Some Remarks on Steiner Systems

Hendrik Van Maldeghem

Abstract

The main purpose of this paper is to introduce Steiner systems obtained from the finite classical generalized hexagons of order q. We show that we can take the blocks of the Steiner systems amongst the lines and the traces of the hexagon, and we prove some facts about the automorphism groups. Also, we make a remark concerning the geometric construction of a known class (KW) of Steiner systems and we deduce some properties of the automorphism group.

Mathematics Subject Classification 1991: 51E10.

Key words and phrases: normal rational curve, Steiner's theorem, generalized hexagon, split Cayley hexagon

1 Introduction and Notation

In this note, I want to contribute to the geometric study of some particular classes of Steiner systems, with an eye on the automorphism groups. I present some refining of a class called (KW) in Table A5.1 of [1], based on an alternative (but well known) geometric description of the Steiner system $S(3, q+1, q^n+1)$ obtained from the projective line $\mathbf{PG}(1, q^n)$ by considering as blocks all projective sublines isomorphic to $\mathbf{PG}(1, q)$. Also, I construct some Steiner systems — in fact, linear spaces with constant block size — from the known classes of finite generalized hexagons and show how this gives rise to interesting questions and connections in both the theory of generalized hexagons and Galois geometry. All Steiner systems involved are t-designs for $t \in \{2, 3\}$. Let us fix some notation now.

An $S(t, k, v)$ *Steiner system* $(\mathcal{P}, \mathcal{B})$ consists of a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of subsets, all of size k, called *blocks*, such that every set of t points is contained in a unique block. We will also call this a t*-Steiner system*. An exhaustive list of all parameters of known infinite classes of t-Steiner systems, $t \geq 2$, is presented in Table A5.1 of [1]. Here, we just mention the infinite classes that we will deal with in the present paper, and, for the sake of convenience later on, we introduce some (non-standard) notation for it.

(AG) The point set is equal to the point set of the n-dimensional affine space $\mathbf{AG}(n,q)$ over the Galois field $\mathbf{GF}(q)$ of q elements. The blocks are the lines of that space. This is an $S(2,q,q^n)$ Steiner system which we denote by $\mathsf{SS}_2(\mathbf{AG}(n,q))$.

(PG) The point set is equal to the point set of the n-dimensional projective space $\mathbf{PG}(n,q)$ over $\mathbf{GF}(q)$. The blocks are the lines of that space. This is an $S(2,q+1,\frac{q^{n+1}-1}{q-1})$ Steiner system which we denote by $\mathsf{SS}_2(\mathbf{PG}(n,q))$.

(SD) The point set is equal to the point set of the projective line $\mathbf{PG}(1,q^n)$; the blocks are the projective sublines isomorphic to $\mathbf{PG}(1,q)$. This is an $S(3,q+1,q^n+1)$ Steiner system, and we denote it by $\mathsf{SS}_3(\mathbf{PG}(1,q^n),q)$. If $n=2$, then $\mathsf{SS}_3(\mathbf{PG}(1,q^2),q)$ can also be constructed as follows. The point set is the set of points of a non degenerate elliptic quadric $Q^-(3,q)$ of $\mathbf{PG}(3,q)$ and the blocks are the intersections with non tangent planes. Projecting this from a point x of $Q^-(3,q)$, we obtain as point set the set of points of $\mathbf{AG}(2,q)$ together with a new symbol ∞ (corresponding to x), and the blocks are the lines of $\mathbf{AG}(2,q)$ completed with ∞, together with certain (elliptic) conics in $\mathbf{AG}(2,q)$. We refer to this construction as the *plane model* of $\mathsf{SS}_3(\mathbf{PG}(1,q^2),q)$.

(KW) Here, the point set is the point set of $\mathbf{AG}(n,q)$ completed with the symbol ∞. For each plane π of $\mathbf{AG}(n,q)$, we consider a plane model of $\mathsf{SS}_3(\mathbf{PG}(1,q^2),q)$ (always using the same symbol ∞). Then the block set is the union of the block sets of all these plane models. This is again an $S(3,q+1,q^n+1)$ Steiner system, not isomorphic to a member of the class (SD) if $n>2$. Since this Steiner system depends on the (many) choices of the plane models, we do not introduce a notation for it, yet. Below we will make (a slight generalization of) this construction more explicit and we will be able to give it a compact unambiguous name.

The paper is organized as follows. In Section 2 we make a slightly more general construction of the class (KW) more explicit and look at some consequences with respect to automorphism groups of such Steiner systems. Our motivation here is to provide examples of Steiner systems together with automorphism groups in order to anticipate on possible future characterizations of certain classes of Steiner systems. For instance, it will be clear from our discussion that the members of (SD) with $n=3$ are not characterized by their parameters and the action of some nontrivial and rather big cyclic group, as we will see that members of (KW) also have this property. In Section 3 we deal with examples from hexagons. In Section 4, finally, we formulate a general construction method for Steiner systems out of old ones (with no claim on originality) and derive some interesting questions from it.

2 Spherical designs and related Steiner systems

2.1 A theorem belonging to folklore

Consider $\mathbf{AG}(n,q)$ and extend this affine space to $\mathbf{AG}(n,q^n)$ by extending the field $\mathbf{GF}(q)$ to $\mathbf{GF}(q^n)$. Let $\mathbf{PG}(n-1,q)$ be the projective space at infinity of $\mathbf{AG}(n,q)$ and denote by $\mathbf{PG}(n,q)$ the projective space obtained by adjoining $\mathbf{PG}(n-1,q)$ to $\mathbf{AG}(n,q)$. Let $\mathbf{PG}(n-1,q^n)$ and $\mathbf{PG}(n,q^n)$ be the respective corresponding extensions. There is a unique collineation group of $\mathbf{PG}(n,q^n)$ of order n fixing $\mathbf{PG}(n,q)$ pointwise. Two elements of the same orbit will be called *conjugate*; an entire orbit will also be called a self conjugate set. We now choose a self conjugate set I of n different points in $\mathbf{PG}(n-1,q^n)$ that generates $\mathbf{PG}(n-1,q^n)$ (over $\mathbf{GF}(q^n)$). To see that this exists, we take a generator g of the field $\mathbf{GF}(q^n)$ and consider all points conjugate to the point of $\mathbf{PG}(n-1,q^n)$ with coordinates $(1,g,g^2,g^3,\ldots,g^{n-1})$ (with respect to a suitable coordinate system).

Now consider three points p_1,p_2,p_3 of $\mathbf{AG}(n,q)$. For $x\in I$, denote by H_x the $(n-2)$-dimensional projective subspace over $\mathbf{GF}(q^n)$ generated by $I\setminus\{x\}$. The pencil of hyperplanes of $\mathbf{PG}(n,q^n)$ which all contain H_x carries — in a natural and obvious way — the structure of a projective line that we shall denote by $\mathbf{PG}^{(x)}(1,q^n)$. For $x,y\in I$, there is a unique projectivity $\theta_{x,y}:\mathbf{PG}^{(x)}(1,q^n)\rightarrow\mathbf{PG}^{(y)}(1,q^n)$ which maps the unique hyperplane spanned by H_x and p_i, $i=1,2,3$, to the unique hyperplane spanned by H_y and p_i (if $x=y$, this is the identity). By uniqueness, it follows that $\theta_{x,y}\theta_{y,z}=\theta_{x,z}$, $x,y,z\in I$, hence the following set of points is independent of $x\in I$: for such x set

$$[\{p_1,p_2,p_3\}]_{q^n}=\{p=\bigcap_{y\in I}H^{\theta_{x,y}}\mid H\in\mathbf{PG}^{(x)}(1,q^n), I\not\subseteq H^{\theta_{x,y}},\forall y\in I\}.$$

We denote the intersection of $[\{p_1,p_2,p_3\}]_{q^n}$ with the point set of $\mathbf{AG}(n,q)$ by $[\{p_1,p_2,p_3\}]_q$.

