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Research Article

Some Nonlinear Gronwall-Bellman-Gamidov Integral Inequalities and Their Weakly Singular Analogues with Applications

Kelong Cheng,[1] Chunxiang Guo,[2] and Min Tang[3]

[1] *School of Science, Southwest University of Science and Technology, Mianyang 621010, China*
[2] *School of Business, Sichuan University, Chengdu 610064, China*
[3] *School of Preparatory Education, Southwest University for Nationalities, Chengdu 610041, China*

Correspondence should be addressed to Chunxiang Guo; guocx70@163.com

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Some Gronwall-Bellman-Gamidov type integral inequalities with power nonlinearity and their weakly singular analogues are established, which can give the explicit bound on solution of a class of nonlinear fractional integral equations. An example is presented to show the application for the qualitative study of solutions of a fractional integral equation with the Riemann-Liouville fractional operator.

1. Introduction

Integral inequalities, which provide explicit bounds on unknown functions, play a fundamental role in the development of the theory of linear and nonlinear differential equation and integral equation. One of the best known and widely used inequalities is the so-called Gronwall-Bellman integral inequality. In view of the important applications of the Gronwall-Bellman inequality, in the past few years, Pachpatte [1–3] established a number of new generalizations of such inequality which can be used as powerful tools in the study of certain new classes of differential and integral equations. Meanwhile, many authors have researched various generalizations of the Gronwall-Bellman inequality; for example, we refer the reader to [4–12].

In [6], Baĭnov and Simeonov discussed the following useful integral inequality:

$$u(t) \le c + \int_0^t f(s)\,u(s)\,ds + \int_0^T g(s)\,u(s)\,ds, \tag{1}$$

which came from the study of the boundary value problem for higher order differential equations by Gamidov [13] and was extended by Pachpatte [2] as follows:

$$u(t) \le c + \int_0^t f(t,s)\,u(s)\,ds + \int_0^T g(t,s)\,u(s)\,ds. \tag{2}$$

Remark 1. It should be noted that the derived result in [2] is not right. In the proof, it involves the definition of $z(\alpha)$, which was treated as a constant by mistake. For example, consider the following integral equation:

$$u(t) = 1 + \int_0^1 tsu(s)\,ds, \tag{3}$$

in which $c = 1$, $f(t,s) = 0$, $g(t,s) = ts$, and $T = 1$. We can obtain the solution of the equation above; that is, $u(t) = (3/4)t + 1$. According to the formula of its upper bound reported in [2], one gets that $u(t) = (3/4)t+1 \le 1/(1-(1/2)t)$ for $t \in [0,1]$. Clearly, the result does not hold for $t \in [0,1]$. Hence, a revised one will be provided in later section.

On the other hand, Zheng [14] also established a weakly singular version of the Gronwall-Bellman-Gamidov inequality as follows:

$$\begin{aligned} u(t) \le C &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,u(s)\,ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s)\,u(s)\,ds, \end{aligned} \tag{4}$$

and discussed its application in a fractional integral equation with the modified Riemann-Liouville derivative. As for weakly singular inequalities and their applications, more results can be found (e.g., see [15–23] and the references therein).

In this paper, motivated by the work in [2, 6, 14], we consider a Gronwall-Bellman-Gamidov integral inequality with power nonlinearity,

$$u^m(t) \le a(t) + b(t)\int_0^t f(s)u^n(s)\,ds + c(t)\int_0^T g(s)u^r(s)\,ds, \tag{5}$$

and its weakly singular analogue

$$u^m(t) \le a(t) + b(t)\int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1-1}s^{\gamma_1-1}f(s)u^n(s)\,ds + c(t)\int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2-1}s^{\gamma_2-1}g(s)u^r(s)\,ds, \tag{6}$$

where $m \ge n \ge 0$, $m \ge r \ge 0$, and $[\alpha_i, \beta_i, \gamma_i]$ $(i = 1, 2)$ is the ordered parameter group. The presented inequalities can be used as a handy tool in the qualitative as well as quantitative analysis of solutions of certain fractional differential equation and integral equation. Furthermore, an application of our result to certain fractional integral equation with the Riemann-Liouville (R-L) fractional operator is also involved.

2. Nonlinear Gronwall-Bellman-Gamidov Inequalities

Throughout this paper, R denotes the set of real numbers, $R_+ = [0, +\infty)$, $R_0 = (0, +\infty)$, and $I = [0, T]$ ($T \ge 0$ is a constant). $C(X, Y)$ denotes the collection of continuous functions from X to Y.

We firstly give some lemmas, which will be used in the proof of the main results.

Lemma 2 (see [10]). *Let $a \ge 0$, $m \ge n \ge 0$, and $m \ne 0$. Then*

$$a^{n/m} \le \frac{n}{m}K^{(n-m)/m}a + \frac{m-n}{m}K^{n/m} \tag{7}$$

for any $K > 0$.

