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KANNAN TYPE MAPPING IN TVS-VALUED CONE METRIC SPACES AND THEIR APPLICATION TO URYSOHN INTEGRAL EQUATIONS

AKBAR AZAM AND ISMAT BEG

Abstract. We obtain sufficient conditions for the existence of a common fixed point of three mappings satisfying Kannan type conditions in TVS valued cone metric spaces. We also give an application by finding the solution for a system of two Urysohn integral equations. Our results generalize several well-known recent results in the literature.

1. Introduction and preliminaries

A system $x = T_i x$ $(i \in \Omega)$, of operator equations has one or more simultaneous solutions obtained by using the common fixed point technique. Recently Beg et al [5, 3, 8, 11, 12], studied common fixed points of a pair of maps on topological vector space (TVS) valued cone metric spaces. The class of TVS cone metric spaces is larger than class of cone metric spaces, used in [1, 2, 9, 10, 14, 15, 16, 17]. In this paper we obtain common fixed points and points of coincidence of three mappings in TVS-valued cone metric spaces without the assumption of normality. As an application we prove the existence of the unique solution of a system of two Urysohn integral equations. Our results improve and generalize several contemporary and recent results in the literature (e.g., see [1, 6, 9, 13, 15, 19]).

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction [4] if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y), \tag{1}$$

for all $x, y \in X$. Mapping T is called Kannan [13] if there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \alpha \left[d(x, Tx) + d(y, Ty)\right], \tag{2}$$

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for all $x, y \in X$. The main difference between contraction and Kannan mappings is that "contractions are always continuous where as Kannan mappings are not necessarily continuous. The Banach contraction theorem [4] is an extremely dynamic tool in mathematical analysis. However, the Kannan fixed point theorem [13] is imperative because it characterizes completeness of metric spaces [18], while Banach theorem cannot characterize the metric completeness of X [7]. the Banach type contractive condition (i.e. $d(Sx,Ty) \leq kd(x,y)$), for a pair $S, T : X \to X$ of mappings implies that both S and T are equal, whereas, the condition

$$d(Sx,Ty) \leq k_1 \left[d(x,Sx) + d(y,Ty)\right],$$

does not assert that $S = T$. Thus Kannan type conditions are useful to find common fixed point of a pair of nonlinear operators. An other type of contractive condition, due to Chatterjea [6] , is based on an assumption analogous to Kannan condition (2): there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that

$$d\left(Tx,Ty\right) \leq \alpha \left[d\left(x,Ty\right) + d\left(y,Tx\right)\right], \tag{3}$$

for all $x, y \in X$. It is well-known that the Banach contractions, Kannan mappings and Chatterjea mappings are independent in general.

Let (E, τ) be a topological vector space (TVS) and P a subset of E. Then, P is called a cone whenever

(i) P is closed, non-empty and $P \neq \{\theta\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b.
(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. A cone P is called solid if $intP$ is nonempty.

Definition 1. [3, 5] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d_1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,
(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(d_3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a TVS-valued cone metric on X and (X, d) is called a TVS-valued cone metric space.

If E is a real Banach space then (X, d) is called cone metric space [9].

Definition 2. [5] Let (X, d) be a TVS-valued cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete TVS valued cone metric space if every Cauchy sequence is convergent.

A pair (F, T) of self-mappings on X is said to be weakly compatible if $FTx = TFx$ whenever $Fx = Tx$. A point $y \in X$ is called point of coincidence of a family T_j, $j \in J$, of self-mappings on X if there exists a point $x \in X$ such that $y = T_j x$ for all $j \in J$.

Lemma 3. [2] *Let X be a nonempty set and the mappings $S, T, F : X \to X$ have a unique point of coincidence $v \in X$. If (S, F) and (T, F) are weakly compatible, then S, T and F have a unique common fixed point.*

2. Common fixed point

Theorem 4. *Let (X, d) be a complete TVS-valued cone metric space, P be a solid cone, and mappings $S, T, F : X \to X$ satisfy:*

$$d(Sx, Ty) \preceq Ad(Fx, Sx) + Bd(Fy, Ty), \tag{4}$$

for all $x, y \in X$, where A, B are non-negative real numbers with $A + B < 1$. If

$$S(X) \cup T(X) \subseteq F(X),$$

and $F(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S, T and F have a unique point of coincidence. Moreover if (S, F) and (T, F) are weakly compatible, then S, T and F have a unique common fixed point.

