $x_{L} C y_{L}$.

Now, let $C_{L}$ denote the cycle $y_{L} C x_{L} P_{L} y_{L}$, that is, the shortening of $C$ obtained via $P_{L}$. Recall that every chordal vertex in $x_{L} C y_{L}$ must have a neighbor in $y_{L}^{+} C x_{L}^{-}$. Thus, as $G$ is claw-free and $C$ has no hops, any chordal vertex $x$ in $x_{L} C y_{L}$ must be adjacent to some vertices $y$ and $y^{+}$in $y_{L} C x_{L}$. Thus, it is possible to increase the length of $C_{L}$ by one by inserting $x$ between $y$ and $y^{+}$. Inserting all chordal vertices from $x_{L} C y_{L}$ into $C_{L}$ allows the creation of cycles of lengths $\left|C_{L}\right|$ to $t-\left|x_{L} C y_{L} \cap X\right|$. If no vertices in $x_{L} C y_{L}$ are also in $X$, then this allows us to construct a $t-1$ cycle in $G$. Thus, we may assume that $x_{L} C y_{L} \cap X$ is nonempty, and recall that since $x_{R} C y_{R}$ contains at least as many vertices of $X$ as $x_{L} C y_{L},\left|C_{L} \cap X\right| \geq\left|x_{L} C y_{L} \cap X\right|$.

We now proceed to extend $C_{L}$ using vertices in $G-C$. Since $G$ has minimum degree at least four, each vertex in $X$ has at least two neighbors in $G-C$. We also claim that by the minimality conditions placed on $P_{L}$ and $P_{R}$, every vertex of $G-C$ can be adjacent to at most three vertices in $x_{R} C y_{R}$ as otherwise there would be a shortening path with one of $x_{R} C y_{R} \cap X$ or $x_{R} C y_{R}$ having smaller cardinality. Further, suppose $v \in G-C$ has three neighbors in $X$ covered by $x_{R} C y_{R}$. Either these three neighbors are consecutive or there is a shortening path that contradicts the minimality of $P_{R}$. Furthermore, $v$ has no other neighbors in $C$ since otherwise $G$ contains an induced claw or $v$ has four consecutive neighbors on $C$.

Let $X \cap x_{R} C y_{R}=\left\{x_{1}, x_{2}, \ldots, x_{l}\right\}$ for some $l$ which by assumption satisfies $l \geq\left|x_{L} C y_{L} \cap X\right|$. Note that each vertex $x_{i}$ has at least two neighbors in $G-C$ and each of these neighbors is adjacent to either $x_{i}^{-}$or $x_{i}^{+}$. We claim that one vertex from $G-C$ can be inserted into $C_{L}$ for each vertex of $X \cap x_{R} C y_{R}$ (which allows us to find cycles of all lengths from $t-\left|x_{L} C y_{L}\right|$ up to our desired length of $t-1$ ). Since each $x_{i}$ has at least two neighbors in $G-C$ that could be inserted, the only way that it is not possible to insert distinct vertices for each $x_{i}$ is if there are consecutive vertices $x^{-}, x$ and $x^{+}$on $C$ such that $N_{G-C}\left(x^{-}, x, x^{+}\right)=\{u, v\}$ for some $u, v$ in $G-C$. Since $G$ is claw-free, we immediately have that $u v$ is an edge in $G$, and that $u$ and $v$ have no other neighbors on $C$. Now, assume that without loss of generality $u$ has some neighbor $u^{\prime} \neq v$ in $G-C$. As $C$ is hop-free, the claw $\left\langle u+u^{\prime} x^{-} x^{+}\right\rangle$implies that $u^{\prime} x^{-}$or $u^{\prime} v^{+}$is an edge in $G$, which contradicts our assumption that $N_{G-C}\left(x^{-}, x, x^{+}\right)=\{u, v\}$. Consequently, the set $\left\{x^{-}, x, x^{+}\right\}$is a cut of size three in $G$, which contradicts our assumption that $G$ is 4 -connected. This completes the proof.

From this result we immediately get the following corollary.
Corollary 2.5. If $G$ is 4 -connected and $\left\{B, K_{1,3}\right\}$-free where $B$ is one of $B(6,0), B(5,1), B(4,2)$ or $B(3,3)$ then $G$ is pancyclic provided all cycles of length at least four contain chords.

We now present some results which will allow us to focus strictly on finding short cycles in order to prove that $G$ is pancyclic.

The first lemma takes advantage of the fact that, via Corollary $2.5, G$ must contain induced cycles. We omit the proof as it is standard.
Lemma 2.6. Let $C=C_{t}$ be an induced cycle in a $K_{1,3}$-free graph $G$ with $t \geq 9$. If there exists $a$ vertex $w \in G-C$ with exactly two neighbors on $C$ then $G$ contains an induced $B(6,0), B(5,1)$, $B(4,2)$ and $B(3,3)$.

The following lemma allows us to find a shorter cycle when a vertex has three or more neighbors on an induced cycle.

Lemma 2.7. Let $C=C_{t}$ for $t \geq 6$ be an induced cycle in a 4-connected $K_{1,3}$-free graph $G$ and suppose that all vertices $v \in G-C$ with $d_{C}(v) \geq 1$ have $d_{C}(v) \geq 3$. Then $G$ contains a cycle of length $t-1$.

Proof. Assume that $G$ does not contain a cycle of length $t-1$, and choose a vertex $w \in G-C$ with $d_{C}(w) \geq 1$. By assumption $w$ must have three neighbors on $C$ and since $G$ is $K_{1,3}$-free and $G$ has no $(t-1)$ cycle, these neighbors must all be consecutive on $C$. Let $v_{1} v_{2} \ldots v_{t}$ denote the vertices of $C$ in order, and let $V_{i}$ denote the set of vertices in $G-C$ which are adjacent to $\left\{v_{i-1}, v_{i}, v_{i+1}\right\}$ where these indices are taken modulo $t$. For $v, w \in V_{i}$, the claw $\left\langle v_{i-1}+v_{i-2} v w\right\rangle$ for $v, w \in V_{i}$ implies that the sets $V_{i}$ must all be complete.
Claim 1. For $w_{i} \in V_{i}, N\left(w_{i}\right) \subseteq\left\{v_{i-1}, v_{i}, v_{i+1}\right\} \cup V_{i-1} \cup V_{i} \cup V_{i+1}$.