Lemma 3. *Suppose $u(t)$, $m(t)$, $n(t)$, and $l(t) \in C(I, R_+)$. If*

$$u(t) \le m(t) + n(t)\int_0^T l(s)u(s)\,ds, \tag{8}$$

then

$$u(t) \le m(t) + \frac{n(t)\int_0^T m(s)l(s)\,ds}{1 - \int_0^T n(s)l(s)\,ds}, \tag{9}$$

for $t \in I$, provided that $\int_0^T n(s)l(s)ds < 1$.

Proof. Let $k = \int_0^T l(s)u(s)ds$. Obviously, k is a constant. It follows from (8) that

$$u(t) \le m(t) + n(t)k, \tag{10}$$

which yields

$$l(t)u(t) \le m(t)l(t) + n(t)l(t)k. \tag{11}$$

Integrating (11) with respect to t from 0 to T, we have

$$k = \int_0^T l(s)u(s)\,ds \le \int_0^T m(s)l(s)\,ds + k\int_0^T n(s)l(s)\,ds. \tag{12}$$

It is easy to observe that

$$k \le \frac{\int_0^T m(s)l(s)\,ds}{1 - \int_0^T n(s)l(s)\,ds}, \tag{13}$$

provided that $\int_0^T n(s)l(s)ds < 1$. Substituting the inequality above into (10), we get (9). □

Lemma 4. *Suppose $u(t)$, $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t) \in C(I, R_+)$. If $a(t)$, $b(t)$, and $c(t)$ are nondecreasing and $u(t)$ satisfies*

$$u(t) \le a(t) + b(t)\int_0^t f(s)u(s)\,ds + c(t)\int_0^T g(s)u(s)\,ds, \tag{14}$$

then

$$u(t) \le \left[a(t) + c(t) \times \frac{\int_0^T a(s)g(s)\exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds}{1 - \int_0^T c(s)g(s)\exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds}\right] \times \exp\left\{b(t)\int_0^t f(\sigma)\,d\sigma\right\}, \tag{15}$$

for $t \in I$, provided that

$$\int_0^T c(s)g(s)\exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds < 1. \tag{16}$$

Proof. Fix any T^*, $0 < T^* < T$; then for $0 \le t \le T^*$, we have

$$u(t) \le a(T^*) + b(T^*)\int_0^t f(s)u(s)\,ds + c(T^*)\int_0^T g(s)u(s)\,ds, \tag{17}$$

since $a(t)$, $b(t)$, and $c(t)$ are nondecreasing. Define $z(t)$ by the right side of (17); then $u(t) \le z(t)$,

$$z(0) = a(T^*) + c(T^*)\int_0^T g(s)u(s)\,ds := \mathscr{Z}(0, T^*), \tag{18}$$

$$z'(t) = b(T^*)f(t)u(t) \le b(T^*)f(t)z(t) \tag{19}$$

for $0 \le t \le T^*$. From (19), we have

$$\frac{z'(t)}{z(t)} \le b(T^*) f(t). \quad (20)$$

Letting $t = \sigma$ in (20) and integrating it with respect to σ from 0 to T^*, we get

$$\begin{aligned} z(T^*) &\le z(0)\exp\left\{b(T^*)\int_0^{T^*} f(\sigma)\,d\sigma\right\} \\ &= \mathscr{Z}(0,T^*)\exp\left\{b(T^*)\int_0^{T^*} f(\sigma)\,d\sigma\right\}. \end{aligned} \quad (21)$$

Since T^* is arbitrary, from (21) with T^* replaced by t and $u(t) \le z(t)$, we have

$$\begin{aligned} u(t) \le z(t) &\le z(0)\exp\left\{b(t)\int_0^{t} f(\sigma)\,d\sigma\right\} \\ &= \mathscr{Z}(0,t)\exp\left\{b(t)\int_0^{t} f(\sigma)\,d\sigma\right\}, \end{aligned} \quad (22)$$

where

$$z(0) = \mathscr{Z}(0,t) = a(t) + c(t)\int_0^T g(s)u(s)\,ds. \quad (23)$$

According to (22), it follows from (23) that

$$\begin{aligned} \mathscr{Z}(0,t) &= a(t) + c(t)\int_0^T g(s)u(s)\,ds \\ &\le a(t) + c(t)\int_0^T g(s)\mathscr{Z}(0,s) \\ &\quad \times \exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds. \end{aligned} \quad (24)$$

Applying Lemma 3, we have

$$\mathscr{Z}(0,t) \le a(t) + c(t)\frac{\int_0^T a(s)g(s)\exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds}{1-\int_0^T c(s)g(s)\exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds}. \quad (25)$$

Substituting the inequality above into (20), we can get (13). The proof is complete. □

Remark 5. Even if $a(t)$, $b(t)$, and $c(t)$ are not nondecreasing, the result also holds, since we can replace it by $\overline{a}(t) = \max_{0\le s\le t} a(s)$, $\overline{b}(t) = \max_{0\le s\le t} b(s)$, and $\overline{c}(t) = \max_{0\le s\le t} c(s)$.

Remark 6. Pachpatte [24] also discussed inequality (14) and derived a slightly complicated bound, but the formula of bound of $u(t)$ in our lemma is quite simple and can be extended easily.