Proof. We shall first show that, if S, T and F have a point of coincidence, then it is unique. For this, assume that there exist two distinct points of coincidence v, v^* of mappings S, T and F in X. It follows that there exists $u, u^* \in X$ such that

$$v = Fu = Su = Tu,$$

and

$$v^* = Fu^* = Su^* = Tu^*.$$

From (4), we obtain

$$\begin{aligned} d(v, v^*) &= d(Su, Tu^*) \\ &\preceq Ad(Fu, Su) + Bd(Fu, Tu) \\ &\preceq (A + B)\, d(v, v^*), \end{aligned}$$

it implies that

$$v = v^*, \text{ a contradiction.}$$

Now, we prove the existence of a point of coincidence of the mappings S, T and F. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $Fx_1 = Tx_0$. This can be done since $S(X) \cup T(X) \subseteq F(X)$. Similarly choose a point x_2 in X such that $Fx_2 = Sx_1$. Continuing this process having chosen x_n in X, we obtain x_{n+1} in X such that

$$\begin{aligned} Fx_{2n+1} &= Tx_{2n} \\ Fx_{2n+2} &= Sx_{2n+1}, \quad n \geq 0. \end{aligned}$$

Suppose there exists n such that $Fx_{2n} = Fx_{2n+1}$. Then $Fx_{2n} = Tx_{2n}$ and from (4)

$$\begin{aligned} d\left(Fx_{2n}, Sx_{2n}\right) &= d\left(Fx_{2n+1}, Sx_{2n}\right) \\ &= d\left(Tx_{2n}, Sx_{2n}\right) \\ &\preceq Ad(Fx_{2n}, Sx_{2n}) + Bd(Fx_{2n}, Tx_{2n}) \\ &\preceq Ad(Fx_{2n}, Sx_{2n}) + Bd(Fx_{2n}, Fx_{2n+1}) \\ &\preceq Ad(Fx_{2n}, Sx_{2n}), \end{aligned}$$

which yields $Fx_{2n} = Sx_{2n}$ and so, $Fx_{2n} = Sx_{2n} = Tx_{2n} = y$ (say) is the required unique point of coincidence of F, S and T. Similarly, if $Fx_{2n+1} = Fx_{2n+2}$ for some n. Then $Fx_{2n+1} = Sx_{2n+1} = Tx_{2n+1} = y$ is the required point. Thus in this sequel of proof we can suppose that $Fx_n \neq Fx_{n+1}$ for all n. From (4)

$$\begin{aligned} d(Fx_{2n}, Fx_{2n+1}) &= d(Sx_{2n-1}, Tx_{2n}) \\ &\preceq Ad(Fx_{2n-1}, Sx_{2n-1}) + Bd(Fx_{2n}, Tx_{2n}) \\ &\preceq Ad(Fx_{2n-1}, Fx_{2n}) + Bd(Fx_{2n}, Fx_{2n+1}) \\ &\preceq \frac{A}{1-B} d(Fx_{2n-1}, Fx_{2n}) \\ &\preceq \max\left\{\frac{B}{1-A}, \frac{A}{1-B}\right\} d(Fx_{2n-1}, Fx_{2n}), \end{aligned}$$

and

$$\begin{aligned} d(Fx_{2n-1}, Fx_{2n}) &= d(Tx_{2n-2}, Sx_{2n-1}) \\ &\preceq A(Fx_{2n-1}, Sx_{2n-1}) + Bd\left(Fx_{2n-2}, Tx_{2n-2}\right) \\ &\preceq A(Fx_{2n-1}, Fx_{2n}) + Bd\left(Fx_{2n-2}, Fx_{2n-1}\right) \\ &\preceq \frac{B}{1-A} d\left(Fx_{2n-2}, Fx_{2n-1}\right) \end{aligned}$$

$$\preceq \max\left\{\frac{B}{1-A}, \frac{A}{1-B}\right\} d\left(Fx_{2n-2}, Fx_{2n-1}\right).$$