Now we define the following point-block system $(\mathcal{P},\mathcal{B})$, which we denote by $\mathsf{SS}_3(\mathbf{AG}(n,q),I)$. The set $\mathcal{P}$ is equal to the point set of $\mathbf{AG}(n,q)$ completed with the symbol ∞, while $\mathcal{B}$ contains all sets $[\{p_1,p_2,p_3\}]_q$ such that p_1,p_2,p_3 are not on a line, and all affine lines of $\mathbf{AG}(n,q)$ completed with ∞ (or, equivalently, all sets $[\{p_1,p_2,p_3\}]_q$ with p_1,p_2,p_3 on a line, completed with ∞). The following theorem belongs to folklore, but we could not find an explicit proof in the literature. A rather short proof using Segre varieties exists (J. A. Thas, personal communication, November 2001), but we prefer a slightly longer proof, because it relies on the seemingly unrelated Theorem of Steiner in the theory of conics, and we find it rather amusing to see the Theorem of Steiner meet the theory of Steiner systems.

Theorem 2.1 *For each legible choice of I, the system $\mathsf{SS}_3(\mathbf{AG}(n,q),I)$ is an $S(3,q+1,q^n+1)$ Steiner system isomorphic to $\mathsf{SS}_3(\mathbf{PG}(1,q^n),q)$.*

Proof. We consider the system $\mathsf{SS}_3(\mathbf{AG}(n,q),I)$ and we will show that it is isomorphic to $\mathsf{SS}_3(\mathbf{PG}(1,q^n),q)$, which will then also prove that $\mathsf{SS}_3(\mathbf{AG}(n,q),I)$ is really a Steiner system. Let $\mathbf{PG}(n-1,q^n)$, $\mathbf{PG}(n,q^n)$, $\mathbf{AG}(n,q^n)$, $\mathbf{PG}(n-1,q)$ and $\mathbf{PG}(n,q)$ be as above. Using obvious self explaining notation, we denote $I=\{\omega,\omega^q,\omega^{q^2},\ldots,\omega^{q^{n-1}}\}$, and we recall that I generates $\mathbf{PG}(n-1,q^n)$. Denote by H_i, $i=0,1,2,\ldots,n-1$, the $(n-2)$-dimensional subspace of $\mathbf{PG}(n,q^n)$ (over $\mathbf{GF}(q^n)$) generated by $I\setminus\{\omega^{q^i}\}$. We denote by $\mathbf{PG}^{(i)}(1,q^n)$ the pencil of hyperplanes of $\mathbf{PG}(n,q^n)$ containing H_i, $i=0,1,2,\ldots,n-1$. Now we define the following mapping $\varphi:\mathbf{AG}(n,q)\cup\{\infty\}\to\mathbf{PG}^{(0)}(1,q^n)$. For a point $x\in\mathbf{AG}(n,q)$, the image x^φ is by definition the hyperplane spanned by H_0 and x; for the symbol ∞, we define ∞^φ to be $\mathbf{PG}(n-1,q^n)$. We claim that φ is bijective. Indeed, if, by way of contradiction, $x^\varphi=y^\varphi$ with $x\neq y$, then $x\neq\infty\neq y$ and H_0 contains the intersection point p of the line xy and the hyperplane $\mathbf{PG}(n-1,q^n)$, which belongs to $\mathbf{PG}(n-1,q)$. Hence p belongs to H_i, for every $i\in\{0,1,\ldots,n-1\}$, and so I cannot generate an $(n-1)$-dimensional subspace, a contradiction.

So, it is clear that, if we show that φ maps the blocks of $\mathsf{SS}_3(\mathbf{AG}(n,q),I)$ onto blocks of $\mathsf{SS}_3(\mathbf{PG}^{(0)}(1,q^n),q)$, then the theorem is proved. We will do that as follows. For each block B of $\mathsf{SS}_3(\mathbf{AG}(n,q),I)$, we will find an $(n-2)$-dimensional subspace H_B of $\mathbf{PG}(n,q)$ and a projectivity θ from the pencil of hyperplanes in $\mathbf{PG}(n,q^n)$ containing H_B to the pencil $\mathbf{PG}^{(0)}(1,q^n)$ of hyperplanes through H_0 such that the hyperplanes of $\mathbf{PG}(n,q)$ through H_B correspond, after first extending to hyperplanes of $\mathbf{PG}(n,q^n)$ and then applying θ, to hyperplanes x^φ, with $x\in B$.

First, let B be equal to $L\cup\{\infty\}$, with L a line of $\mathbf{AG}(n,q)$. Then we may take for H_B any $(n-2)$-dimensional subspace of $\mathbf{PG}(n,q)$ not intersecting the projective extension of L to $\mathbf{PG}(n,q)$. Let $\overline{L}$ be the unique line of $\mathbf{PG}(n,q^n)$ containing all points of L. We define the projectivity θ as the map which takes the hyperplane H containing H_B to the unique hyperplane H^θ containing H_0 with the property that $H^\theta\cap\overline{L}=H\cap\overline{L}$. Note that, in fact, θ is a perspectivity.

Next, we let B be equal to a set $[\{p_1,p_2,p_3\}]_q$ with p_1,p_2,p_3 non collinear points of $\mathbf{AG}(n,q)$, and we denote the projectivity $\theta_{\omega_0,\omega_i}$ (defined in the second paragraph of the present section) briefly by θ_i. We denote by L_i the line in $\mathbf{PG}(n,q^n)$ generated by p_1 and ω^{q^i}. The projectivity θ_i induces a projectivity $\theta'_i:L_0\to L_i$ mapping a point z on L_0 to the intersection of L_i with the hyperplane $\langle z,H_0\rangle^{\theta_i}$. Clearly we have $p_1^{\theta'_i}=p_1$ and so, by a theorem of Steiner, θ_i is a perspectivity, i.e. there is a point r_i in the plane $\langle L_0,L_i\rangle$ such that r_i is incident with every line $zz^{\theta'_i}$, for all z on L_0, $z\neq p_1$. Since I generates an $(n-1)$-dimensional space, the points $r_1,r_2,\ldots,r_{n-1}$ generate an $(n-2)$-dimensional space $\overline{H}_B$ which does not meet L_j, for all $j\in\{0,1,\ldots,n-1\}$. Furthermore, by construction, the space $\overline{H}_B$ is contained in the hyperplane H_z generated by $z,z^{\theta'_1},z^{\theta'_2},\ldots,z^{\theta'_{n-1}}$, for every z on L_0, $z\neq p_1$. Hence we have a projectivity from L_0 to the pencil of hyperplanes through $\overline{H}_B$, and composing projectivities, we obtain a projectivity θ from the pencil of hyperplanes containing $\overline{H}_B$ to the pencil of hyperplanes through H_0. Now let z_2 ad z_3 be the intersection of L_0 with the hyperplane through H_0 containing p_2 and p_3, respectively. Since, with obvious notation, $p_2^q = p_2$ (p_2 is a "real" point, i.e. it belongs to $\mathbf{AG}(n,q)$), we easily see that $z_2^{\theta'_i} = z_2^{q^i}$, hence the hyperplane H_{z_2}, and by the same token also the hyperplane H_{z_3}, is a "real" hyperplane, i.e. it contains the point set of a hyperplane of $\mathbf{PG}(n,q)$. But $\overline{H}_B = H_{z_2} \cap H_{z_3}$, and so $\overline{H}_B$ contains an $(n-2)$-dimensional subspace H_B of $\mathbf{PG}(n,q)$. We also have shown (see the argument involving p_2 and p_3) that the image under θ^{-1} of a hyperplane containing an element of $B\setminus\{p\}$ is a "real" hyperplane, and this also holds almost trivially true for p_1. It can also be easily checked (using intersections of mutually conjugate hyperplanes) that "real" hyperplanes correspond under θ to "real" points of $[\{p_1,p_2,p_3\}]_{q^n}$, hence to elements of B (which, by the way, shows that $|B| = q+1$). This completes the proof of the theorem. $\square$