Theorem 7. *Let $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t)$ be defined as in Lemma 4. Suppose that $u(t) \in C(I, R_+)$ satisfies* (5). *If*

$$\int_0^T \mathscr{C}(s)\mathscr{G}(s)\exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds < 1, \quad (26)$$

then

$$\begin{aligned} u(t) &\le \Bigg\{a(t) + \Bigg[\mathscr{A}(t) + \mathscr{C}(t) \\ &\quad \times \left(\int_0^T \mathscr{A}(s)\mathscr{G}(s) \exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds\right) \\ &\quad \times \left(1 - \int_0^T \mathscr{C}(s)\mathscr{G}(s) \exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds\right)^{-1}\Bigg] \\ &\quad \times \exp\left\{\mathscr{B}(t)\int_0^t \mathscr{F}(\sigma)\,d\sigma\right\}\Bigg\}^{1/m}, \end{aligned} \quad (27)$$

for $t \in I$, where $m \ge n \ge 0$, $m \ge r \ge 0$, m, n, and r are constants, and

$$\begin{aligned} \mathscr{A}(t) &= b(t)\int_0^t f(s)\left[\frac{n}{m}K_1^{(n-m)/m}(s)a(s) + \frac{m-n}{m}K_1^{n/m}(s)\right]ds \\ &\quad + c(t)\int_0^T g(s)\left[\frac{r}{m}K_2^{(r-m)/m}(s)a(s) + \frac{m-r}{m}K_2^{r/m}(s)\right]ds, \\ \mathscr{B}(t) &= \frac{n}{m}b(t), \qquad \mathscr{C}(t) = \frac{r}{m}c(t), \\ \mathscr{F}(t) &= f(t)K_1^{(n-m)/m}(t), \qquad \mathscr{G}(t) = g(t)K_2^{(r-m)/m}(t), \end{aligned} \quad (28)$$

for any $K_i(t) \in C(I, R_0)$ $(i = 1, 2)$.

Proof. Letting

$$v(t) = b(t)\int_0^t f(s)u^n(s)\,ds + c(t)\int_0^T g(s)u^r(s)\,ds, \quad (29)$$

from (5), we have

$$u^m(t) \le a(t) + v(t), \quad (30)$$

or

$$u(t) \le (a(t) + v(t))^{1/m}. \quad (31)$$

Applying Lemma 2, for any $K_i(t) \in C(I, R_0)$ $(i = 1, 2)$, we get

$$\begin{aligned} u^n(t) &\le (a(t)+v(t))^{n/m} \\ &\le \frac{n}{m}K_1^{(n-m)/m}(t)(a(t)+v(t)) + \frac{m-n}{m}K_1^{n/m}(t), \\ u^r(t) &\le (a(t)+v(t))^{r/m} \\ &\le \frac{r}{m}K_2^{(r-m)/m}(t)(a(t)+v(t)) + \frac{m-r}{m}K_2^{r/m}(t). \end{aligned} \tag{32}$$

Substituting (32) into (29), we get

$$\begin{aligned} v(t) &\le b(t)\int_0^t f(s)\left[\frac{n}{m}K_1^{(n-m)/m}(s)(a(s)+v(s)) + \frac{m-n}{m}K_1^{n/m}(s)\right]ds \\ &\quad + c(t)\int_0^T g(s)\left[\frac{r}{m}K_2^{(r-m)/m}(s)(a(s)+v(s)) + \frac{m-r}{m}K_2^{r/m}(s)\right]ds \\ &= b(t)\int_0^t f(s)\left[\frac{n}{m}K_1^{(n-m)/m}(s)a(s) + \frac{m-n}{m}K_1^{n/m}(s)\right]ds \\ &\quad + c(t)\int_0^T g(s)\left[\frac{r}{m}K_2^{(r-m)/m}(s)a(s) + \frac{m-r}{m}K_2^{r/m}(s)\right]ds \\ &\quad + \frac{n}{m}b(t)\int_0^t f(s)K_1^{(n-m)/m}(s)v(s)\,ds \\ &\quad + \frac{r}{m}c(t)\int_0^T g(s)K_2^{(r-m)/m}(s)v(s)\,ds \\ &= \mathscr{A}(t) + \mathscr{B}(t)\int_0^t \mathscr{F}(s)v(s)\,ds \\ &\quad + \mathscr{C}(t)\int_0^T \mathscr{G}(s)v(s)\,ds, \end{aligned} \tag{33}$$

which is similar to (14), where $\mathscr{A}(t)$, $\mathscr{B}(t)$, $\mathscr{C}(t)$, $\mathscr{F}(t)$, and $\mathscr{G}(t)$ are defined as in (28). Clearly, $\mathscr{A}(t)$, $\mathscr{B}(t)$, $\mathscr{C}(t)$, $\mathscr{F}(t)$, $\mathscr{G}(t) \in C(I, R_+)$ and $\mathscr{A}(t)$, $\mathscr{B}(t)$, $\mathscr{C}(t)$ are nondecreasing since $a(t)$, $b(t)$, $c(t)$ are nondecreasing, respectively.

