Optimization: A Journal of Mathematical Programming and Operations Research
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gopt20

Stochastic programming problems with generalized integrated chance constraints

Martin Branda [a] [b]

[a] Department of Probability and Mathematical Statistics, Charles University in Prague, Sokolovská 83, Prague 8, 18675, Czech Republic

[b] Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, Prague 8, 182 08, Czech Republic

Version of record first published: 20 Jul 2012

To cite this article: Martin Branda (2012): Stochastic programming problems with generalized integrated chance constraints, Optimization: A Journal of Mathematical Programming and Operations Research, 61:8, 949-968

To link to this article: http://dx.doi.org/10.1080/02331934.2011.587007

Optimization
Vol. 61, No. 8, August 2012, 949–968

Stochastic programming problems with generalized integrated chance constraints

Martin Branda[ab*]

[a]*Department of Probability and Mathematical Statistics, Charles University in Prague, Sokolovská 83, Prague 8, 18675, Czech Republic;* [b]*Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, Prague 8, 182 08, Czech Republic*

(*Received 22 October 2010; final version received 3 May 2011*)

If the constraints in an optimization problem are dependent on a random parameter, we would like to ensure that they are fulfilled with a high level of reliability. The most natural way is to employ chance constraints. However, the resulting problem is very hard to solve. We propose an alternative formulation of stochastic programs using penalty functions. The expectations of penalties can be left as constraints leading to generalized integrated chance constraints, or incorporated into the objective as a penalty term. We show that the penalty problems are asymptotically equivalent under quite mild conditions. We discuss applications of sample-approximation techniques to the problems with generalized integrated chance constraints and propose rates of convergence for the set of feasible solutions. We will direct our attention to the case when the set of feasible solutions is finite, which can appear in integer programming. The results are then extended to the bounded sets with continuous variables. Additional binary variables are necessary to solve sample-approximated chance-constrained problems, leading to a large mixed-integer non-linear program. On the other hand, the problems with penalties can be solved without adding binary variables; just continuous variables are necessary to model the penalties. The introduced approaches are applied to the blending problem leading to comparably reliable solutions.

Keywords: chance constraints; integrated chance constraints; penalty functions; sample approximations; blending problem

AMS Subject Classification: 90C15

1. Introduction

In practical optimization problems, the constraints need not be known precisely but can depend on a realization of a random vector. Our goal is then to ensure that the constraints are fulfilled with a high level of reliability, i.e. for realizations of the

*Email: branda@karlin.mff.cuni.cz

ISSN 0233–1934 print/ISSN 1029–4945 online

http://dx.doi.org/10.1080/02331934.2011.587007
http://www.tandfonline.com

random vector with at least a prescribed probability. The most natural way to deal with this problem is to employ chance constraints. However, solving the resulting chance-constrained problems is not easy. In general, the feasible region is not convex even if the functions are convex, and in many cases it is not even easy to check feasibility because it leads to computations of multivariate integrals. On the other hand, there are some special conditions under which the convexity is preserved, e.g. the log-concave distributions [19], or it is relatively easy to check the feasibility of a point, e.g. for the normal distribution. Numerical methods for computation of multivariate integrals can be helpful for verifying feasibility or solving the problems, see [10] and the references therein. There are various methods for numerically solving chance-constrained problems, as in [19,20]. For the problems with discretely distributed random variables and separable random parts, p-efficient points can be used, cf. [18]. For continuously distributed random variables, the methods based on supporting hyperplanes and reduced gradients are available. In a case where the underlying distribution is continuous or discrete with many realizations, the sample-approximation techniques and the mixed-integer programming reformulation can help us to solve the problem approximately, see [1,16,17]. With an increased sample size we can even approximate the true solution of the chance-constrained problem. However, the resulting problems are NP-hard in general because additional binary variables are necessary to model the chance constraints leading to large mixed-integer problems. A linear problem with random right-hand side is studied, e.g. in [14], where valid inequalities are derived for the mixed-integer linear set of feasible solutions, see also [9] for a review of various methods for solving general mixed-integer linear problems.

We will study other possible formulations of stochastic programming problems which are based on penalty functions. Their main advantage over the chance-constrained problems is that they can be solved easier without any additional binary variables just using continuous variables to model the penalties. On the other hand, this approach does not address finding or approximating an optimal solution of the chance constrained problem. The penalty functions help us to penalize possible violations of the random constraints with respect to the decision vector and the random parameter leading to a reliable solution.