Remark 2.2 Using the above mentioned Theorem of Steiner (namely, the assertion that in a Pappian projective plane the intersections of corresponding lines belonging to two different line pencils that correspond under a bijection β form a (not necessarily non-degenerate) conic if and only if β is a projectivity), it is easy to see that the blocks of $\mathsf{SS}_3(\mathbf{AG}(n,q),I)$ that do not contain ∞ are normal rational curves in the (projective) subspace they generate (in $\mathbf{PG}(n,q)$). Recall that a *normal rational curve* in a projective space $\mathbf{PG}(k,q)$ is a set of points that is projectively equivalent to the set $\{(1,t,t^2,\ldots,t^k)\,|\,t\in\mathbf{GF}(q)\}\cup\{(0,0,\ldots,0,1)\}$, with respect to some coordinate system. If n is prime, then it is readily checked that, with above notation, every subset of size $n+1$ of the set $I\cup\{p_1,p_2,p_3\}$ generates $\mathbf{PG}(n,q^n)$ (where we assume that p_1,p_2,p_3 are not on one affine line), and hence, according to Theorem 21.1.1(v) of [5], there is a unique normal rational curve C containing $I\cup\{p_1,p_2,p_3\}$. The intersection of C with $\mathbf{AG}(n,q)$ is precisely $[\{p_1,p_2,p_3\}]_q$ and is itself a normal rational curve in $\mathbf{PG}(n,q)$ disjoint from $\mathbf{PG}(n-1,q)$ (actually, this is only completely correct in the technical sense if $q\geq n$; in the other case we have a normal rational curve in a subspace). For n not a prime number, one can also formulate such a geometric construction, but it is more involved and we leave it to the interested reader (one must consider a normal rational curve in the smallest subspace $\mathbf{PG}(m,q)$ of $\mathbf{PG}(n,q)$ containing p_1,p_2,p_3 whose extension $\mathbf{PG}(m,q^n)$ inside $\mathbf{PG}(n,q^n)$ is not disjoint from I).

Remark 2.3 Our proof and the previous remark reflects the well known fact that the Theorem of Steiner can be generalized to normal rational curves (normal rational curves are essentially equivalent to projectivities between pencils of hyperplanes).

2.2 Some $S(3,q+1,q^n+1)$ Steiner systems

Using Theorem 2.1 we now present a construction of 3-Steiner systems which makes well known constructions of Assmus and Key (see [8]) more explicit. We show with a few examples that non isomorphic systems can arise from the "same situation" and show a way to find some automorphisms of the designs. We shall not repeat here the general extension theorem of [8] but immediately apply it to our situation.

General Construction. Let $\mathbf{AG}(n, q)$, $n \geq 3$, $q \geq 3$, be the n-dimensional affine space over $\mathbf{GF}(q)$ viewed as a Steiner system. We choose an arbitrary set $\mathcal{S}'$ of affine subspaces over $\mathbf{GF}(q)$ with the following properties.

(i) Every member has dimension ≥ 3.

(ii) Two arbitrary members of $\mathcal{S}'$ intersect either in an affine line, or in a point, or in the empty set.

Note that different members of $\mathcal{S}'$ can have different dimensions. Also, $\mathcal{S}'$ may be empty, or contain very few elements. Examples of big sets $\mathcal{S}'$ are provided by k-spreads, $k \geq 2$, of the projective space $\mathbf{PG}(n-1, q)$ at infinity of $\mathbf{AG}(n, q)$ (a k-spread is a partition of the point set of $\mathbf{PG}(n-1, q)$ by k-dimensional subspaces and such spreads exist whenever $k+1$ divides n, see Theorem 4.1 in [6]; to obtain the set $\mathcal{S}'$ join every point of $\mathbf{AG}(n, q)$ to every element of the spread). Two elements of $\mathcal{S}'$ then either are disjoint (parallel) or meet in a unique point.

Let $\mathcal{S}''$ be the set of affine planes of $\mathbf{AG}(n, q)$ that are contained in no member of $\mathcal{S}'$. Put $\mathcal{S} := \mathcal{S}' \cup \mathcal{S}''$. Let ∞ be a new symbol. For each element A of $\mathcal{S}$, we consider the Steiner system $\mathsf{SS}_3(A, I_A)$, for some appropriate set I_A of imaginary points at infinity of A, on the set of points of A and ∞. Let $\mathcal{S}^* = \{(A, I_A) \,|\, A \in \mathcal{S}\}$. Then we define the $S(3, q+1, q^n+1)$ Steiner system $\mathsf{SS}_3(\mathbf{AG}(n, q), \mathcal{S}^*)$ as follows. The points are the points of $\mathbf{AG}(n, q)$ together with the symbol ∞. The blocks are the blocks of $\mathsf{SS}_3(A, I_A)$, for $A \in \mathcal{S}$. It is a trivial exercise to check that we indeed defined an $S(3, q+1, q^n+1)$ Steiner system.

If $\mathcal{S}' = \emptyset$, then this construction is more or less the one given in [9] (and we obtain the examples (KW)), except that the latter does not emphasize the explicit form of $\mathsf{SS}_3(A, I_A)$ for a given plane A, and that (KW) also allows other Steiner systems, such as the ones arising from a Suzuki-Tits ovoid. To illustrate the refining, we consider the smallest case, i.e. we put $n = 3$ (and note we still assume $q \neq 2$). The Steiner system is then defined whenever we choose for each plane A of $\mathbf{AG}(3, q)$ two points at infinity of A which are conjugate to each other in a quadratic extension. If we make this choice such that parallel planes correspond to same pairs of points, then the full affine translation group will act on the resulting Steiner system (if, on the contrary, we choose, for instance, this pair of points different for one plane, then every automorphism of the Steiner system which fixes ∞ must also fix that particular plane). Moreover, we can start by choosing a pair of imaginary points for one parallel class, and then construct the other pairs by letting a

Singer cycle act on the projective plane at infinity. This results in a bigger automorphism group (indeed, remark that the automorphism group fixing ∞ is precisely the affine group of $\mathbf{AG}(n,q)$, since the set of lines must be preserved — as these define the blocks through ∞). Hence one can get a certain control over the automorphism group of the new Steiner system. This is also true when $\mathcal{S}'$ arises from a homogeneous spread (for instance, a classical one) .