Applying Lemma 4 to (33), we have

$$\begin{aligned} v(t) &\le \Bigg[\mathscr{A}(t) + \mathscr{C}(t) \\ &\quad \times \frac{\int_0^T \mathscr{A}(s)\mathscr{G}(s)\exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds}{1-\int_0^T \mathscr{C}(s)\mathscr{G}(s)\exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds}\Bigg] \\ &\quad \times \exp\left\{\mathscr{B}(t)\int_0^t \mathscr{F}(\sigma)\,d\sigma\right\}. \end{aligned} \tag{34}$$

From (31) and (34), we get (27). □

When $m = 2$, $n = r = 1$ in Theorem 7, a Gronwall-Bellman-Pachpatte-Ou-Iang type inequality is obtained as follows.

Corollary 8. *Let $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t)$ be defined as in Lemma 4. Suppose that $u(t) \in C(I, R_+)$ satisfies*

$$u^2(t) \le a(t) + b(t)\int_0^t f(s)u(s)\,ds + c(t)\int_0^T g(s)u(s)\,ds. \tag{35}$$

If

$$\int_0^T \frac{1}{2}c(s)\widetilde{G}(s)\exp\left\{\frac{1}{2}b(s)\int_0^s \widetilde{F}(\sigma)\,d\sigma\right\}ds < 1, \tag{36}$$

then

$$\begin{aligned} u(t) &\le \Bigg\{a(t) + \Bigg[\widetilde{A}(t) + \frac{1}{2}c(t) \\ &\quad \times \left(\int_0^T \widetilde{A}(s)\widetilde{G}(s) \times \exp\left\{\frac{1}{2}b(s)\int_0^s \widetilde{F}(\sigma)\,d\sigma\right\}ds\right) \\ &\quad \times \left(1 - \int_0^T \frac{1}{2}c(s)\widetilde{G}(s) \times \exp\left\{\frac{1}{2}b(s)\int_0^s \widetilde{F}(\sigma)\,d\sigma\right\}ds\right)^{-1}\Bigg] \\ &\quad \times \exp\left\{\frac{1}{2}b(t)\int_0^t \widetilde{F}(\sigma)\,d\sigma\right\}\Bigg\}^{1/2}, \end{aligned} \tag{37}$$

for $t \in I$, where

$$\widetilde{A}(t) = b(t)\int_0^t \frac{1}{2} f(s)\left[K_1^{-1/2}(s)a(s) + K_1^{1/2}(s)\right]ds + c(t)\int_0^T \frac{1}{2} g(s)\left[K_2^{-1/2}(s)a(s) + K_2^{1/2}(s)\right]ds,$$
$$\widetilde{F}(t) = f(t)K_1^{-1/2}(t), \qquad \widetilde{G}(t) = g(t)K_2^{-1/2}(t), \tag{38}$$

for any $K_i(t) \in C(I, R_0)$ $(i = 1, 2)$.

When $m = n = 1$, $0 \le r \le 1$ in Theorem 7, we can also get an interesting result as follows.

Corollary 9. *Let $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t)$ be defined as in Lemma 4. Suppose that $u(t) \in C(I, R_+)$ satisfies*

$$u(t) \le a(t) + b(t)\int_0^t f(s)u(s)\,ds + c(t)\int_0^T g(s)u^r(s)\,ds. \tag{39}$$

If

$$\int_0^T rc(s)\overline{G}(s)\exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds < 1, \tag{40}$$

then

$$u(t) \le a(t) + \left[\overline{A}(t) + rc(t) \times \left(\int_0^T \overline{A}(s)\overline{G}(s)\exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds\right) \times \left(1 - \int_0^T rc(s)\overline{G}(s) \times \exp\left\{b(s)\int_0^s f(\sigma)\,d\sigma\right\}ds\right)^{-1}\right] \times \exp\left\{b(t)\int_0^t f(\sigma)\,d\sigma\right\}, \tag{41}$$

for $t \in I$, where $0 \le r \le 1$ and

$$\overline{A}(t) = b(t)\int_0^t f(s)a(s)\,ds + c(t)\int_0^T g(s)\left[rK^{r-1}(s)a(s) + (1-r)K^r(s)\right]ds,$$
$$\overline{G}(t) = g(t)K^{r-1}(t), \tag{42}$$

for any $K(t) \in C(I, R_0)$.

3. Nonlinear Weakly Singular Integral Inequalities

Lemma 10 (discrete Jensen inequality). *Let $A_1, A_2, \ldots, A_n$ be nonnegative real numbers and $r > 1$ a real number. Then*

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1}\left(A_1^r + A_2^r + \cdots + A_n^r\right). \tag{43}$$

Lemma 11 (see [16]). *Let α, β, γ, and p be positive constants. Then*

$$\int_0^t \left(t^\alpha - s^\alpha\right)^{p(\beta-1)} s^{p(\gamma-1)}ds = \frac{t^\theta}{\alpha}B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+, \tag{44}$$

where $B[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ ($\mathrm{Re}\,\xi > 0$, $\mathrm{Re}\,\eta > 0$) is the well-known B-function and $\theta = p[\alpha(\beta-1) + \gamma - 1] + 1$.