It implies that

$$d(Fx_{2n}, Fx_{2n+1}) \preceq \lambda d(Fx_{2n-1}, Fx_{2n}),$$

where $\lambda = \max\left\{\frac{B}{1-A}, \frac{A}{1-B}\right\}$. As $Fx_n \neq Fx_{n+1}$ and $A + B < 1$, therefore $0 < \lambda < 1$, and for all n,

$$\begin{aligned} d(Fx_n, Fx_{n+1}) &\preceq \lambda d(Fx_{n-1}, Fx_n) \\ &\preceq \lambda^2 d(Fx_{n-2}, Fx_{n-1}) \preceq \cdots \preceq \lambda^n d(Fx_0, Fx_1), \end{aligned}$$

Now for any $m > n$,

$$\begin{aligned} d(Fx_m, Fx_n) &\preceq d(Fx_n, Fx_{n+1}) + d(Fx_{n+1}, Fx_{n+2}) + \cdots + d(Fx_{m-1}, Fx_m) \\ &\preceq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right] d(Fx_0, Fx_1) \\ &\preceq \left[\frac{\lambda^n}{1-\lambda}\right] d(Fx_0, Fx_1). \end{aligned}$$

Let $\theta \ll c$ be given, choose a symmetric neighborhood V of θ such that $c + V \subseteq \text{int}P$. Also, choose a natural number N_1 such that $\left[\frac{\lambda^n}{1-\lambda}\right] d(Fx_0, Fx_1) \in V$, for all $n \geq N_1$. Then, $\frac{\lambda^n}{1-\lambda} d(Fx_1, Fx_0) \ll c$, for all $n \geq N_1$. Thus,

$$d(Fx_m, Fx_n) \preceq \left[\frac{\lambda^n}{1-\lambda}\right] d(Fx_0, Fx_1) \ll c,$$

for all $m > n$. Therefore, $\{Fx_n\}_{n\geq 1}$ is a Cauchy sequence. Since FX is complete, there exist $u \in X, v \in FX$ such that $Fx_n \to v = Fu$ (this holds also if $S(X) \cup T(X)$ is complete with $v \in S(X) \cup T(X)$). Choose a natural number N_2 such that for all $n \geq N_2$

$$d(Fx_{n+1}, Fx_n) \ll \frac{c\,(1-B)}{2A} \text{ and } d(Fx_{n+1}, Fu) \ll \frac{c\,(1-B)}{2}.$$

Then for all $n \geq N_2$

$$\begin{aligned} d(Fu, Tu) &\preceq d(Fu, Fx_{2n+2}) + d(Fx_{2n+2}, Tu) \\ &\preceq d(Fu, Fx_{2n+2}) + d(Sx_{2n+1}, Tu) \\ &\preceq d(Fu, Fx_{2n+2}) + Ad(Fx_{2n+1},\ Fx_{2n+2}) + Bd(Fu, Tu) \\ &\preceq \frac{1}{1-B} d(Fu, Fx_{2n+2}) + \frac{A}{1-B} d(Fx_{2n+1},\ Fx_{2n+2}) \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Thus

$$d(Fu, Tu) \ll \frac{c}{m}, \text{ for all } \ m \geq 1.$$

So, $\frac{c}{m} - d(Fu, Tu) \in P$, for all $m \geq 1$. Since $\frac{c}{m} \to \theta$ (as $m \to \infty$) and P is closed, $-d(Fu, Su) \in P$. But $d(Fu, Tu) \in P$, therefore, $d(Fu, Tu) = \theta$. Hence

$$v = Fu = Tu,$$

and

$$d\,(Fu, Su) = d\,(Tu, Su) \preceq Ad(Fu, Su) + Bd(Fu, Tu) = Ad(Fu, Su),$$

implies that v is a unique point of coincidence of F, S and T. If (S, F) and $(T, F\ )$ are weakly compatible, then by Lemma 3, v is a unique common fixed point of S, T and F. □