Remark 2.4 As certain potential characterizations and related conjectures deal with the derivation of Steiner systems, I make a small comment on that subject. I conjecture that the derivation at an affine point of any of the above constructed 3-Steiner systems is isomorphic to $\mathsf{SS}_2(\mathbf{AG}(n,q))$ if and only if $\mathcal{S}^*$ reduces to a unique element corresponding to the space $\mathbf{AG}(n,q)$ itself. An algebraic proof would involve an easy but tiresome computation in order to show that some particular intersecting lines do not generate an affine plane. The following observation might also be useful in this respect. Consider two Steiner systems $\mathsf{SS}_3(\mathbf{AG}(n,q),\mathcal{S}^*)$ and $\mathsf{SS}_3(\mathbf{AG}(n,q),\overline{\mathcal{S}}^*)$, where the set of subspaces of $\mathcal{S}^*$ coincides with the one of $\overline{\mathcal{S}}^*$, and where the sets of imaginary points only differ for one plane. Then the derivations at a point of this plane only differ in an $S(2,q,q^2)$ sub Steiner system, and for $q>2$ and $n\geq 3$, clearly both cannot be affine spaces!

3 2-Steiner systems from generalized hexagons

3.1 The split Cayley hexagons

In this section we will need some notions from the theory of generalized hexagons. We refer to [13] for a detailed account on the subject of generalized polygons. Here, we content ourselves with a brief definition of a generalized n-gon and the explicit description of one class of (finite) generalized hexagons. Recall that a point-line geometry $(\mathcal{P},\mathcal{L},\mathtt{I})$ consists of a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines (disjoint from $\mathcal{P}$), and a symmetric incidence relation $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$ telling which points are incident with which lines.

A *generalized n-gon*, $n\geq 2$, or a *generalized polygon*, is a nonempty point-line geometry $\Gamma=(\mathcal{P},\mathcal{L},\mathtt{I})$ the incidence graph of which has diameter n (i.e. any two elements are at most at distance n) and girth $2n$ (i.e., the length of any shortest circuit is $2n$; in particular we assume that there *is* at least one circuit). Recall that the incidence graph is the graph with $\mathcal{P}\cup\mathcal{L}$ as set of vertices, and two vertices x,y form an edge if $x\mathtt{I}y$; the distance between two vertices x,y in that graph will be denoted by $\delta(x,y)$. A *thick* generalized polygon is a generalized polygon for which each element is incident with at least three elements. In this case, the number of points on a line is a constant, say $s+1$, and the number of lines through a point is also a constant, say $t+1$. The pair (s,t) is called the *order* of the polygon.

Now we define the *split Cayley hexagon*, which is a particular class of thick generalized hexagons. They exist for every field, but we only consider finite fields $\mathbf{GF}(q)$, in which case the corresponding split Cayley hexagon is denoted by $\mathsf{H}(q)$. Consider the quadric Q in $\mathbf{PG}(6,q)$ given by the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. The points of $\mathsf{H}(q)$ are all points of Q. The lines of $\mathsf{H}(q)$ are certain lines of Q, namely, those lines of Q whose *Grassmann coordinates* satisfy the equations $p_{01} = p_{36}$, $p_{12} = p_{34}$, $p_{20} = p_{35}$, $p_{03} = p_{56}$, $p_{13} = p_{64}$ and $p_{23} = p_{45}$. The order of $\mathsf{H}(q)$ is (q,q). In fact, the only known generalized hexagons of order (q,q) are $\mathsf{H}(q)$ and its dual (obtained from $\mathsf{H}(q)$ by interchanging the point set with the line set); $\mathsf{H}(q)$ is isomorphic to its dual if and only if q is a power of 3. We will need the following basic properties of $\mathsf{H}(q)$ (for proofs, see [13]).

(H1) Two points of $\mathsf{H}(q)$ are at distance 6 from each other if and only if the line of $\mathbf{PG}(6,q)$ joining them is not contained in Q.

(H2) For every pair $\{a,b\}$ of points of $\mathsf{H}(q)$ at distance 6 from each other, there are exactly $q+1$ points at distance 3 from every line at distance 3 from both a,b; these $q+1$ points form a conic $C(a,b)$ on Q.

(H3) The type preserving (i.e., not interchanging the set of points and the set of lines) automorphism group of $\mathsf{H}(q)$ is Dickson's group $G_2(q)$ extended with the automorphism group of $\mathbf{GF}(q)$. The full automorphism group is the full automorphism group of $G_2(q)$ and differs from the type preserving automorphism group precisely when q is a power of 3.

We will now embed $\mathsf{H}(q)$ in a Steiner system. We will later on also embed the dual $\mathsf{H}^*(q)$ in a Steiner system. It will turn out that this is in fact very easy. In order to have some interesting geometric input, we first restrict ourselves to Steiner systems obtained by choosing the blocks that are no lines of the hexagon amongst the traces of the generalized hexagon in question (a *trace* in a hexagon is a set of points at distance i from a given point p and at distance $6-i$ from another point p', where p and p' are at distance 6 from each other, and where $i \in \{2,3\}$).

3.2 The Steiner system $\mathsf{SS}_2(\mathsf{H}(q))$

We can now define the $S(2, q+1, \frac{q^6-1}{q-1})$ Steiner system $\mathsf{SS}_2(\mathsf{H}(q))$ as follows. The points of $\mathsf{SS}_2(\mathsf{H}(q))$ are the points of $\mathsf{H}(q)$ (or, equivalently, the points of Q); the blocks are the lines of Q together with the sets $C(a,b)$, for a and b two points of $\mathsf{H}(q)$ at distance 6 from each other.

This is indeed a Steiner system because if two points x,y are at distance 2 or 4 from each other in $\mathsf{H}(q)$, then there is a unique line of Q joining them by property (H1); if they are at distance 6 from each other, then there is a unique conic $C(x, y)$ containing them by property (H2).

Theorem 3.1 *If q is even, then $\mathsf{SS}_2(\mathsf{H}(q))$ is isomorphic to $\mathsf{SS}_2(\mathbf{PG}(5, q))$. If q is odd, then the automorphism group of $\mathsf{SS}_2(\mathsf{H}(q))$ is isomorphic to the (type preserving) automorphism group of $\mathsf{H}(q)$. In particular, if q is odd, then $\mathsf{SS}_2(\mathsf{H}(q))$ is not isomorphic to $\mathsf{SS}_2(\mathbf{PG}(5, q))$.*

Proof. If q is even, then one can project from the nucleus of Q and we obtain the Steiner system of points and lines of $\mathbf{PG}(5, q)$, see Section 2.4.14 of [13]. Now suppose q is odd. We prove that the full automorphism group G of $\mathsf{SS}_2(\mathsf{H}(q))$ coincides with the type preserving automorphism group G^* of $\mathsf{H}(q)$. We do that with a number of claims.