Assume that

(H) for the parameter group $[\alpha_i, \beta_i, \gamma_i]$, $\alpha_i \in (0, 1]$, $\beta_i \in (0, 1)$ and $\gamma_i > 1 - 1/p$ such that $1/p + \alpha_i(\beta_i - 1) + \gamma_i - 1 \ge 0$, ($p > 1$, $i = 1, 2$).

Theorem 12. *Under assumption (H), let $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t)$ be defined as in Lemma 4. Suppose that $u(t) \in C(I, R_+)$ satisfies (6). If*

$$\int_0^T \mathscr{C}(s)\mathscr{G}(s)\exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds < 1, \tag{45}$$

then

$$w(t) \le \left\{\tilde{a}(t) + \left[\mathscr{A}(t) + \mathscr{C}(t) \times \left(\int_0^t \mathscr{A}(s)\mathscr{G}(s)\exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds\right) \times \left(1 - \int_0^t \mathscr{C}(s)\mathscr{G}(s) \times \exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\,d\sigma\right\}ds\right)^{-1}\right] \times \exp\left\{\mathscr{B}(t)\int_0^t \mathscr{F}(\sigma)\,d\sigma\right\}\right\}^{1/qm}, \tag{46}$$

for $t \in I$, where $m \geq n \geq 0$, $m \geq r \geq 0$, m, n, p, q, and r are constants, $1/p + 1/q = 1$, and

$$\begin{gathered}
M_i = \frac{1}{\alpha_i} B\left[\frac{p(\gamma_i - 1) + 1}{\alpha_i}, p(\beta_i - 1) + 1\right], \\
\theta_i = p\left[\alpha_i(\beta_i - 1) + \gamma_i - 1\right] + 1, \quad i = 1, 2, \\
\tilde{a}(t) = 3^{q-1} a^q(t), \qquad \tilde{b}(t) = 3^{q-1} b^q(t)\left(M_1 t^{\theta_1}\right)^{q/p}, \\
\tilde{c}(t) = 3^{q-1} c^q(t)\left(M_2 T^{\theta_2}\right)^{q/p}, \\
\mathscr{A}(t) = \tilde{b}(t) \int_0^t f^q(s) \times \left[\frac{n}{m} K_1^{(n-m)/m}(s)\, \tilde{a}(s) + \frac{m-n}{m} K_1^{n/m}(s)\right] ds \\
+ \tilde{c}(t) \int_0^T g^q(s) \times \left[\frac{r}{m} K_2^{(r-m)/m}(s)\, \tilde{a}(s) + \frac{m-r}{m} K_2^{r/m}(s)\right] ds, \\
\mathscr{B}(t) = \frac{n}{m} \tilde{b}(t), \mathscr{C}(t) = \frac{r}{m} \tilde{c}(t), \\
\mathscr{F}(t) = f^q(t)\, K_1^{(n-m)/m}(t), \mathscr{G}(t) = g^q(t)\, K_2^{(r-m)/m}(t),
\end{gathered} \tag{47}$$

for any $K_i(t) \in C(I, R_0)$ $(i = 1, 2)$.

Remark 13. When $g(t) = 0$, inequality (6) can be reduced to the case discussed by Ma and Pečarić [12]. But their result is based on the assumption that the ordered parameter group $[\alpha_i, \beta_i, \gamma_i]$ $(i = 1, 2)$ obeys distribution I or II (for details, see [16]), which leads up to slightly complicated formula of bound on solutions.

Proof. From assumption (H), using the Hölder inequality with indices p, q to (6), we get

$$\begin{aligned}
u^m(t) \leq a(t) &+ b(t)\left(\int_0^t \left(t^{\alpha_1} - s^{\alpha_1}\right)^{p(\beta_1 - 1)} s^{p(\gamma_1 - 1)} ds\right)^{1/p} \\
&\times \left(\int_0^t f^q(s)\, u^{qn}(s)\, ds\right)^{1/q} \\
&+ c(t)\left(\int_0^T \left(T^{\alpha_2} - s^{\alpha_2}\right)^{p(\beta_2 - 1)} s^{p(\gamma_2 - 1)} ds\right)^{1/p} \\
&\times \left(\int_0^T g^q(s)\, u^{qr}(s)\, ds\right)^{1/q}.
\end{aligned} \tag{48}$$