Proof. For a contradiction, suppose $z \in N\left(w_{i}\right)$ and $z \notin\left\{v_{i-1}, v_{i}, v_{i+1}\right\} \cup V_{i-1} \cup V_{i} \cup V_{i+1}$. Considering the claw $\left\langle w_{i}+z v_{i-1} v_{i+1}\right\rangle$, we must have either $z v_{i-1}$ or $z v_{i+1}$ in $G$. Without loss of generality, suppose $z v_{i+1} \in E(G)$. By assumption, $z$ must have three consecutive edges to $C$ but since $z \notin V_{i} \cup V_{i+1}$, we must have $z \in V_{i+2}$. Then the cycle $v_{i-1} w_{i} z v_{i+3} C v_{i-1}$ is a $(t-1)$-cycle, a contradiction.

Next we claim that there are at most two sets $V_{i}$ which are empty and furthermore, if $V_{i}$ and $V_{j}$ are both empty with $i<j$, then $j=i+1$. Suppose that the sets $V_{i}$ and $V_{j}$ are empty and $j \neq i+1$. By Claim 1 and the fact that $C$ is induced, the set $\left\{v_{i}, v_{j}\right\}$ forms a 2 -cut of $G$, a contradiction to the assumption that $G$ is 4 -connected. Hence, $j=i+1$ and there can be at most two empty sets.

Since $t \geq 6$ and at most two $V_{i}$ are empty, we may assume without loss of generality that $V_{s} \neq \emptyset$ for $1 \leq s \leq t-2$. Choose a vertex $x_{i}$ in $V_{i}$ for each $1 \leq i \leq t-2$. If $t=2 m$ and $m$ is odd, then $v_{t} x_{1} v_{2} x_{3} v_{4} \ldots v_{\frac{t-2}{2}} x_{\frac{t-2}{2}} v_{\frac{t-4}{2}} \ldots x_{2} v_{1} v_{t}$ is a cycle of length $t-1$ in $G$. If $t=2 m$ and $m$ is even, then $v_{t} x_{1} v_{2} x_{3} v_{4} \ldots x_{\frac{t-2}{2}} v_{\frac{t-2}{2}} v_{\frac{t-4}{2}} \ldots x_{2} v_{1} v_{t}$ is a cycle of length $t-1$ in $G$. Now, if $t=2 m+1$ and $m$ is odd, then $v_{1} x_{2} v_{3} x_{4} \ldots x_{\frac{t-1}{2}} v_{\frac{t-1}{2}} x_{\frac{t-3}{2}} \ldots v_{2} x_{1} v_{1}$ is a cycle of length $t-1$ in $G$. Finally, if $t=2 m+1$ and $m$ is even, then $v_{1} x_{2} v_{3} x_{4} \ldots v_{\frac{t-1}{2}} x_{\frac{t-1}{2}} x_{\frac{t-3}{2}} \ldots v_{2} x_{1} v_{1}$ is a cycle of length $t-1$ in $G$, completing the proof.

From these lemmas we get the following corollary.
Corollary 2.8. If $G$ is a 4-connected $\left\{K_{1,3}, B\right\}$-free graph where $B$ is one of $B(6,0), B(5,1)$, $B(4,2)$ or $B(3,3)$, then $G$ is pancyclic as long as it contains cycles of length four, five, six and seven.

Proof. By Corollary 2.3, $G$ is hamiltonian and since $G$ is 4-connected, no hamiltonian cycle is induced. So, this hamiltonian cycle has a chord, and by Lemma 2.4, $G$ contains a $(n-1)$-cycle. Let $C$ be a $t$-cycle of $G$ for some $9 \leq t \leq n-1$. If $C$ is not induced, then Lemma 2.4 implies the existence of a $(t-1)$-cycle so suppose $C$ is induced and there exists no $(t-1)$-cycle in $G$. Then by Lemmas 2.6-2.7, we obtain an induced copy of $B$, contradiction. Since $G$ is 4 -connected and $K_{1,3}$-free, $G$ clearly contains a triangle and the result follows.

### 2.1 Proof of Theorem 1.5

We first make some general observations which will be used heavily. Let $G$ be a 4 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph where $i+j=6$ and suppose $G$ contains no $C_{t}$ for $4 \leq t \leq 7$. By Theorem 1.4, since the line graph of the Petersen graph contains $B(i, j)$, we may assume there is an induced $P_{10}$ say $P$, in $G$, with vertices $p_{1}, p_{2}, \ldots, p_{10}$.

We also prove another small fact for use in the first few cases.
Fact 2.9. If there is a vertex $v \in G-P$ with three consecutive neighbors on $P$, then $G$ contains $C_{4}, C_{5}$ and $C_{6}$.

Proof. Let $v$ be a vertex in $G-P$ and assume that $p_{i}, p_{i+1}$ and $p_{i+2}$ are elements of $N_{P}(v)$. Further, let $w$ be a neighbor of $p_{i+1}$ in $G-P$ that is distinct from $v$. As $G$ is claw-free, $w$ must also be adjacent to either $p_{i}$ or $p_{i+2}$ and hence if $v$ is also adjacent to either $p_{i+3}$ or $p_{i-1}$ then we obtain cycles of length four, five and six. Thus, we may assume that no vertex in $G-P$ is adjacent to 4 consecutive vertices on $P$.

Without loss of generality, suppose $w p_{i} \in E(G)$. Since $G$ is 4 -connected, $v$ must be adjacent to some vertex $x$ that, as outlined above, does not lie on $P$. To avoid an induced claw centered at $v$, we must have either $x p_{i} \in E(G)$ or $x p_{i+2} \in E(G)$. Either case produces all desired cycles unless $x=w$ so we therefore conclude that $v w \in E(G)$.

At this point, $\left\{p_{i+2}, p_{i}, w\right\}$ comprises a 3 -cut that separates $v$ and $p_{i+1}$ from the rest of the graph. Since $G$ is 4 -connected, there must be another edge from either $v$ or $p_{i+1}$ to a vertex $x \notin\left\{p_{i+2}, p_{i}, w\right\}$. If $x p_{i+1} \in E(G)$ then since $P$ is induced we have that $x \notin P$. Hence either $x p_{i}$ or $x p_{i+2}$ must be in $G$ to avoid a claw, in either case producing all desired cycles. Similarly if $x v \in E(G)$, we also get that either $x p_{i}$ or $x p_{i+2}$ is an edge in $G$, again producing all desired cycles.