Step 1: *The only collineations of Q in G are those that belong to G^*.*

Let g be an automorphism of Q preserving the sets $C(a, b)$, with a, b points of $\mathsf{H}(q)$ at distance 6 from each other. Let L be a line of $\mathsf{H}(q)$. We show that L^g is again a line of $\mathsf{H}(q)$. Choose an arbitrary point x on L and choose two arbitrary points y, y' at distance 2 from x such that y and y' are at mutual distance 4 and neither y nor y' is incident with L. Further, choose two points u, u' at distance 2 from y, y', respectively, both at distance 4 from x and such that u and u' are at mutual distance 6 (it is easy to see that this is possible). Let M and M' be two lines of $\mathsf{H}(q)$ incident with u and u', respectively, and both at distance 6 from L. Let $z \neq y$ and $z' \neq y'$ be two points at distance 3 from L and from M and M', respectively. If we denote by ρ the polarity associated with Q, and with $\pi(a, b)$ the plane containing $C(a, b)$, for all points a, b at mutual distance 6, then it is easy to see (and well known) that $\pi(y, z)^\rho = \langle L, M \rangle$ (where the notation $\langle X \rangle$ means the projective subspace of $\mathbf{PG}(6, q)$ generated by X) and $\pi(y', z')^\rho = \langle L, M' \rangle$. Since g is an automorphism of Q, we have $\langle C(y, z)^g \rangle^\rho = \pi(y^g, z^g)^\rho = \langle L^g, M^g \rangle$ and $\langle C(y', z')^g \rangle^\rho = \pi(y'^g, z'^g)^\rho = \langle L^g, M'^g \rangle$. Now note that $\langle L^g, M^g \rangle$ intersects the quadric Q in a hyperbolic quadric consisting of two reguli, one of which consists entirely of lines of $\mathsf{H}(q)$, the other regulus does not contain any line of $\mathsf{H}(q)$. Similarly for $\langle L^g, M'^g \rangle$. So, if L^g is not a line of $\mathsf{H}(q)$, then the lines $x^g u^g$ and $x^g u'^g$ are, implying that u^g and u'^g are at distance 4 from each other. But since u and u' are not collinear on Q, their images u^g and u'^g are neither, and so these cannot be at distance 4 from each other in $\mathsf{H}(q)$ by (H1), a contradiction. We conclude that L^g is a line of $\mathsf{H}(q)$.

Step 2: *If $G \neq G^*$, then there exists a line L of $\mathsf{H}(q)$ and a conic $C(a, b)$ contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$.*

The lines of $\mathsf{SS}_2(\mathsf{H}(q))$ are of two types, which we call, with obvious notation, *quadric type* and *conic type*. A line of quadric type can also be of *hexagon type* (if and only if it belongs to $\mathsf{H}(q)$). A line of quadric type which is not of hexagon type will be called of *ideal type.* We will usually delete the word type and simply write *quadric line, ideal line, hexagon line,* etc.

The group G^*, which is a subgroup of G, acts transitively on the set of hexagon lines, ideal lines and conic lines, respectively. Also, if some element of G preserves the set of conic lines (in particular if it preserves all types), then it belongs to G^* by Step 1. There are now three possibilities. Either G has two orbits on the set of lines — and then (1) either the ideal lines form an orbit or (2) the hexagon lines form an orbit — or (3) G acts transitively on the set of lines of $\mathsf{SS}_2(\mathsf{H}(q))$. In case (2) every element of G belongs to G^*, so $G = G^*$, a contradiction. In case (1) we consider a point x and its stabilizer G_x in G. There are $q+1$ hexagon lines through x and q^4 conic lines. Since $q+1$ does not divide q^4, the hexagon lines do not form a block of imprimitivity. Hence there exists $g \in G_x$ mapping a hexagon line L through x onto a hexagon line and mapping another hexagon line L' through x onto a conic line. Since L and L' are contained in a projective plane entirely contained in Q (as follows from (H1), the hexagon line L^g and the conic line L'^g are contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$. This proves the claim of Step 2 in this case.

So we may assume that G is transitive on the set of blocks of $\mathsf{SS}_2(\mathsf{H}(q))$. Similarly as above this implies that there exists $g \in G_x$ mapping a quadric line L through some point x onto a quadric line, and mapping another quadric line L' through x onto a conic line. Moreover, it is clear that we may assume that L and L' are contained in a plane of Q (indeed, if not, then we consider a quadric line L'' through x lying is some plane of Q together with L, and lying in some other plane of Q together with L'; we then switch the roles of L and L'' or of L' and L'' according as L''^g is a quadric line or a conic line, respectively). So the quadric line L^g is contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$ different from a plane of Q. This implies that it is contained in at least $q+2$ $S(2, q+1, q^2+q+1)$ Steiner subsystems, as there are precisely $q+1$ $S(2, q+1, q^2+q+1)$ Steiner subsystems consisting of only quadric lines By the transitivity of G on the quadric lines, there are at least $q+2$ $S(2, q+1, q^2+q+1)$ Steiner subsystems containing any hexagon line, and hence we conclude that there is an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$ different from any plane of Q and containing a hexagon line. If this subsystem did not contain a conic line, then it is easy to see that it would be a plane of Q. This completes the proof of Step 2.

Step 3: *If $G \neq G^*$, then every two blocks of $\mathsf{SS}_2(\mathsf{H}(q))$ one of which is a hexagon line, and which meet in a point, are contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$.*

Let L be a hexagon line and x a point on L. By Step 2 and the transitivity of G^* on incident point-line pairs of $\mathsf{H}(q)$, there is a conic line C through x such that L and C are contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$. But now $G^*_{x,L}$ acts transitively on the points of $\mathsf{H}(q)$ at distance 6 from x (see e.g. 4.5.6 of [13]), hence any conic line through x is contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$ containing also L. Now let T be an ideal line through x and choose a second point $y \neq x$ on L. If all points of T were at distance ≤ 4 from y, then L and T are contained in a plane of Q. If there is a point z on T at distance 6 from y, then T must belong to the $S(2, q+1, q^2+q+1)$ Steiner subsystem containing L and $C(y, z)$ (which exists by the above part of the proof, substituting y for x). The claim of Step 3 is proved.

Step 4: *If $G \neq G^*$, then every pair of intersecting blocks of $\mathsf{SS}_2(\mathsf{H}(q))$ is contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$.*

As in the proof of Step 2 there are two possibilities for the orbits of G on the set of blocks of $\mathsf{SS}_2(\mathsf{H}(q))$. Either G acts transitively on the block set — in which case Step 4 follows immediately from Step 3 — or the set of ideal lines forms one of the two orbits. By Step 3 we only have to check whether every pair of intersecting ideal lines is contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$. But this follows from a similar argument as the one at the end of the proof of Step 3. Namely, if a certain pair of intersecting ideal lines is not contained in a plane of Q, then there is a conic line meeting both lines in different points, and hence these three lines are contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem of $\mathsf{SS}_2(\mathsf{H}(q))$.