Applying Lemma 10 to (48), we have

$$\begin{aligned}
u^{qm}(t) &\leq 3^{q-1} a^q(t) + 3^{q-1} b^q(t) \\
&\quad \times \left(\int_0^t \left(t^{\alpha_1} - s^{\alpha_1}\right)^{p(\beta_1 - 1)} s^{p(\gamma_1 - 1)} ds\right)^{q/p} \\
&\quad \times \left(\int_0^t f^q(s)\, u^{qn}(s)\, ds\right) \\
&\quad + 3^{q-1} c^q(t)\left(\int_0^T \left(T^{\alpha_2} - s^{\alpha_2}\right)^{p(\beta_2 - 1)} s^{p(\gamma_2 - 1)} ds\right)^{q/p} \\
&\quad \times \left(\int_0^T g^q(s)\, u^{qr}(s)\, ds\right) \\
&= 3^{q-1} a^q(t) + 3^{q-1} b^q(t)\left(M_1 t^{\theta_1}\right)^{q/p} \\
&\quad \times \left(\int_0^t f^q(s)\, u^{qn}(s)\, ds\right) \\
&\quad + 3^{q-1} c^q(t)\left(M_2 T^{\theta_2}\right)^{q/p}\left(\int_0^T g^q(s)\, u^{qr}(s)\, ds\right),
\end{aligned} \tag{49}$$

where M_i, θ_i $(i = 1, 2)$ are given in (47).

Letting $u^q(t) = w(t)$, we have

$$\begin{aligned}
w^m(t) \leq \tilde{a}(t) &+ \tilde{b}(t) \int_0^t f^q(s)\, w^n(s)\, ds \\
&+ \tilde{c}(t) \int_0^T g^q(s)\, w^r(s)\, ds,
\end{aligned} \tag{50}$$

which is similar to inequality (5), where $\tilde{a}(t)$, $\tilde{b}(t)$, and $\tilde{c}(t)$ are also given in (47).

An application of Theorem 7 to the inequality above gives that

$$\begin{aligned}
w(t) \leq \Bigg\{ \tilde{a}(t) &+ \Bigg[\mathscr{A}(t) + \mathscr{C}(t) \\
&\times \left(\int_0^T \mathscr{A}(s)\, \mathscr{G}(s) \times \exp\left\{\mathscr{B}(s) \int_0^s \mathscr{F}(\sigma)\, d\sigma\right\} ds\right) \\
&\times \left(1 - \int_0^T \mathscr{C}(s)\, \mathscr{G}(s) \times \exp\left\{\mathscr{B}(s) \int_0^s \mathscr{F}(\sigma)\, d\sigma\right\} ds\right)^{-1}\Bigg] \\
&\times \exp\left\{\mathscr{B}(t) \int_0^t \mathscr{F}(\sigma)\, d\sigma\right\}\Bigg\}^{1/m}
\end{aligned} \tag{51}$$

holds for $t \in I$, where $\mathscr{A}(t)$, $\mathscr{B}(t)$, $\mathscr{C}(t)$, $\mathscr{F}(t)$, and $\mathscr{G}(t)$ are also given in (47). Since $u(t) = w^{1/q}(t)$, we can get (46). □

Similarly, if we take $m = n = 1$, $0 \le r \le 1$ in (5), the following result is obtained.

Corollary 14. *Under assumption (H), let $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t)$ be defined as in Lemma 4. Suppose that $u(t) \in C(I, R_+)$ satisfies*

$$u(t) \le a(t) + b(t)\int_0^t \left(t^{\alpha_1} - s^{\alpha_1}\right)^{\beta_1 - 1} s^{\gamma_1 - 1} f(s) u(s)\, ds + c(t)\int_0^T \left(T^{\alpha_2} - s^{\alpha_2}\right)^{\beta_2 - 1} s^{\gamma_2 - 1} g(s) u^r(s)\, ds. \tag{52}$$

If

$$\int_0^T r\widetilde{c}(s)\,\mathbb{G}(s)\exp\left\{\widetilde{b}(s)\int_0^s \mathbb{F}(\sigma)\, d\sigma\right\} ds < 1, \tag{53}$$

then

$$\begin{aligned} u(t) \le \Bigg\{ &\widetilde{a}(t) + \Bigg[\mathbb{A}(t) + r\widetilde{c}(t) \times \left(\int_0^T \mathbb{A}(s)\,\mathbb{G}(s)\exp\left\{\widetilde{b}(s)\int_0^s \mathbb{F}(\sigma)\, d\sigma\right\} ds\right) \\ &\times \left(1 - \int_0^T r\widetilde{c}(s)\,\mathbb{G}(s) \times \exp\left\{\widetilde{b}(s)\int_0^s \mathbb{F}(\sigma)\, d\sigma\right\} ds\right)^{-1}\Bigg] \times \exp\left\{\widetilde{b}(t)\int_0^t \mathbb{F}(\sigma)\, d\sigma\right\}\Bigg\}^{1/q}, \end{aligned} \tag{54}$$

for $t \in I$, where $0 \le r \le 1$, p and q are constants, $1/p + 1/q = 1$, $\widetilde{a}(t)$, $\widetilde{b}(t)$, $\widetilde{c}(t)$ are defined as in Theorem 12, and

$$\begin{aligned} \mathbb{A}(t) &= \widetilde{b}(t)\int_0^t f^q(s)\,\widetilde{a}(s)\, ds + \widetilde{c}(t)\int_0^T g^q(s)\left[rK^{r-1}(s)\,\widetilde{a}(s) + (1-r)K^r(s)\right] ds, \\ \mathbb{F}(t) &= f^q(t), \qquad \mathbb{G}(t) = g^q(t)\,K^{r-1}(t), \end{aligned} \tag{55}$$

for any $K(t) \in C(I, R_0)$.