The remainder of the proof of Theorem 1.5 is broken into Lemmas 2.10-2.13, each showing the existence of a small cycle.
Lemma 2.10. Every 4 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$, contains a $C_{4}$.
Proof. Let $G$ be a 4 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$ and $i \geq j$, and suppose that there is no $C_{4}$ in $G$. Note that since $G$ is 4 -connected, $K_{1,3}$-free and contains no $C_{4}, G$ must be 4-regular.

As $P$ is induced, each $p_{\ell}, 2 \leq \ell \leq 9$, has at least two neighbors in $G-P$. Since $G$ is $K_{1,3}$-free, each of these neighbors must be adjacent to either $p_{\ell-1}$ or $p_{\ell+1}$. To avoid a $C_{4}$, for each $1 \leq t \leq 9$ there is a vertex $v_{t}$ adjacent to both $p_{t}$ and $p_{t+1}$. Note that these $v_{t}$ may not be distinct. Certainly $v_{t} \neq v_{t+1}, v_{t+2}$, or $v_{t+3}$ as each of these equalities would imply the existence of a $C_{4}$.

The remainder of the proof is broken into cases in which each $B(i, j)$ with $i+j=6$ is forbidden.
Case 1. $i=j=3$.
The bull $B=B\left(p_{5} p_{6} v_{5} ; p_{4} p_{3} p_{2}, p_{7} p_{8} p_{9}\right)$ cannot be induced, and therefore implies that either $v_{5}=v_{1}$ or $v_{5}=v_{9}$, as any other edge in $B$ would result in a $C_{4}$. Suppose without loss of generality that $v_{5}=v_{9}$, so that $v_{5} p_{9}$ and $v_{5} p_{10}$ are edges. As $v_{6} \notin\left\{v_{5}, v_{7}, v_{8}, v_{9}\right\}$ the bull $B_{1}=$
$B\left(p_{6} p_{7} v_{6} ; p_{5} p_{4} p_{3}, p_{8} p_{9} p_{10}\right)$ implies that $v_{6}=v_{2}$. Finally, to avoid a $C_{4}, v_{7}$ is not adjacent to any vertex in $\left\{p_{2}, p_{3}, p_{5}, p_{6}, p_{9}, v_{5}, v_{6}\right\}$. Now, as $v_{7} p_{3}$ and $v_{7} p_{5}$ are not in $G$, we also know that $v_{7} p_{4} \notin E(G)$. However, this means the bull $B\left(p_{5} v_{5} p_{6} ; p_{4} p_{3} p_{2}, p_{9} p_{8} v_{7}\right)$ is induced, a contradiction.

Case 2. $i=4$ and $j=2$.
As the bull $B\left(p_{5} p_{6} v_{5} ; p_{4} p_{3} p_{2} p_{1}, p_{7} p_{8}\right)$ cannot be induced and neither $v_{5} p_{7}$ nor $v_{5} p_{8}$ is in $E(G)$, as either edge would create a $C_{4}$, we have that $v_{5} p_{1}$ (and possibly $v_{5} p_{2}$ ) is in $E(G)$. Similarly, $B\left(p_{6} p_{5} v_{5} ; p_{7} p_{8} p_{9} p_{10}, p_{4} p_{3}\right)$ implies that $v_{5} p_{10}$ (and possibly $\left.v_{5} p_{9}\right)$ is in $E(G)$. However, then $\left\langle v_{5}+\right.$ $\left.p_{1}, p_{5}, p_{10}\right\rangle$ is an induced claw, a contradiction.
Case 3. $i=5$ and $j=1$.
Consider the bull $B\left(p_{4} p_{3} v_{3} ; p_{5} p_{6} p_{7} p_{8} p_{9}, p_{2}\right)$, and note that $v_{3} p_{5}, v_{3} p_{6} \notin E(G)$ as either of these would create a $C_{4}$. We now consider several possible cases. First, if $v_{3}=v_{7}$, then $B\left(p_{3} p_{2} v_{2}\right.$;
$p_{4} p_{5} p_{6} p_{7} p_{8}, p_{1}$ ) must be induced, as any additional edges would create a $C_{4}$ in $G$, a contradiction. If $v_{3}=v_{8}$, then $v_{4} p_{\ell} \notin E(G)$ for all $6 \leq \ell \leq 9$ so that $v_{4} p_{10}$ must be in $E(G)$ lest the bull $B\left(p_{5} p_{4} v_{4} ; p_{6} p_{7} p_{8} p_{9} p_{10}, p_{3}\right)$ is induced. Now since $B\left(p_{3} p_{2} v_{2} ; p_{4} p_{5} p_{6} p_{7} p_{8}, p_{1}\right)$ is not induced, $v_{2}=v_{6}$ since all other edges would produce a $C_{4}$. Then $B\left(v_{2} p_{2} p_{3} ; p_{7} p_{8} p_{9} p_{10} v_{4}, p_{1}\right)$ is necessarily induced, as all edges within this structure would either produce an induced $K_{1,3}$ or a $C_{4}$. Finally, if $v_{3}=v_{9}$, then the bull $B\left(p_{8} p_{9} v_{8} ; p_{7} p_{6} p_{5} p_{4} p_{3}, p_{10}\right)$ is necessarily induced, as otherwise we would again contradict the assumption that $G$ is claw-free and does not contain a $C_{4}$.