Example 5. Let $X = 1, 2, 3$ and E be the set of all real valued functions on $[0, 1]$ which also have continuous derivatives on X. Then E is a vector space over $\mathbb{R}$ under the following operations:

$$(x + y)\,(t) = x\,(t) + y\,(t)\,, \quad (\alpha x)\,(t) = \alpha x\,(t)\,,$$

for all $x, y \in E, \alpha \in \mathbb{R}$. Let τ be the strongest vector (locally convex) topology on E. Then (X, τ) is a topological vector space which is not normable and is not even metrizable. Define $d : X \times X \to E$ as follows:

$$d(x, y)\,(t) = \begin{cases} 0 & \text{if } x = y \\ e^{\left(\ln \frac{4}{7} + t\right)} & \text{if } x \neq y \text{ and } x, y \in X - \{2\} \\ e^{t} & \text{if } x \neq y \text{ and } x, y \in X - \{3\} \\ e^{(t - \ln 2)} & \text{if } x \neq y \text{ and } x, y \in X - \{1\}. \end{cases}$$

Let $P = \{x \in E : x\,(t) \geq 0 \text{ for all } t\}$. Then (X, d) is a TVS-valued cone metric space. Define a mappings $F, T : X \to X$ as follows:

$$T(x) = \begin{cases} 3 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}\ , \ F\,(x) = x$$

Note that, for all $t \in [0, 1]$ and for $\alpha < \frac{1}{2}$

$$d(T(3), T(2)\,(t) = d(3, 1)\,(t) = e^{\left(\ln \frac{4}{7} + t\right)},$$

$$\begin{aligned} &\alpha\,[d(F\,(3)\,, T(3))\,(t) + d(F\,(2)\,, T(2))\,(t)] \\ &= \alpha\,[d(3, T(3))\,(t) + d(2, T(2))\,(t)] \\ &= \alpha\,[d(2, 1)\,(t)] = \alpha e^{t} \\ &< d(T(3), T(2))\,(t)\,. \end{aligned}$$

Therefore the previous relevant results on fixed points [9, 13, 15, 19] and on common fixed points [1] are not applicable to obtain fixed point of T and common fixed point of F and T. In order to apply Theorem 4, consider the mapping $Sx = 3$ for each $x \in X$. Then,

$$d(Sx, Ty)\,(t) = \begin{cases} 0 & \text{if } y \neq 2 \\ e^{\left(\ln \frac{4}{7} + t\right)} & \text{if } y = 2 \end{cases}$$

and for $B = \frac{4}{7}$

$$Bd(Fy, Ty)\,(t) = \frac{4}{7}e^{t} \text{ if } y = 2.$$

It follows that F, S and T satisfy all conditions of Theorem 4 for $A = 0, B = \frac{4}{7}$ and we obtain $F\,(3) = T(3) = S(3) = 3$.

In the following we use a Chatterjea type condition to obtain point of coincidence and common fixed point of three mappings on a TVS-valued cone metric space.

Theorem 6. *Let (X, d) be a complete TVS-valued cone metric space, P be a solid cone, and mappings $S, T, F : X \to X$ satisfy:*

$$d(Sx, Ty) \preceq Cd(Fy, Sx) + Dd(Fx, Ty), \tag{5}$$

for all $x, y \in X$, where C, D are non-negative real numbers with $C + D < 1$. If

$$S(X) \cup T(X) \subseteq F(X),$$

and $F(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S, T and F have a unique point of coincidence. Moreover if (S, F) and (T, F) are weakly compatible, then S, T and F have a unique common fixed point.