Step 5: *Conclusion. Either $G = G^*$ (and q is odd) or $\mathsf{SS}_2(\mathsf{H}(q))$ is isomorphic to $\mathsf{SS}_2(\mathbf{PG}(5, q))$ (and q is even).*

If $G \neq G^*$, then by Step 4 above the Steiner system $\mathsf{SS}_2(\mathsf{H}(q))$ satisfies the axiom of Pasch-Veblen-Young, and hence is the point-line system of a projective space $\mathbf{PG}(5, q)$. This implies that $\mathsf{H}(q)$ is embedded in $\mathbf{PG}(5, q)$ and that the lines of $\mathsf{H}(q)$ through a point are contained in a plane of $\mathbf{PG}(5, q)$. By the Main Result (ii) of [11] q is even. □

Remark 3.2 The 2-Steiner system $\mathsf{SS}_2(\mathsf{H}(q))$ is in fact well known among experts in the field of generalized polygons, although to my knowledge, it never appeared in the literature as such. The fact that the automorphism group is as stated, is, again to the best of my knowledge, new (certainly considering the literature).

3.3 The Steiner system $\mathsf{SS}_2(\mathsf{H}^*(q), \beta)$

Now we consider the dual $\mathsf{H}^*(q)$ of $\mathsf{H}(q)$. As already mentioned, we would like to embed this hexagon in a Steiner system in such a way that the new blocks are traces. We first do this for $q \equiv 2 \bmod 3$. More precisely, we construct in this case an $S(2, q+1, \frac{q^6-1}{q-1})$ Steiner system $\mathsf{SS}_2(\mathsf{H}^*(q), \beta)$, where β is an arbitrary mapping from the point set of $\mathsf{H}^*(q)$ to itself with the only property that any point gets mapped onto a point at distance 6, as follows. The point set of $\mathsf{SS}_2(\mathsf{H}^*(q), \beta)$ is the point set of $\mathsf{H}^*(q)$. The block set is defined in the following manner. First, the lines of $\mathsf{H}^*(q)$ are blocks of $\mathsf{SS}_2(\mathsf{H}^*(q), \beta)$ (called of *hexagon* *type*). Secondly, the dual of property (H2) is in fact equivalent to (H2) itself, and so we can define sets $C(a,b)$ in completely the same way as before, for a,b points at distance 6. These are also blocks of $\mathsf{SS}_2(\mathsf{H}^*(q),\beta)$, called of *regulus type*, since these sets of points correspond in $\mathsf{H}(q)$ to a set of $q+1$ lines which forms a regulus of a hyperbolic quadric arising as the intersection of Q with a 3-dimensional space of $\mathbf{PG}(6,q)$. Finally, let x be an arbitrary point of $\mathsf{H}^*(q)$ and consider any point y of $\mathsf{H}^*(q)$ at distance 6 from x and at distance 0 or 2 from x^β. Then the set of points, denoted by x^y, at distance 2 from x and at distance 4 from y contains $q+1$ elements and is by definition a block of $\mathsf{SS}_2(\mathsf{H}^*(q),\beta)$, called of *exceptional type*. We now show that we indeed have a 2-Steiner system. To that end, we first remark that all points of a block of hexagon type are at mutual distance 0 or 2 in $\mathsf{H}^*(q)$, all points of a block of exceptional type are at mutual distance 0 or 4, and all points of a block of regulus type are at mutual distance 0 or 6 in $\mathsf{H}^*(q)$. Let a,b be two distinct points of $\mathsf{H}^*(q)$ and let $2k$ be their mutual distance, $k\in\{1,2,3\}$. If $k=1$, then there is a unique line of $\mathsf{H}^*(q)$ joining them, and no other block can contain both by our remark. Similarly, if $k=3$, then there is a unique block $C(a,b)$ through them and it is of regulus type. Now let $k=2$. There is a unique point x of $\mathsf{H}^*(q)$ at distance 2 from both a,b. If both a,b are at distance 4 from x^β, then $a,b\in x^y$ in a unique way, namely for $y=x^\beta$. Similarly, if a is at distance 4 from x^β, and b is at distance 6 from x^β, then, denoting by y the unique point of $(x^\beta)^b$ at distance 4 from a, both a,b are contained in x^y in a unique way. Suppose now both a,b are at distance 6 from x^β and let a' and b' be the unique points at distance 2 from x^β and at distance 3 from the lines ax and bx, respectively. Let c be an arbitrary point on the line bx, with $c\neq x$ and $c\notin x^{x^\beta}$. There are $q-1$ choices for c. If $|(x^\beta)^a\cap(x^\beta)^c|>1$, then two points of that intersection together with a and c induce the dual of a "Kantor configuration", see 3.7.13 of [13], and such a configuration does not exist in $\mathsf{H}^*(q)$ for $q\equiv 2 \bmod 3$ by [7]. Hence, every of the $q-1$ points c is at distance 4 from at most one of the $q-1$ points of $(x^\beta)^a\setminus\{a',b'\}$. But clearly, for each point $d\in(x^\beta)^a\setminus\{a',b'\}$, there is a unique point c_d on bx at distance 4 from d. We conclude that for each choice of c above, $|(x^\beta)^a\cap(x^\beta)^c)|=1$, and so, in particular, there is a unique point $z\in(x^\beta)^a\cap(x^\beta)^b$. Hence $a,b\in x^z$, with z at distance 2 from x^β, and z is unique with that property. So we indeed have a 2-Steiner system.

Considering two such Steiner systems $\mathsf{SS}_2(\mathsf{H}^*(q),\beta)$ and $\mathsf{SS}_2(\mathsf{H}^*(q),\beta')$, where β differs from β' in exactly one point x, in such a way that $x^{x^{\beta'}}$ is not a block of $\mathsf{SS}_2(\mathsf{H}^*(q),\beta)$ (which is always possible, see for instance [4]), we see that not both can be isomorphic to $\mathsf{SS}_2(\mathbf{PG}(5,q))$, or to $\mathsf{SS}_2(\mathsf{H}(q))$, because they differ from each other in only one plane (and the blocks in one plane of both $\mathsf{SS}_2(\mathbf{PG}(5,q))$ and $\mathsf{SS}_2(\mathsf{H}(q))$ are determined by all other blocks).

Note that we can take special choices for β to obtain certain isomorphisms of $\mathsf{SS}_2(\mathsf{H}^*(q),\beta)$. For instance, if p is a prime that does not divide $(q+1)q(q-1)$, but that divides $|G_2(q)|$, then it is easy to see that any element g of $G_2(q)$ of order p acts fixed point freely on $\mathsf{H}^*(q)$. Hence we may arbitrarily choose x^β for one fixed point x in every point orbit, and then define y^β as $(x^\beta)^{g^m}$, where $y = x^{g^m}$. Then the centralizer of g in $G_2(q)$ acts as an automorphism group on $\mathsf{SS}_2(\mathsf{H}^*(q), \beta)$. In general, $\mathsf{SS}_2(\mathsf{H}^*(q), \beta)$ does not seem to have a big automorphism group.

Note also that the results of this section settles the question whether $\mathsf{H}(q)$ and $\mathsf{H}^*(q)$, $q \equiv 2 \bmod 3$, are embeddable in Steiner systems by just adding traces as blocks. A similar question has been considered for partial geometries in [2]. We will remove the condition $q \equiv 2$ mod 3 in the next subsection.

Finally, note that two intersecting blocks of hexagon type are contained in an $S(2, q+1, q^2+q+1)$ Steiner subsystem (projective plane). This projective plane is Desarguesian, and a proof of this will lead to some interesting questions and observations in Galois geometry.