Remark 15. If we take $m = 2$, $n = r = 1$, similar to Corollary 8, we can get a general Ou-Iang type singular inequality of (6). Here we leave the details to the reader.

When we take $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = \gamma_2 = 1$ in (6), we also get the following result.

Corollary 16. *Let $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t)$ be defined as in Lemma 4. Suppose that $u(t) \in C(I, R_+)$ satisfies*

$$u^m(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta_1 - 1} f(s)\, u^n(s)\, ds + c(t)\int_0^T (T-s)^{\beta_2 - 1} g(s)\, u^r(s)\, ds. \tag{56}$$

If

$$\int_0^T \mathscr{C}(s)\,\mathscr{G}(s)\exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\, d\sigma\right\} ds < 1, \tag{57}$$

then

$$\begin{aligned} w(t) \le \Bigg\{ &\widetilde{a}(t) + \Bigg[\mathscr{A}(t) + \mathscr{C}(t) \times \left(\int_0^T \mathscr{A}(s)\,\mathscr{G}(s) \times \exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\, d\sigma\right\} ds\right) \\ &\times \left(1 - \int_0^T \mathscr{C}(s)\,\mathscr{G}(s) \times \exp\left\{\mathscr{B}(s)\int_0^s \mathscr{F}(\sigma)\, d\sigma\right\} ds\right)^{-1}\Bigg] \times \exp\left\{\mathscr{B}(t)\int_0^t \mathscr{F}(\sigma)\, d\sigma\right\}\Bigg\}^{1/qm}, \end{aligned} \tag{58}$$

for $t \in I$, where $m, n, r, p, q, \widetilde{a}(t), \widetilde{b}(t), \widetilde{c}(t), \mathscr{A}(t), \mathscr{B}(t), \mathscr{C}(t), \mathscr{F}(t), \mathscr{G}(t)$ are defined as in Theorem 12, $0 \le \beta_i \le 1$ $(i = 1, 2)$, the choice of p satisfies that $p > 1$ and $1/p + (\beta_i - 1) > 0$, and M_i, θ_i are replaced by

$$M_i = B\left[1, p(\beta_i - 1) + 1\right], \qquad \theta_i = p(\beta_i - 1) + 1, \quad i = 1, 2. \tag{59}$$

Remark 17. If we take $m = n = r = 1$, $\beta_1 = \beta_2 = \beta$, $a(t) = C$, $b(t) = c(t) = 1/\Gamma(\beta)$, and $f(t) = g(t)$, inequality (6) becomes inequality (4). So, Zheng's result [14] is the special case of our result.

Furthermore, if we take $m = 2$, $n = 1$, and $r = 1$, the weakly singular case of the Gronwall-Bellman-Gamidov-Ou-Iang type inequality also can be obtained.

Corollary 18. *Let $a(t)$, $b(t)$, $c(t)$, $f(t)$, and $g(t)$ be defined as in Lemma 4. Suppose that $u(t) \in C(I, R_+)$ satisfies*

$$u^2(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta_1 - 1} f(s)\, u(s)\, ds + c(t)\int_0^T (T-s)^{\beta_2 - 1} g(s)\, u(s)\, ds. \tag{60}$$

If

$$\int_0^T \frac{1}{2}\widetilde{c}(s)\,\mathbf{G}(s)\exp\left\{\frac{1}{2}\widetilde{b}(s)\int_0^s \mathbf{F}(\sigma)\,d\sigma\right\}ds < 1, \quad (61)$$

then

$$\begin{aligned} u(t) &\leq \left\{\widetilde{a}(t) + \left[\mathbf{A}(t) + \frac{1}{2}\widetilde{c}(t)\right.\right.\\ &\quad \times \left(\int_0^T \mathbf{A}(s)\,\mathbf{G}(s)\right.\\ &\quad \left.\times \exp\left\{\frac{1}{2}\widetilde{b}(s)\int_0^s \mathbf{F}(\sigma)\,d\sigma\right\}ds\right)\\ &\quad \times \left(1 - \int_0^T \frac{1}{2}\widetilde{c}(s)\,\mathbf{G}(s)\right.\\ &\quad \left.\left.\times \exp\left\{\frac{1}{2}\widetilde{b}(s)\int_0^s \mathbf{F}(\sigma)\,d\sigma\right\}ds\right)^{-1}\right]\\ &\quad \left.\times \exp\left\{\frac{1}{2}\widetilde{b}(s)\int_0^T \mathbf{F}(\sigma)\,d\sigma\right\}\right\}^{1/2q}, \end{aligned} \quad (62)$$

for $t \in I$, *where* $\widetilde{a}(t)$, $\widetilde{b}(t)$, *and* $\widetilde{c}(t)$ *are defined as in Theorem 12,* β_i, M_i, θ_i, p, q *are defined as in Corollary 16, and*

$$\begin{aligned} \mathbf{A}(t) &= \widetilde{b}(t)\int_0^t \frac{1}{2}f^q(s)\left[K_1^{-1/2}(s)\,\widetilde{a}(s) + K_1^{1/2}(s)\right]ds\\ &\quad + \widetilde{c}(t)\int_0^T \frac{1}{2}g^q(s)\left[K_2^{-1/2}(s)\,\widetilde{a}(s) + K_2^{1/2}(s)\right]ds,\\ \mathbf{F}(t) &= f^q(t)\,K_1^{-1/2}(t), \qquad \mathbf{G}(t) = g^q(t)\,K_2^{-1/2}(t), \end{aligned} \quad (63)$$

for any $K_i(t) \in C(I, R_0)$ $(i = 1, 2)$.