Proof. It can be easily seen that if S, T and F have a point of coincidence, then it is unique. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $Fx_1 = Tx_0$. This can be done since $S(X) \cup T(X) \subseteq F(X)$. Similarly choose a point x_2 in X such that $Fx_2 = Sx_1$. Continuing this process having chosen x_n in X, we obtain x_{n+1} in X such that

$$\begin{aligned} Fx_{2n+1} &= Tx_{2n} \\ Fx_{2n+2} &= Sx_{2n+1}, \quad n \geq 0. \end{aligned}$$

Suppose there exists n such that $Fx_{2n} = Fx_{2n+1}$. Then using (5), we obtain $Fx_{2n} = Sx_{2n} = Tx_{2n} = y$ (say) is the required unique point of coincidence of F, S and T. Similarly, if $Fx_{2n+1} = Fx_{2n+2}$ for some n. Then $Fx_{2n+1} = Sx_{2n+1} = Tx_{2n+1} = y$ is the required point. Thus in this sequel of proof we can suppose that $Fx_n \neq Fx_{n+1}$. From (5), we obtain

$$\begin{aligned} d(Fx_{2n}, Fx_{2n+1}) &= d(Sx_{2n-1}, Tx_{2n}) \\ &\preceq Cd(Fx_{2n}, Sx_{2n-1}) + Dd(Fx_{2n-1}, Tx_{2n}) \\ &\preceq D\left[d(Fx_{2n-1}, Fx_{2n}) + d(Fx_{2n}, Fx_{2n+1})\right] \\ &\quad \frac{D}{1-D} d(Fx_{2n-1}, Fx_{2n}), \end{aligned}$$

and

$$\begin{aligned} d(Fx_{2n-1}, Fx_{2n}) &= d(Tx_{2n-2}, Sx_{2n-1}) \\ &\preceq Cd(Fx_{2n-2}, Sx_{2n-1}) + Dd\left(Fx_{2n-1}, Tx_{2n-2}\right) \\ &\preceq Cd(Fx_{2n-2}, Fx_{2n}) \\ &\preceq C\left[d(Fx_{2n-2}, Fx_{2n-1}) + d(Fx_{2n-1}, Fx_{2n})\right] \\ &\preceq \frac{C}{1-C} d(Fx_{2n-2}, Fx_{2n-1}). \end{aligned}$$

It follows that

$$\begin{aligned} d(Fx_{2n}, Fx_{2n+1}) &\preceq \frac{D}{1-D} d(Fx_{2n-1}, Fx_{2n}) \\ &\preceq \frac{D}{1-D} \frac{C}{1-C} d(Fx_{2n-2}, Fx_{2n-1}) \\ &\preceq \left[\frac{D}{1-D} \frac{C}{1-C}\right]^n d(Fx_o, Fx_1), \end{aligned}$$

and

$$\begin{aligned} d(Fx_{2n+1}, Fx_{2n+2}) &\preceq \frac{C}{1-C} d(Fx_{2n}, Fx_{2n+1}) \\ &\preceq \frac{C}{1-C}\left[\frac{D}{1-D} \frac{C}{1-C}\right]^n d(Fx_o, Fx_1). \end{aligned}$$

Let

$$\alpha = \frac{C}{1-C}, \quad \beta \frac{C}{1-C},$$

then, as $Fx_n \neq Fx_{n+1}$ and $C + D < 1$,

$$0 < \alpha\beta = \frac{C}{1-C} \frac{D}{1-D} = \frac{D}{1-C} \frac{C}{1-D} < 1.$$

Now for $p < q$ we have,

$$\begin{aligned}
d(Fx_{2p+1}, Fx_{2q+1}) &\preceq d(Fx_{2p+1}, Fx_{2p+2}) + d(Fx_{2p+2}, Fx_{2p+3}) \\
&\quad + \cdots + d(Fx_{2q}, Fx_{2q+1}) \\
&\preceq \alpha\,[\alpha\beta]^p\, d(Fx_o, Fx_1) + [\alpha\beta]^{p+1}\, d(Fx_o, Fx_1) \\
&\quad + \cdots + [\alpha\beta]^q\, d(Fx_o, Fx_1) \\
&\preceq \left[\alpha \sum_{i=p}^{q-1} (\alpha\beta)^i + \sum_{i=p+1}^{q} (\alpha\beta)^i\right] d(Fx_0, Fx_1) \\
&\preceq \left[\begin{array}{c} \frac{\alpha(\alpha\beta)^p[1-\alpha\beta]^{q-p}}{1-\alpha\beta} \\ +\frac{(\alpha\beta)^{p+1}[1-\alpha\beta]^{q-p}}{1-\alpha\beta} \end{array}\right] d(Fx_0, Fx_1) \\
&\preceq \left[\frac{\alpha(\alpha\beta)^p}{1-\alpha\beta} + \frac{(\alpha\beta)^{p+1}}{1-\alpha\beta}\right] d(Fx_0, Fx_1) \\
&\preceq (1+\beta)\left[\frac{\alpha(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1),
\end{aligned}$$