Case 4. $i=6$ and $j=0$.
Recall that $v_{1}, v_{2}$, and $v_{3}$ are distinct, and note that for $t \leq 3$ the bulls $B_{t}=B\left(p_{t} p_{t+1} v_{t}\right.$;
$\left.p_{t+2} \ldots p_{t+7}\right)$ imply that $v_{t}$ is adjacent to one of $p_{t+4}, p_{t+5}, p_{t+6}$, or $p_{t+7}$. In particular, we have that $v_{1} \in\left\{v_{5}, v_{6}, v_{7}, v_{8}\right\}, v_{2} \in\left\{v_{6}, v_{7}, v_{8}, v_{9}\right\}$, and also that $v_{3} \in\left\{v_{7}, v_{8}, v_{9}\right\}$ or $v_{3} p_{10} \in E(G)$ but $v_{3} p_{9}$ is not. Note that $v_{1}$ and $v_{2}$ can have no common neighbor on $P$ except for $p_{2}$ (such a neighbor would force a $C_{4}$ ), and similarly $v_{2}$ and $v_{3}$ can have no common neighbor on $P$ except for $p_{3}$. With this in mind, there are several possibilities. We will consider cases based on $v_{3}$. If $v_{3} p_{10} \in E(G)$ but $v_{3} p_{9} \notin E(G)$, then either (i) $v_{1}=v_{5}$ and $v_{2}=v_{7}$, (ii) $v_{1}=v_{5}$ and $v_{2}=v_{8}$, or (iii) $v_{1}=v_{6}$ and $v_{2}=v_{8}$. In (i) and (ii), the bull $B\left(v_{1} p_{1} p_{2} ; p_{6} p_{7} p_{8} p_{9} p_{10} v_{3}\right)$ is induced (as otherwise we get a $C_{4}$ or an induced claw). In (iii), the bull $B\left(v_{3} p_{3} p_{4} ; p_{10} p_{9} p_{8} p_{7} v_{1} p_{1}\right)$ is similarly induced. Now, if $v_{3}=v_{9}$, then either $v_{1}=v_{5}$ and $v_{2}=v_{7}$, which leads to the induced bull $B\left(v_{1} p_{1} p_{2} ; p_{6} p_{7} p_{8} p_{9} v_{3} p_{4}\right)$, or $v_{1}=v_{8}$ and $v_{2}=v_{6}$, which leads to the induced bull $B\left(v_{1} p_{1} p_{2} ; p_{8} p_{7} p_{6} p_{5} p_{4} v_{3}\right)$. The restrictions above on common neighbors between $v_{i}$ and $v_{i+1}$ for $i \in\{1,2\}$ implies that $v_{3} \neq v_{8}$ unless $v_{1}=v_{8}$ as well. However, this immediately leads to a $C_{4}$. Thus, the only remaining possibility is that $v_{3}=v_{7}$. Suppose $v_{3}=v_{7}$. Now, if $v_{1}=v_{5}$ and $v_{2}=v_{9}$, then the bull $B\left(v_{3} p_{7} p_{8} ; p_{4} p_{5} v_{1} p_{2} v_{2} p_{10}\right)$ is induced. If $v_{1}=v_{6}$ and $v_{2}=v_{9}$, then the bull $B\left(p_{6} v_{1} p_{7} ; p_{5} p_{4} p_{3} v_{2} p_{9} p_{8}\right)$ is induced. This final contradiction completes the proof.

Lemma 2.11. Every 4-connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$, contains a $C_{5}$.
Proof. Let $G$ be a 4-connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$ (assume again that $i \geq j$ ) and suppose there is no $C_{5}$ in $G$. As above consider an induced $P_{10}, P=p_{1} \ldots p_{10}$, but note that
we cannot assure the existence of the vertices $v_{1}, \ldots, v_{9}$ here, as we are not prohibiting $C_{4}$ as a subgraph of $G$.

Case 1. $i=j=3$.
We note first that since $P$ is induced, $d_{G-P}\left(p_{5}\right) \geq 2$; let $v$ be one such vertex so that $v$ is also adjacent to either $p_{4}$ or $p_{6}$. Without loss of generality, suppose $v p_{4}$ is an edge of $G$ and observe that by Fact 2.9 neither $v p_{3}$ nor $v p_{6}$ is an edge in $G$. Also, the edges $v p_{i}$ with $i \in\{1,2,7,8\}$ are forbidden as each of these creates a $C_{5}$ in $G$. All other edges between vertices in $B\left(p_{4} p_{5} v ; p_{3} p_{2} p_{1}, p_{6} p_{7} p_{8}\right)$ are forbidden as $P$ is induced, forcing a contradiction.

Case 2. $i=4$ and $j=2$.
Again let $v \in N_{G-P}\left(p_{5}\right)$, and assume first that $v p_{6} \in E(G)$ so that by Fact $2.9, v p_{7}$ and $v p_{4}$ are not in $E(G)$. In order to avoid a $C_{5}$, we also know that $v p_{i} \notin E(G)$ for $i \in\{2,3,8,9\}$. Consideration of $B\left(p_{5} p_{6} v ; p_{4} p_{3} p_{2} p_{1}, p_{7} p_{8}\right)$ implies that we must have $v p_{1} \in E(G)$ and symmetrically, we must also have $v p_{10} \in E(G)$ but this gives us an induced claw centered at $v$ using $p_{1}, p_{5}$ and $p_{10}$, a contradiction. Thus, we may assume $v p_{4} \in E(G)$.

Now, as $B\left(p_{5} p_{4} v ; p_{6} p_{7} p_{8} p_{9}, p_{3} p_{2}\right)$ is not induced, it follows that $v p_{9}$, and hence $v p_{10}$ in $G$. However, then $B\left(v p_{9} p_{10} ; p_{4} p_{3} p_{2} p_{1}, p_{8} p_{7}\right)$ is necessarily induced by Fact 2.9 and the fact that $G$ contains no $C_{5}$.

Case 3. $i=5$ and $j=1$.
Let $v$ and $w$ be vertices in $N_{G-P}\left(p_{3}\right)$, and note that both of $v$ and $w$ are also adjacent to either $p_{2}$ or $p_{4}$. Suppose first that both $v p_{2}$ and $w p_{2}$ are edges in $G$, so that by Fact $2.9, v p_{4} \notin E(G)$ and, to avoid a $C_{5}$, we also do not have $v p_{5}$ or $v p_{6}$ in $G$. Consequently, the bull $B\left(p_{3} p_{2} v ; p_{4} p_{5} p_{6} p_{7} p_{8}, p_{1}\right)$ implies that $v p_{8}$ (and possibly also $v p_{7}$ ) must be an edge of $G$. Similarly, we have that $w p_{8}$ is in $E(G)$ so that $v p_{3} p_{2} w p_{8} v$ is a $C_{5}$ in $G$. The case where $w p_{4}$ and $v p_{4}$ are in $E(G)$ is handled in a nearly identical fashion.

Thus, assume that $v p_{2}$ and $w p_{4}$ are in $G$. As above, we have that $v p_{8}$ is an edge in $G$, and similarly that $w p_{9}$ is as well. Then, $v p_{3} w p_{9} p_{8} v$ is a $C_{5}$ in $G$.

Case 4. $i=6$ and $j=0$.
Let $v$ and $w$ be vertices in $N_{G-P}\left(p_{2}\right)$, and assume first that $w p_{3}$ and $v p_{3}$ are both in $G$. Examination of the bull $B\left(p_{3} p_{2} w ; p_{4} p_{5} p_{6} p_{7} p_{8} p_{9}\right)$ implies that $w$ is adjacent to $p_{7}$ and $p_{8}, p_{8}$ and $p_{9}$, or $p_{9}$ and $p_{10}$, and the bull $B\left(p_{3} p_{2} v ; p_{4} p_{5} p_{6} p_{7} p_{8} p_{9}\right)$ allows us to reach a similar conclusion about $v$. However, $v$ and $w$ must have either common or consecutive neighbors in the subpath of $P$ from $p_{7}$ to $p_{10}$, and this leads to a $C_{5}$ in $G$, a contradiction. If $v p_{1}$ and $w p_{1}$ are edges in $G$, then we reach a similar conclusion and contradiction.