3.4 Projective plane representations

It is well known that $\mathsf{H}^*(q)$ has a representation (embedding) in $\mathbf{PG}(13, q)$ (related to the Lie algebra of type G_2), see for instance [12]. The set of lines of $\mathsf{H}^*(q)$ through a fixed point x of $\mathsf{H}^*(q)$ is a cone $\mathcal{K}$ in a subspace $\mathbf{PG}(4, q)$ of $\mathbf{PG}(13, q)$ with base curve a normal rational curve $\mathcal{C}$ in some subspace $\mathbf{PG}(3, q)$ of $\mathbf{PG}(4, q)$, where x is a point of $\mathbf{PG}(4, q) \setminus \mathbf{PG}(3, q)$ (see [12]). Now consider any point y of $\mathsf{H}^*(q)$ opposite x. Without loss of generality we may assume that $x^y = \mathcal{C}$. It takes an elementary but easy calculation, which we shall not perform explicitly, to see that, in case $q \equiv 2$ mod 3, if z is opposite x but collinear with y, with $y \neq z$, then x^z is a normal rational curve $\mathcal{C}'$ on $\mathcal{K}$ with the property that the intersection $\langle \mathcal{C} \rangle \cap \langle \mathcal{C}' \rangle$ is the osculating plane to both $\mathcal{C}$ and $\mathcal{C}'$ at the intersection point $\mathcal{C} \cap \mathcal{C}'$. It now follows immediately from the example on page 316 of [3] that the projective plane with point set the set of points collinear in $\mathsf{H}^*(q)$ with x, and blocks the lines through x together with the traces x^u, with u collinear with y and opposite x, is a Desarguesian projective plane. It also follows that, if we replace y by some other element of $C(x, y) \setminus \{x\}$, then we obtain exactly the same projective plane. This can also be proved only using the regularity properties of $\mathsf{H}^*(q)$. On the other hand, if $q \equiv 0$ mod 3, then, with the above notation, $\langle \mathcal{C} \rangle \cap \langle \mathcal{C}' \rangle$ contains a fixed line (independent of z) in $\mathbf{PG}(4, q)$.

Also, it is easy to see that, if $q \not\equiv 0$ mod 3, then every normal rational curve on $\mathcal{K}$ in some 3-dimensional subspace of $\mathbf{PG}(4, q)$ arises as the trace x^w, for some point w of $\mathsf{H}^*(q)$ opposite x. Hence, in order to embed $\mathsf{H}^*(q)$ in a Steiner system where additional blocks are traces, it is sufficient to construct a representation of a projective plane of order q on the cone $\mathcal{K}$ with point set $\mathcal{K}$ and set of lines the generators together with a well chosen set of normal rational curves on $\mathcal{K}$ in 3-dimensional subspaces of $\mathbf{PG}(4, q)$. We call this a *projective system of* 3*-spaces of* $\mathcal{K}$. One possibility is to look for a *linear* such system, i.e., all such subspaces have a line in common. So we ask ourselves if there exists a line L of $\mathbf{PG}(4,q)$ with the property that, given two arbitrary points $a, b \in \mathcal{K} \setminus \{x\}$, the space $\langle a, b, L\rangle$ is 3-dimensional and does not contain x. It is rather easy to see that this problem is equivalent to finding a line L in $\mathbf{PG}(3,q)$ such that every plane through L meets the normal rational curve $\mathcal{C}$ in a unique point. Hence, no point of L can be contained in a chord of $\mathcal{C}$. Also, obviously, every imaginary chord (i.e., a line of $\mathbf{PG}(3,q)$ containing two conjugate imaginary points of $\mathcal{C}$) satisfies this condition. By projecting from L onto a plane in $\mathbf{PG}(4,q)$ skew to L, we see that the corresponding projective plane is Desarguesian. Hence we have proved the following theorem.

Theorem 3.3 *For every prime power q, the generalized hexagon $\mathsf{H}^*(q)$ is embedded in an $S(2, q+1, \frac{q^6-1}{q-1})$ Steiner system whose blocks are lines of $\mathsf{H}^*(q)$ and traces of $\mathsf{H}^*(q)$, and such that two intersecting lines of $\mathsf{H}^*(q)$ are contained in a unique subsystem isomorphic to a Desarguesian projective plane.*

The question arises whether other projective systems of 3-spaces of $\mathcal{K}$ exist. Besides the examples mentioned in the first paragraph of this subsection, I have no idea about non-linear ones, but one can classify all linear such systems. Indeed, the line L (see above) does not contain any point on a chord of $\mathcal{C}$. If it contains a point on a tangent, then an elementary calculation shows that, if $\mathcal{C} = \{(1, r, r^2, r^3) \mid r \in \mathbf{GF}(q)\} \cup \{(0,0,0,1)\}$, we can take without loss of generality for L the line spanned by $(0,1,0,0)$ and $(0,0,1,0)$. This line satisfies our conditions if and only if $q-1$ is relatively prime to 3. If L does not contain any point on a tangent line, then Theorem 1 of [10] implies that L is either an imaginary chord — and then we have the above example — or a so-called imaginary axis, and then $q \equiv 1$ mod 3. Hence we have proved the following theorem.

Theorem 3.4 *For $q \equiv 2$ mod 3, there are at least 3 non-isomorphic projective systems of 3-spaces of $\mathcal{K}$, 2 of which are linear. No other linear systems exist in this case. If $q \not\equiv 2$ mod 3, then exactly two non-isomorphic linear projective systems of 3-spaces of $\mathcal{K}$ exist.*

The existence of non-linear projective systems for $q \not\equiv 2$ mod 3 is an open question. It may be related to flocks of quadratic cones by the observation that projection from a point $y \in \mathcal{K} \setminus \{x\}$ onto some 3-dimensional space of $\mathbf{PG}(4,q)$ not containing x maps $\mathcal{K}$ and the normal rational curves of the system through y onto a system of conics on a quadratic cone which partitions the point set of the cone up to the points on one generator. Such a system is hence "almost" a flock and one could study the case where, varying y on the line xy, one has q flocks.

Let us finally remark that the projective systems corresponding to $\mathsf{SS}_2(\mathsf{H}^*(q), \beta)$, for arbitrary but suitable β, and for $q \equiv 0$ mod 3, are all linear.

3.5 Some general construction

Now we consider any finite classical generalized hexagon Γ of order (q, q) (hence $\mathsf{H}(q)$ or $\mathsf{H}^*(q)$). We define a Steiner system as follows. The points of Γ are the points of the Steiner system. The lines and the sets $C(a, b)$ are again blocks of the Steiner system. For each point p of Γ, we construct an arbitrary (not necessarily Desarguesian) projective plane with point set the set of points collinear with p, and such that the lines of the hexagon through p are lines of the projective plane. Since q is a prime power, this is always possible. The lines of all these planes are also blocks of our Steiner system. Hence we have embedded every classical hexagon of order (q, q) into a 2-Steiner system. In order to do a similar thing with the other classical hexagons (of order (q, q^3) and (q^3, q)), one would first have to construct $S(2, q+1, q^4+q+1)$ and $S(2, q+1, q^4+q^3+1)$ Steiner systems, respectively. This is necessary and sufficient.