$$d(Fx_{2p},\ Fx_{2q+1}) \preceq (1+\alpha)\left[\frac{(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1),$$

$$d(Fx_{2p}, Fx_{2q}) \preceq (1+\alpha)\left[\frac{(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1),$$

and

$$d(Fx_{2p+1}, Fx_{2q}) \preceq (1+\beta)\left[\frac{\alpha(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1).$$

Hence, for $0 < n < m$

$$d(Fx_n, Fx_m) \preceq \left[\frac{2(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1)$$

where p is the integer part of $\frac{n}{2}$. Let $\theta \ll c$ be given, choose a symmetric neighborhood V of θ such that $c + V \subseteq \operatorname{int}P$. Since

$$\lim_{p\to\infty} \left[\frac{2(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1) = \theta,$$

there exists a natural number N_1 such that

$$\left[\frac{2(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1) \in V,$$

for all $p \geq N_1$ and so

$$\left[\frac{2(\alpha\beta)^p}{1-\alpha\beta}\right] d(Fx_0, Fx_1) \ll c, \text{ for all } p \geq N_1.$$

Consequently, for all $n, m \in \mathbb{N}$, with $2N_1 < n < m$, we have

$$d(Fx_n, Fx_m) \ll c.$$

Therefore, $\{Fx_n\}_{n\geq 1}$ is a Cauchy sequence. Since FX is complete, there exist $u \in X, v \in FX$ such that $Fx_n \to v = Fu$ (this hold also if $S(X) \cup T(X)$ is complete with $v \in S(X) \cup T(X)$). Choose a natural number N_2 such that for all $n \geq N_2$

$$d(Fx_{n+1}, Fu) \ll \frac{c}{2M},$$

where $M = \max\left\{\frac{1+C}{1-D}, \frac{D}{1-D}\right\}$. Then for all $n \geq N_2$

$$\begin{aligned}
d(Fu, Tu) &\preceq d(Fu, Fx_{2n+2}) + d(Fx_{2n+2}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + d(Sx_{2n+1}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + Cd(Fu, Sx_{2n+1}) + Dd(Fx_{2n+1}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + Cd(Fu, Fx_{2n+2}) + Dd(Fx_{2n+1}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + Cd(Fu, Fx_{2n+2}) \\
&+ D\left[d\left(Fx_{2n+1}, Fu\right) + d\left(Fu, Tu\right)\right] \\
&\preceq \left(\frac{1+C}{1-D}\right) d(Fu, Fx_{2n+2}) + \frac{D}{1-D} d(Fx_{2n+1}, Fu) \\
&\preceq Md(Fu, Fx_{2n+2}) + Md(Fx_{2n+1}, Fu) \ll \frac{c}{2} + \frac{c}{2} = c.
\end{aligned}$$

By a similar argument $Fu = Tu = Su$, which implies that v is a unique point of coincidence of F, S and T. If (S, F) and (T, F) are weakly compatible, then by Lemma 3, v is a unique common fixed point of S, T and F.

□

The following example shows that the above theorem is an improvement and a real generalization of results [1, 6, 9, 13, 15, 19].