If $v p_{1}$ and $w p_{3}$ are edges in $G$, then $w$ is adjacent to $p_{7}$ and $p_{8}, p_{8}$ and $p_{9}$, or $p_{9}$ and $p_{10}$, and $v$ is adjacent to $p_{6}$ and $p_{7}, p_{7}$ and $p_{8}$, or $p_{8}$ and $p_{9}$. This implies that $v$ and $w$ have either common or consecutive neighbors in the subpath of $P$ from $p_{6}$ to $p_{10}$ unless $v$ is adjacent to $p_{6}$ and $p_{7}$ and $w$ is adjacent to $p_{9}$ and $p_{10}$.

We therefore examine the neighbors of $p_{9}$ in $G-P$, and similarly conclude that there are vertices $v^{\prime}$ and $w^{\prime}$ in $N_{G-P}\left(p_{9}\right)$ such that $w^{\prime}$ is adjacent to $p_{8}, p_{1}$ and $p_{2}$, and $v^{\prime}$ is adjacent to $p_{10}, p_{4}$ and $p_{5}$.

However, as no vertex in $G-P$ has five neighbors on $P, v, w, v^{\prime}$ and $w^{\prime}$ must be distinct vertices so that $w p_{3} p_{2} w^{\prime} p_{9} w$ is a $C_{5}$ in $G$.

Lemma 2.12. Every 4 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$, contains a $C_{6}$.
Proof. Let $G$ be a 4 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$ (assume $i \geq j$ ) and suppose there is no $C_{6}$ in $G$.

Case 1. Either $i=j=3$ or $i=4$ and $j=2$.
Choose $v \in N_{G-P}\left(p_{5}\right)$ so that $v$ must also be adjacent to either $p_{4}$ or $p_{6}$. We may assume $v p_{4} \in E(G)$ as the case where $v p_{6}$ is in $G$ is handled in a nearly identical manner. Since neither $B\left(p_{5} p_{4} v ; p_{6} p_{7} p_{8}, p_{3} p_{2} p_{1}\right)$ nor $B\left(p_{5} p_{4} v ; p_{6} p_{7} p_{8} p_{9}, p_{3} p_{2}\right)$ may be induced, we must get that either $v p_{2} \in E(G)$ or $v p_{7} \in E(G)$ since all other edges would produce a $C_{6}$. However, by Fact $2.9, v$ is adjacent to neither $p_{3}$ nor $p_{6}$, which implies (as $P$ is induced and $G$ is claw-free) that either $v$ is adjacent to $p_{1}$ and $p_{2}$, or is adjacent to $p_{7}$ and $p_{8}$. In both cases, $C_{6} \in G$, a contradiction.
Case 2. $i=5$ and $j=1$.
Let $v$ be a neighbor of $p_{3}$ in $G-P$ and suppose that $v p_{2} \in E(G)$ (the case where $v p_{4} \in E(G)$ is identical). Fact 2.9 and the assumption that $G$ has no $C_{6}$ imply that $v$ is not adjacent to any vertex in $\left\{p_{1}, p_{4}, p_{5}, p_{6}, p_{7}\right\}$. Since $B\left(p_{3} p_{2} v ; p_{4} p_{5} p_{6} p_{7} p_{8}, p_{1}\right)$ is not induced, we must have the $v p_{8}$, and hence $v p_{9}$ in $E(G)$. Now let $w \neq v$ be another vertex in $N_{G-P}\left(p_{3}\right)$ so that again $w$ must be adjacent to either $p_{2}$ or $p_{4}$. If $w p_{2} \in E(G)$, then by the same argument, $w p_{8}, w p_{9} \in E(G)$ and hence $w p_{8} p_{9} v p_{2} p_{3} w$ is a $C_{6}$ in $G$. If $w p_{4} \in E(G)$, then $w p_{9}$ and $w p_{10}$ are edges in $G$, so that $v p_{3} w p_{10} p_{9} p_{8} v$ is a $C_{6}$.

Case 3. $i=6$ and $j=0$.
Let $v \in N_{G-P}\left(p_{2}\right)$ and assume that $v p_{1} \in E(G)$. The case when $v p_{3} \in E(G)$ can be handled in a similar manner. Fact 2.9 and the assumption that $G$ contains no $C_{6}$ imply that $v$ also cannot be adjacent to any vertex in $\left\{p_{3}, p_{4}, p_{5}, p_{6}\right\}$.

Since the bull $B\left(p_{2} p_{1} v ; p_{3} p_{4} p_{5} p_{6} p_{7} p_{8}\right)$ cannot be induced, we must have $v p_{8}$ (and possibly also $v p_{7}$ ) in $E(G)$. Now let $w \neq v$ be another neighbor of $p_{8}$ in $G-P$. Then $w$ is also adjacent to either $p_{9}$ or $p_{7}$. Suppose that $w p_{9} \in E(G)$. An argument similar to the above yields that $w p_{2} \in G$, implying the existence of the $C_{6}$ given by $w p_{2} p_{1} v p_{8} p_{9} w$. As the case when $w p_{7} \in E(G)$ is similar, this completes the proof.

Lemma 2.13. Every 4 -connected $\left\{K_{1,3}, B(i, j)\right\}$-free graph, where $i+j=6$, contains a $C_{7}$.
Proof. Suppose that $G$ is a 4 -connected, claw-free graph that does not contain a $C_{7}$. We once again consider an induced $P_{10}, P=p_{1} \ldots p_{10}$.

Claim 2. If a vertex $v$ in $G-P$ is adjacent to vertices $p_{\ell}, p_{\ell+1}, p_{t}$ and $p_{t+1}$ with $\ell+1<t$, then $7 \leq|\ell-t| \leq 8$.