4 Some further considerations

There is a rather easy way to construct "new" Steiner systems out of old ones, in a rather free way, especially if the old system has some subsystems. Let us state this construction as a theorem. Even though this construction must be well known, I could not find a proof in the literature, so, for the sake of completeness, I include it (it is very short anyway).

Theorem 4.1 *Let $(\mathcal{P}, \mathcal{B})$ and $(\mathcal{Q}', \mathcal{B}')$ be two t-Steiner systems, $t \geq 2$, with block size k. Let $\mathcal{P}'' \subseteq \mathcal{P}' \subseteq \mathcal{P}$ satisfy the following properties:*

- *(i) every block having at least t points in $\mathcal{P}'$ and at least one point in $\mathcal{P}' \setminus \mathcal{P}''$ is entirely contained in $\mathcal{P}'$,*
- *(ii) every block having at least t points in $\mathcal{P}''$ is disjoint from $\mathcal{P}' \setminus \mathcal{P}''$.*

Suppose there is a bijection $\beta : \mathcal{P}' \to \mathcal{Q}'$ mapping $\mathcal{P}''$ onto the point set of a t-Steiner subsystem $(\mathcal{Q}'', \mathcal{B}'')$ of $(\mathcal{Q}', \mathcal{B}')$ with block size k. We now define a new set $\mathcal{C}$ of blocks in the point set $\mathcal{P}$ as follows. Every member of $\mathcal{B}$ which has at least one point outside $\mathcal{P}'$ or which is contained in $\mathcal{P}''$ belongs to $\mathcal{C}$ (blocks of type (OLD)*). Also, if $B \in \mathcal{B}' \setminus \mathcal{B}''$, then $B^{\beta^{-1}}$ belongs to $\mathcal{C}$ (blocks of type* (NEW)*). There are no other elements of $\mathcal{C}$. Then $(\mathcal{P}, \mathcal{C})$ is a t-Steiner system.*

Proof. Let $A = \{x_1, x_2, \ldots, x_t\}$ be a set of distinct points of $\mathcal{P}$. We show that there is a unique member $C \in \mathcal{C}$ containing A. If $A \not\subseteq \mathcal{P}'$ or if $A \subseteq \mathcal{P}''$, then there is a unique block of type (OLD) — since $(\mathcal{P}, \mathcal{B})$ is a t-Steiner system — and obviously no block of type (NEW) containing A (this is true for $A \subseteq \mathcal{P}''$ by (ii) and the fact that $(\mathcal{Q}'', \mathcal{B}'')$ is a t-Steiner system with block size k). Suppose now $A \subseteq \mathcal{P}'$ with $A \not\subseteq \mathcal{P}''$. By (i), the unique block in $\mathcal{B}$ containing A does not belong to $\mathcal{C}$. Hence, if there is a block containing A, then it must be of type (NEW). The unique block B in $\mathcal{B}'$ containing A^β does not belong to $\mathcal{B}''$ (because $A^\beta \not\subseteq \mathcal{Q}''$) and hence $B^{\beta^{-1}}$ is the unique block of type (NEW) containing A.

The theorem is proved. □

The question I want to raise here is whether the Steiner systems constructed from $\mathsf{H}(q)$ and $\mathsf{H}^*(q)$ in the previous section can also be obtained from $\mathbf{PG}(5, q)$ be applying Theorem 4.1 a number of times. I conjecture that this is not possible. If we call two Steiner systems *equivalent* if they can be obtained from each other by applying Theorem 4.1 a finite number of times, then it seems reasonable to look for different equivalence classes with respect to fixed parameters. I conjecture, but cannot prove, that the constructions in the present paper add two new equivalence classes to $S(2, q+1, \frac{q^6-1}{q-1})$ Steiner systems (at most 2 is obvious).

Remark 4.2 We can also start with an $\mathsf{SS}_2(\mathbf{AG}(n, q))$, construct an $\mathsf{SS}_3(\mathbf{AG}(n, q), \mathcal{S}^*)$ as in Section 2.2, then with the aid of Theorem 4.1, replace one of the Steiner Subsystems isomorphic to $\mathsf{SS}_3(\mathbf{PG}(1, q^m), q)$ with an $\mathsf{SS}_3(\mathbf{AG}(m, q), \overline{\mathcal{S}}^*)$ not isomorphic to $\mathsf{SS}_3(\mathbf{PG}(1, q^m), q)$ in such a way that the two points labelled ∞ do not correspond (and such that the point of $\mathsf{SS}_3(\mathbf{AG}(m, q), \overline{\mathcal{S}}^*)$ that corresponds under the bijection with the point ∞ of $\mathsf{SS}_3(\mathbf{PG}(1, q^m), q)$ has a derived Steiner system not isomorphic to $\mathsf{SS}_2(\mathbf{AG}(m, q))$). This yields $S(3, q+1, q^n+1)$ Steiner systems no derivation of which is isomorphic to an $\mathsf{SS}_2(\mathbf{AG}(n, q))$, for some n, q. We will not prove this here, since it is not important for our purposes. We simply want to point out the usefulness of Theorem 4.1 to construct examples or counterexamples to given conjectures or claims, or to show that certain assumptions in some characterizations are essential.

References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Volume 2, Second Edition, *Encyclopedia of Mathematics and its Applications* **78**, Cambridge University Press, Cambridge, 1999.

[2] A. E. Brouwer, W. H. Haemers and V. D. Tonchev, Embedding partial geometries in Steiner designs, **in** *Geometry, Combinatorial Designs and Related Structures* (J. W. P. Hirschfeld *et al.* (eds.)), Cambridge University Press, Cambridge (1997), 33 – 41.

[3] D. Glynn, A geometrical representation theory for orthogonal arrays, *Bull. Austral. Math. Soc.* **49** (1994), 311 – 324.

[4] E. Govaert and H. Van Maldeghem, Some combinatorial and geometric characterizations of the finite dual classical generalized hexagons, *J. Geom.* **68** (2000), 87 – 95

[5] J. W. P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1985.

[6] J. W. P. Hirschfeld, *Projective Geometry over Finite Fields*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1998.

[7] W. M. Kantor, Generalized quadrangles associated with $G_2(q)$, *J. Combin. Theory Ser.* A **29** (1980), 212 – 219.

[8] J. D. Key, Extendable Steiner designs, *Geom. Dedicata* **41** (1992), 201 – 205.

[9] J. D. Key and A. Wagner, On an infinite class of Steiner systems constructed from affine spaces, *Arch. Math. (Basel)* **47** (1986), 376 – 378.

[10] G. Lunardon and O. Polverino, On the twisted cubic of $\mathrm{PG}(3, q)$, manuscript.

[11] J. A. Thas and H. Van Maldeghem, Flat lax and weak lax embeddings of finite generalized hexagons, *Europ. J. Combin.* **19** (1998), 733 – 751.

[12] J. A. Thas and H. Van Maldeghem, Full embeddings of the dual split Cayley hexagon in $\mathbf{PG}(13, q)$, manuscript submitted for publication.

[13] H. Van Maldeghem, *Generalized Polygons*, Birkhäuser-Verlag, *Monographs in Math.* **93**, Basel, Berlin, Boston, 1998.

Ghent University
Pure Mathematics and Computer Algebra
Galglaan 2,
9000 Gent
BELGIUM
hvm@cage.rug.ac.be