4. Applications

In this section, we give some applications of our result in the study of the boundedness of solutions of a fractional integral equation with the Riemann-Liouville (R-L) fractional operator.

Definition 19 (see [25]). The R-L fractional integral of order α is defined by the following expression:

$$\begin{aligned} I^\alpha f(t) &= \frac{1}{\Gamma(1+\alpha)}\int_0^t f(s)\,(ds)^\alpha\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds. \end{aligned} \quad (64)$$

Consider the following fractional integral equation:

$$\begin{aligned} u^2(t) &= a(t) + I^\alpha(F(t, u(t)))\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} G(s, u(s))\,ds,\\ &\qquad t \in I, \end{aligned} \quad (65)$$

where $0 < \alpha < 1$ and $F, G \in C(R \times R, R)$.

Theorem 20. *If* $a(t) \in C(I, R_+)$ *is nondecreasing,* $|F(t,u)| \leq f(s)|u|$ *and* $G(t,u) \leq g(s)|u|$, *where* $f, g \in C(I, R_+)$. *Under the condition* $\int_0^T (1/2)\widetilde{c}(s)\mathbf{G}(s)\exp\{(1/2)\widetilde{b}(s)\int_0^s \mathbf{F}(\sigma)d\sigma\}ds < 1$, *the following estimate*

$$\begin{aligned} u(t) &\leq \left\{\widetilde{a}(t) + \left[\mathbf{A}(t) + \frac{1}{2}\widetilde{c}(t)\right.\right.\\ &\quad \times \left(\int_0^T \mathbf{A}(s)\,\mathbf{G}(s)\right.\\ &\quad \left.\times \exp\left\{\frac{1}{2}\widetilde{b}(s)\int_0^s \mathbf{F}(\sigma)\,d\sigma\right\}ds\right)\\ &\quad \times \left(1 - \int_0^T \frac{1}{2}\widetilde{c}(s)\,\mathbf{G}(s)\right.\\ &\quad \left.\left.\times \exp\left\{\frac{1}{2}\widetilde{b}(s)\int_0^s \mathbf{F}(\sigma)\,d\sigma\right\}ds\right)^{-1}\right]\\ &\quad \left.\times \exp\left\{\frac{1}{2}\widetilde{b}(s)\int_0^t \mathbf{F}(\sigma)\,d\sigma\right\}\right\}^{1/2q} \end{aligned} \quad (66)$$

holds, where $\widetilde{a}(t) = 3^{q-1}a^q(t)$, $\theta = p(\alpha-1)+1$, $M = B[1, p(\alpha-1)+1]$, $\widetilde{b}(t) = 3^{q-1}(1/\Gamma^q(\alpha))(Mt^\theta)^{q/p}$, *and* $\widetilde{c}(t) = 3^{q-1}(1/\Gamma^q(\alpha))(MT^\theta)^{q/p}$ *and* $\mathbf{A}(t)$, $\mathbf{F}(t)$, *and* $\mathbf{G}(t)$ *are defined as in Corollary 18.*

Proof. According to Definition 19, from (65), we have

$$\begin{aligned} u^2(t) &= a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s, u(s))\,ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} G(s, u(s))\,ds, \end{aligned} \quad (67)$$

for $t \in I$. Hence

$$\begin{aligned} \left|u^2(t)\right| &\leq a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|F(s, u(s))|\,ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\,|G(s, u(s))|\,ds\\ &\leq a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,|u(s)|\,ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)\,|u(s)|\,ds. \end{aligned} \quad (68)$$

Letting $\beta_1 = \beta_2 = \alpha$, $b(t) = 1/\Gamma(\alpha)$, and $c(t) = 1/\Gamma(\alpha)$, and applying Corollary 18, we have

$$\begin{aligned}
\tilde{a}(t) &= 3^{q-1}a^q(t), \\
\theta_1 &= \theta_2 = p(\alpha-1)+1 := \theta, \\
M_1 &= M_2 = B[1, p(\alpha-1)+1] := M, \\
\tilde{b}(t) &= 3^{q-1}\frac{1}{\Gamma^q(\alpha)}\left(Mt^\theta\right)^{q/p}, \\
\tilde{c}(t) &= 3^{q-1}\frac{1}{\Gamma^q(\alpha)}\left(MT^\theta\right)^{q/p}.
\end{aligned} \tag{69}$$

From (62), we get the desired estimate (66) which implies that $u(t)$ in (65) is bounded. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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