Proof. Let $v$ be a vertex in $G-P$ adjacent to $p_{\ell}, p_{\ell+1}, p_{t}$, and $p_{t+1}$ with $\ell<t$, and assume to the contrary that $2 \leq|\ell-t| \leq 6$. If $4 \leq|\ell-t| \leq 7$, then $G$ immediately contains a $C_{7}$, so we may suppose that $2 \leq|\ell-t| \leq 3$. If $|\ell-t|=3$, then since $G$ is 4 -connected, there is some vertex $x \neq v$ in $N_{G-P}\left(p_{\ell+2}\right)$. Since $G$ is claw-free and $P$ is induced, $x$ is either adjacent to $p_{\ell+1}$ or $p_{t}$, so that either $v p_{\ell} p_{\ell+1} x p_{\ell+2} p_{t} p_{t+1} v$ or $v p_{\ell} p_{\ell+1} p_{\ell+2} x p_{t} p_{t+1} v$ is a $C_{7}$ in $G$.

Thus, we may assume that $t=\ell+2$, namely that $v$ is adjacent to $p_{\ell}, p_{\ell+1}, p_{\ell+2}$ and $p_{\ell+3}$. Since $G$ is 4 -connected, $p_{\ell}$ and $p_{\ell+3}$ cannot separate $v, p_{\ell+1}$ and $p_{\ell+2}$ from the remainder of $G$. We therefore have that there are distinct vertices $u_{1}, u_{2} \in\left\{v, p_{\ell+1}, p_{\ell+2}\right\}$ and distinct vertices $y_{1}$ and $y_{2}$ in $G-P-v$ such that $u_{1} y_{1}$ and $u_{2} y_{2}$ are edges in $G$. Since $G$ is claw-free, if $u_{1}=p_{\ell+1}$, then $y_{1}$ is adjacent to either $p_{\ell}$ or $p_{\ell+2}$ and if $u_{1}=p_{\ell+2}$, then $y_{1}$ is adjacent to $p_{\ell+1}$ or $p_{\ell+3}$. Similarly, if $u_{1}=v$, then $y_{1}$ is adjacent to at least one vertex in each of $\left\{p_{\ell}, p_{\ell+2}\right\},\left\{p_{\ell}, p_{\ell+3}\right\}$, and $\left\{p_{\ell+1}, p_{\ell+3}\right\}$. We reach identical conclusions if $u_{2}$ is each of $p_{\ell+1}, p_{\ell+2}$ or $v$.

For any choices of $u_{1}$ and $u_{2}$, these additional edges immediately imply that $G$ contains a $C_{7}$, except in the case where, without loss of generality, $u_{1}=p_{\ell+1}, u_{2}=p_{\ell+2}$ and both $y_{1} p_{\ell+2}$ and $y_{2} p_{\ell+1}$ are edges in $G$. However, in this case, the claw $\left\langle p_{\ell+1}+y_{1} y_{2} v\right\rangle$ implies that either $y_{1} y_{2} \in E(G)$ or, without loss of generality, $y_{1} v \in E(G)$. If $y_{1} y_{2} \in E(G)$, then $v p_{\ell} p_{\ell+1} y_{1} y_{2} p_{\ell+2} p_{\ell+3} v$ is a $C_{7}$ in $G$. If $y_{1} v$ is an edge in $G$, then $\left\langle v+p_{\ell} y_{1} p_{\ell+3}\right\rangle$ implies that $y_{1}$ is either adjacent to $p_{\ell}$ or $p_{\ell+3}$. Either possibility implies the existence of a $C_{7}$ in $G$.

Claim 3. If there are vertices $v$ and $x$ in $G$ such that $v$ is adjacent to $p_{\ell}, p_{\ell+1}$ and $p_{\ell+2}$, and $x$ is adjacent to $p_{\ell}$ and $p_{\ell+2}$, then $G$ contains a $C_{7}$.

Proof. By symmetry, we may assume that $\ell>1$. Claim 2 and the claw $\left\langle p_{\ell}+v x p_{\ell-1}\right\rangle$ then together imply that $v x$ is an edge in $G$. As $G$ is 4 -connected, $p_{\ell}$ and $p_{\ell+2}$ cannot separate $\left\{v, x, p_{\ell+1}\right\}$ from the remainder of $G$. Therefore, there are distinct vertices $y_{1}$ and $y_{2}$ in $G-(P \cup\{v, x\})$ and distinct vertices $u_{1}, u_{2} \in\left\{x, v, p_{\ell+1}\right\}$ such that $u_{1} y_{1}, u_{2} y_{2} \in E(G)$. Since each of $x, v$, and $p_{\ell+1}$ are adjacent to $p_{\ell}$ and $p_{\ell+2}$, each of $y_{1}$ and $y_{2}$ is adjacent to at least one of $p_{\ell}$ and $p_{\ell+2}$ as well. Subject to these observations, it is straightforward to check that any way the neighbors of $y_{1}$ and $y_{2}$ are chosen from $\left\{p_{\ell}, p_{\ell+1}, p_{\ell+2}, x, v\right\}$, we obtain a $C_{7}$ in $G$.

Case 1. $i=6$ and $j=0$
By Claim 2, no vertex in $G-P$ has four consecutive neighbors on $P$. We now claim that there is no vertex $v$ in $G-P$ that is adjacent to $p_{1}, p_{2}$, and $p_{3}$. Indeed, assume otherwise, and consider the bull $B\left(p_{3} p_{2} v ; p_{4} p_{5} p_{6} p_{7} p_{8} p_{9}\right)$ which, since $G$ contains no $C_{7}$ and $v$ cannot be adjacent to $p_{4}$, must be induced unless $v p_{9}$ is in $G$. However, then $\left\langle v+p_{1} p_{3} p_{9}\right\rangle$ is necessarily induced, a contradiction.

As $P$ is induced, $p_{1}$ has three neighbors in $G-P$, call them $v_{1}, v_{2}$ and $v_{3}$. Suppose first that none of $v_{1}, v_{2}$ or $v_{3}$ is adjacent to $p_{2}$, which implies that $v_{1} v_{2} v_{3}$ must be a triangle in $G$. Now, consider the bull $B\left(p_{1} v_{1} v_{2}\right.$;
$p_{2} p_{3} p_{4} p_{5} p_{6} p_{7}$ ), which, since neither $v_{1}$ nor $v_{2}$ is adjacent to $p_{2}$, would imply that $G$ contains a $C_{7}$ unless (without loss of generality) $v_{1}$ is adjacent to $p_{7}$. To avoid an induced claw or a $C_{7}$ in $G, v_{1}$ must also be adjacent to $p_{8}$. Now the bull $B\left(p_{1} v_{2} v_{3} ; p_{2} p_{3} p_{4} p_{5} p_{6} p_{7}\right)$ also implies that (without loss of generality) $v_{2}$ is adjacent to $p_{7}$ and $p_{8}$.