Example 7. Let (X, d) be the TVS-valued cone metric space of Example 5. Define a mappings $F, T : X \to X$ as follows:

$$T(x) = \begin{cases} 1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2. \end{cases}, \quad F(x) = x.$$

Note that,for all $t \in [0, 1]$ and for $\alpha < \frac{1}{2}$

$$d(T(3), T(2)(t) = d(1, 3)(t) = e^{\left(\ln \frac{4}{7} + t\right)},$$

$$\begin{aligned}
&\alpha\left[d(F(3), T(2))(t) + d(F(2), T(3))(t)\right] \\
&= \alpha\left[d(3, T(2))(t) + d(2, T(3))(t)\right] = \alpha e^t \\
&< d(T(3), T(2))(t),
\end{aligned}$$

Therefore the previous relevant results on fixed points [6, 9, 15, 19] and on common fixed points [1] are not applicable to obtain fixed point of T and common fixed point of F and T. In order to apply the Theorem 6, consider the mapping $Sx = 1$ for each $x \in X$. Then,

$$d(Sx, Ty)(t) = \begin{cases} 0 & \text{if } y \neq 2 \\ e^{\left(\ln \frac{4}{7} + t\right)} & \text{if } y = 2 \end{cases}$$

and for $D = \frac{4}{7}$

$$Dd(Fx, Ty)(t) = \frac{4}{7} e^t \quad \text{if } y = 2.$$

It follows that S and T satisfy all conditions of Theorem 6 and we obtain $F(1) = T(1) = S(1) = 1$.

3. Application

In this section we prove an existence theorem for the common solution for two Urysohn integral equations. Let $X = C([a, b], \mathbb{R}^n)$,$E$ is a topological vector space of Example 5

$$P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [0, 1]\},$$

and $d : X \times X \to E$ is defined as follows:

$$d(x, y)(t) = (\|x - y\|_\infty) e^t.$$

It is easily seen that (X, d) is a complete TVS-valued cone metric space.

Theorem 8. *Consider the Urysohn integral equations*

$$x(t) = \int_a^b K_1(t, s, x(s))ds + g(t), \tag{6}$$

$$x(t) = \int_a^b K_2(t, s, x(s))ds + h(t), \tag{7}$$

where $t \in [a, b] \subset \mathbb{R}$, $x, g, h \in X$.

Suppose that $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for each $x \in X$, where

$$F_x(t) = \int_a^b K_1(t, s, x(s))ds, \quad G_x(t) = \int_a^b K_2(t, s, x(s))ds \text{ for all } t \in [a, b].$$

If there exist $0 < h < 1$ such that for every $x, y \in X$

$$\left(\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty\right) e^t \leq hM(x, y) e^t,$$

for all $x, y \in X$, where

$$M(x, y) = \max \left\{ \begin{array}{c} \|F_x(t) + g(t) - x(t\|_\infty, \|G_y(t) + h(t) - y(t)\|_\infty, \\ \|F_x(t) + g(t) - y(t\|_\infty, \|G_y(t) + h(t) - x(t)\|_\infty \end{array} \right\}.$$

Then the system of integral equations (6) and (7) have a unique common solution.

Proof. Define $F, S, T : X \to X$ by

$$F\,(x) = x, \quad S(x) = F_x + g, \quad T((x) = G_x + h.$$

If

$$M(x, y) = \|F_x\,(t) + g(t) - x(t\|_\infty\,,$$

it is easily seen that

$$(\|S - T\|_\infty)e^t \leq h\,(\|S\,(x) - x\|_\infty)\,e^t$$

for every $x, y \in X$. By Theorem 4 if $A = h, B = 0$, the Urysohn integral equations (6) and (7) have a unique common solution. If

$$M(x, y) = \|G_y\,(t) + h(t) - y(t)\|_\infty\,,$$

then

$$(\|S - T\|_\infty)e^t \leq h\,(\|T\,(y) - y\|_\infty)\,e^t$$

for every $x, y \in X$. Again by Theorem 4 if $A = 0, B = h$, the Urysohn integral equations (6) and (7) have a unique common solution. Similarly in other cases the result follows by Theorem 6. □

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Akbar Azam
COMSATS Institute of
Information Technology
Islamabad- 44000
Pakistan
akbarazam@yahoo.com

Ismat Beg
Lahore School of Economics
Lahore-53200
Pakistan
begismat@yahoo.com