Symmetrically, $p_{10}$ must also have three neighbors in $G-P$, call them $x_{1}, x_{2}$, and $x_{3}$. Note that $x_{i} \neq v_{1}$ for any $i$, as then $v_{1}$ would be adjacent to $p_{1}, p_{6}$, and $p_{10}$, forming an induced claw in $G$.

As $x_{i}$ is similarly not equal to $v_{2}$ for any $i$, we may assume without loss of generality that $v_{1}$ and $v_{2}$ are not any of $x_{1}, x_{2}$, or $x_{3}$. Since $G$ contains no $C_{7}, x_{1}$ and $x_{2}$ are immediately not adjacent to $p_{5}$. If $x_{1}$ (or equivalently $x_{2}$ ) is adjacent to $p_{6}$, then $x_{1} p_{6} p_{7} v_{1} p_{8} p_{9} p_{10} x_{1}$ is a $C_{7}$ in $G$.

Assume that either $x_{1}$ or $x_{2}$ is adjacent to $p_{9}$, say $x_{1}$, and consider the bull $B\left(p_{9} p_{10} x_{1}\right.$; $\left.p_{8} p_{7} p_{6} p_{5} p_{4} p_{3}\right)$. Recall that no vertex in $G-P$ is adjacent to $p_{1}, p_{2}$, and $p_{3}$. Since $p_{1}$ and $p_{10}$ behave symmetrically, there is also no vertex in $G-P$ that is adjacent to $p_{8}, p_{9}$, and $p_{10}$. In particular, as $x_{1} p_{9}, x_{1} p_{10} \in E(G), x_{1}$ cannot be adjacent to $p_{8}$. As $x_{1}$ is also not adjacent to $p_{6}$ and $G$ is claw-free, we conclude that $x_{1} p_{7} \notin E(G)$ as well. Finally, $x_{1} p_{4} \notin E(G)$ as it would create the $C_{7}$ given by $x_{1} p_{4} p_{5} p_{6} p_{7} p_{8} p_{9} x_{1}$. So, we must have $x_{1} p_{3}, x_{1} p_{2} \in E(G)$, but this provides a contradiction as we now have the $C_{7}$ given by $x_{1} p_{2} p_{1} v_{2} p_{7} p_{8} p_{9} x_{1}$.

Thus, we may conclude that neither $x_{1}$ nor $x_{2}$ is adjacent to $p_{9}$, so that the claw $\left\langle p_{10}+x_{1} x_{2} p_{9}\right\rangle$ implies that $x_{1} x_{2}$ is an edge in $G$. We now consider the bull $B\left(p_{10} x_{1} x_{2} ; p_{9} p_{8} p_{7} p_{6} p_{5} p_{4}\right)$ which is induced unless, without loss of generality, $x_{1}$ has a neighbor in $\left\{p_{4}, \ldots, p_{9}\right\}$. By assumption, $x_{1}$ is not adjacent to $p_{9}$, and either $x_{1} p_{5}$ or $x_{1} p_{6}$ would form a $C_{7}$ in $G$. Since $v_{1}$ is adjacent to both $p_{7}$ and $p_{8}$, the vertex $x_{1}$ cannot be adjacent to $p_{7}$ and $p_{8}$ as this forms a $C_{7}$. Therefore, $x_{1}$ must be adjacent to $p_{3}$ and $p_{4}$. However, then the bull $B\left(x_{1} x_{2} p_{10} ; p_{4} p_{5} p_{6} p_{7} v_{1} p_{1}\right)$ is necessarily induced, as every possible edge within this substructure either creates a $C_{7}$ or an induced claw.

We may therefore suppose that some vertex in $N_{G-P}\left(p_{1}\right)$, say $v_{1}$, is adjacent to $p_{2}$. As we have already ruled out the possibility that $v_{1} p_{3} \in E(G)$, the bull $B\left(p_{2} p_{1} v_{1} ; p_{3} p_{4} p_{5} p_{6} p_{7} p_{8}\right)$ is induced unless $v_{1}$ is adjacent to either $p_{4}$ and $p_{5}$ or to $p_{8}$ and $p_{9}$. Since $p_{4}$ and $p_{5}$ would contradict Claim 2 , we may assume $v_{1}$ is adjacent to $p_{8}$ and $p_{9}$.

Note then that $v_{1}$ is not adjacent to $p_{10}$, as then the claw $\left\langle v_{1}+p_{1} p_{8} p_{10}\right\rangle$ is induced. By symmetry, there is some neighbor $v$ of $p_{10}$ that is also adjacent to $p_{9}$ and also by a symmetric argument, $v$ must be adjacent to $p_{3}$ and $p_{2}$. However, then $v_{1} p_{1} p_{2} p_{3} v p_{10} p_{9} v_{1}$ is a $C_{7}$ in $G$, the final contradiction that completes this case.

Case 2. $i=5$ and $j=1$
Again by Claim 2, no vertex in $G$ is adjacent to four consecutive vertices on $P$. We next wish to show that there is no vertex $v$ in $G-P$ such that $N_{P}(v)=\left\{p_{2}, p_{3}, p_{4}\right\}$. Assume otherwise, and let $v$ be such a vertex and, since $G$ is 4 -connected and $v$ cannot have any other neighbors on $P$, there is some vertex $x \in N_{G-P}(v)$. The claw $\left\langle v+x p_{2} p_{4}\right\rangle$ implies that $x$ must be adjacent to $p_{2}$ or $p_{4}$.

Suppose first that $x$ is adjacent to $p_{4}$ but is not adjacent to $p_{2}$ and consider the bull $B\left(v p_{4} x ; p_{2}\right.$, $\left.p_{5} p_{6} p_{7} p_{8} p_{9}\right)$. Now, $x$ cannot be adjacent to any vertex in $\left\{p_{6}, p_{7}, p_{8}, p_{9}\right\}$ by Claim 2 and the assumption that $G$ is claw-free. Since $x$ is not adjacent to $p_{2}$, we have that $B$ is induced unless $x p_{5}$ is an edge in $G$. Given that $G$ is 4 -connected, $p_{2}$ and $p_{5}$ cannot separate $\left\{p_{3}, p_{4}, x, v\right\}$ from the rest of $G$. Thus, there is some vertex $y$, distinct from $p_{2}$ and $p_{5}$, with a neighbor in $\left\{p_{3}, p_{4}, x, v\right\}$. However, since $P$ is induced and $x$ is not adjacent to $p_{2}$, any neighbor of $y$ in this set forces $y$ to be adjacent to consecutive vertices on the $C_{6}$ given by $x p_{5} p_{4} p_{3} p_{2} v x$, forming a $C_{7}$ in $G$. Similarly, if $x$ is adjacent to $p_{2}$ but not $p_{4}$, the bull $B\left(p_{2} v x ; p_{1}, p_{4} p_{5} p_{6} p_{7} p_{8}\right)$ implies that $x$ is either adjacent to $p_{1}$ or $p_{5}$ and again we can use the connectivity of $G$ to demonstrate the existence of a $C_{7}$ in $G$. Thus we have that $x$ is adjacent to both $p_{2}$ and $p_{4}$, contradicting Claim 3 and implying that there is no vertex $v$ in $G-P$ that is adjacent to $p_{2}, p_{3}$ and $p_{4}$. A nearly identical argument yields that there is no vertex $v$ in $G-P$ that is adjacent to $p_{3}, p_{4}$ and $p_{5}$.

Now consider a vertex $w \in G-P$ that is adjacent to $p_{4}$, and note that $w$ is adjacent to either $p_{3}$ or $p_{5}$, but not both. If $w p_{3}$ is in $E(G)$, then the bull $B\left(p_{3} p_{4} w ; p_{2}, p_{5} p_{6} p_{7} p_{8} p_{9}\right)$ is induced unless $w$ is adjacent to $p_{6}$ and $p_{7}$, contradicting Claim 2. If $w p_{5}$ is an edge in $G$, then by Claim 2 and the fact that $w$ is not adjacent to $p_{3}$, the bull $B\left(p_{4} p_{5} w ; p_{3}, p_{6} p_{7} p_{8} p_{9} p_{10}\right)$ is induced unless $w$ is adjacent to $p_{6}$. Symmetrically, we may assume that there is some vertex $w^{\prime}$ in $G-P$ that is adjacent to $p_{7}, p_{6}$ and $p_{5}$. As $G$ is 4 -connected and $\left\{p_{4}, p_{7}\right\}$ would separate $\left\{w, w^{\prime}, p_{5}, p_{6}\right\}$ from the rest of $G$, one of these four vertices must have a neighbor $w^{\prime \prime}$ in $G-P$. As $G$ is claw-free, the vertex $w^{\prime \prime}$ is adjacent to one of the following pairs of vertices: $p_{5}$ and $p_{6}, w$ and $p_{4}, w$ and $p_{6}, w^{\prime}$ and $p_{5}$, or $w^{\prime}$ and $p_{7}$. In each of these cases, $G$ necessarily contains a $C_{7}$ unless $w^{\prime \prime}$ is adjacent to $p_{5}$ and $p_{6}$. However, then the claw $\left\langle p_{5}+w w^{\prime} w^{\prime \prime}\right\rangle$ implies that one of the edges $w w^{\prime}, w w^{\prime \prime}$, or $w^{\prime} w^{\prime \prime}$ is in $G$. Each of these edges implies that $G$ contains a copy of $C_{7}$, as desired.
Case 3. $i=4$ and $j=2$
This case proceeds in a manner nearly identical to that for $B(5,1)$, and so we only provide a sketch here in the interest of concision. Using Claim 3, one can show that there is no vertex in $G-P$ adjacent to $p_{i}, p_{i+1}$ and $p_{i+2}$ for $3 \leq i \leq 6$. We then consider a vertex $v$ in $G-P$ that is adjacent to $p_{5}$, and therefore also to one of either $p_{4}$ or $p_{6}$. By Claim 2, if $v$ is adjacent to $p_{4}$, then $B\left(p_{4} p_{5} v ; p_{3} p_{2}, p_{6} p_{7} p_{8} p_{9}\right)$ is induced, and if $v$ is adjacent to $p_{6}$, then $B\left(p_{6} p_{5} v ; p_{7} p_{8}, p_{4} p_{3} p_{2} p_{1}\right)$ is induced. In both cases, we have a contradiction.
Case 4. $i=j=3$
Using Claims 2 and 3 , along with an argument similar to those in the previous cases, we have that no vertex in $G-P$ is adjacent to $p_{4}, p_{5}$, and $p_{6}$, or adjacent to $p_{5}, p_{6}$, and $p_{7}$. We therefore consider a vertex $v$ in $N_{G-P}\left(p_{5}\right)$, which is necessarily also adjacent to either $p_{4}$ or $p_{6}$. If $v$ is adjacent to $p_{6}$, then, as $v$ cannot also be adjacent to $p_{4}$ or $p_{7}$, the bull $B\left(p_{5} p_{6} v ; p_{4} p_{3} p_{2}, p_{7} p_{8} p_{9}\right)$ is necessarily induced.

Thus, we may assume that $v$ is adjacent to $p_{4}$ and $p_{5}$, and more so that there is no vertex in $G-P$ adjacent to both $p_{5}$ and $p_{6}$. Considering the bull $B\left(p_{4} p_{5} v ; p_{3} p_{2} p_{1}, p_{6} p_{7} p_{8}\right)$, we conclude that $v p_{3}$ is an edge in $G$, and that $v$ has no additional edges on $P$. Thus, since $d_{G}(v) \geq 4$, there is some vertex $x$ in $N_{G-P}(v)$ and as $G$ is claw-free, $x$ is also adjacent to either $p_{3}$ or $p_{5}$. If $x$ is adjacent to $p_{5}$, then since $x$ cannot be adjacent to $p_{6}$, the edge $x p_{4}$ is also in $G$. However, then the bull $B\left(p_{4} p_{5} x ; p_{3} p_{2} p_{1}, p_{6} p_{7} p_{8}\right)$ is necessarily induced, as $x$ cannot be adjacent to $p_{3}$ by Claim 3 .

So, assume $x p_{5} \notin E(G)$ and $x p_{3} \in E(G)$. Since $d_{G}\left(p_{5}\right) \geq 4$, there is some vertex $y \neq v$ in $N_{G-P}\left(p_{5}\right)$. As $y$ cannot be adjacent to both $p_{5}$ and $p_{6}$, we have that $y p_{4}$ is an edge of $G$. However, then the claw $\left\langle p_{5}+p_{6} y v\right\rangle$ implies that $y v$ is an edge of $G$, so that there is some neighbor of $v$ adjacent to $p_{4}$ and $p_{5}$, a possibility that has been prohibited. This is the final contradiction that completes the proof of the lemma.

From Lemmas 2.10-2.13 and Corollary 2.8, we immediately obtain Theorem 1.5.

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