We will direct our attention to the problems with integrated chance constraints which were originally defined as expectation type constraints using maximal penalty function, cf. [11], see also [12,19]. We will define generalized constraints using arbitrary penalty functions. We will show that the resulting problem with several integrated chance constraints is asymptotically equivalent to the problem with penalties in the objective. In [8], the asymptotic equivalence between the problem with one joint chance constraint and the problem with simple penalty function was shown. The approach was recently extended to a whole class of penalty functions in [5], and to the problems with several joint chance constraints in [3] which cover the joint as well as the individual chance-constrained problems as special cases. Note that no assumptions on convexity or linearity of underlying functions were necessary in the proofs. This makes both alternative approaches based on penalty functions more appropriate for stochastic integer and mixed-integer problems, where convexity is usually not present. Stability of the resulting stochastic programming problems with respect to the changes of underlying probability measure can be studied using contamination technique, cf. [5,7].

We will also discuss applications of sample-approximation techniques to the problems with generalized integrated chance constraints. We will direct our attention to the case with a finite set of feasible solutions, which can appear in bounded integer programming, and with an infinite bounded set with continuous variables. We will extend the result to the rates of convergence for a set of feasible solutions defined by sample average approximations of generalized integrated chance constraints to the true set of feasible solutions. First results for expectation-type constraints were proposed in [23].

The introduced approaches will be applied to a particular blending problem, which was formulated for instance in [17] as a joint chance-constrained problem. We will show that the penalty-function approaches and the sample-approximation technique can be helpful in numerical solution of this stochastic optimization problem. We explore how to try to ensure that the chance constraints are fulfilled with a high level of reliability. We compare the ability to generate a feasible solution of the original joint chance-constrained problem using the sample approximations of the chance constraint directly or via sample approximation of the penalty function objective or general integrated chance constraint.

This article is organized as follows. In Section 2, we formulate the chance-constrained problem and propose a possible formulation using the penalty functions. The expectations of the penalized constraints are left in the constraints with prescribed levels or incorporated into the objective as a penalty term. We show that the problems with penalties are asymptotically equivalent with the decreasing levels and the increasing penalty parameter. In Section 3, sample-approximation techniques for solving the problems are discussed and result on rates of convergence for the generalized integrated chance constraint are derived. Numerical comparison of the proposed approaches on a blending problem is included in Section 4. In Section 5, we summarize our results.

2. Relation between chance-constrained and penalty function problems

Let $f(x)$ be a real function on $\mathbb{R}^n$, $g_{ji}(x, \xi)$, $i = 0, \dots, k_j$, $j = 1, \dots, m$, be real functions on $\mathbb{R}^n \times \mathbb{R}^{n'}$ measurable in ξ for all $x \in X$, and ξ be a random vector on $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{R}^{n'}$. The problem with uncertain constraints which depend on a random factor ξ can be formulated as

$$\begin{aligned} &\min_{x \in X} f(x) \\ &\quad \text{s.t.} \\ &g_{11}(x, \xi) \le 0, \dots, g_{1k_1}(x, \xi) \le 0, \\ &\qquad \vdots \qquad\qquad\qquad \vdots \\ &g_{m1}(x, \xi) \le 0, \dots, g_{mk_m}(x, \xi) \le 0. \end{aligned} \tag{1}$$

In this form, it is not obvious as to how to solve the problem. As we claim in Section 1, we would like to ensure that the constraints are fulfilled with a high level of reliability. Hence, if the distribution P of the random vector is known, we can formulate the general chance-constrained problem as follows:

$$
\begin{aligned}
\varphi_{\epsilon}^{CCP} = \min_{x \in X} & f(x) \\
\text{s.t.} & \\
& P\big(g_{11}(x,\xi) \le 0, \ldots, g_{1k_1}(x,\xi) \le 0\big) \ge 1 - \varepsilon_1, \\
& \vdots \\
& P\big(g_{m1}(x,\xi) \le 0, \ldots, g_{mk_m}(x,\xi) \le 0\big) \ge 1 - \varepsilon_m,
\end{aligned} \tag{2}
$$

with an optimal solution x_ϵ, where $\epsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, with given levels $\varepsilon_j \in (0,1)$. The formulation covers the joint ($k_1 > 1$ and $m = 1$) as well as the separate ($k_j = 1$ and $m > 1$) chance-constrained problems as special cases.

Below, we will consider the penalty functions $\vartheta_j : \mathbb{R}^{k_j} \to \mathbb{R}_+,\ j = 1, \ldots, m,$ which are continuous nondecreasing in their components, equal to 0 on $\mathbb{R}^{k_j}_-$ and positive otherwise. Two special penalty functions are readily available: $\vartheta^{1,o}(u) = \sum_{i=1}^{k}([u_i]^+)^o, o > 0$, usually $o = 1$ or $o = 2$, and $\vartheta^2(u) = \max_{1 \le i \le k}[u_i]^+$. Both penalty functions preserve convexity of the random constraints in the decision vector. We denote the penalized constraints

$$p_j(x,\xi) = \vartheta_j(g_{j1}(x,\xi), \ldots, g_{jk_j}(x,\xi)) : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}.$$

Our choice is appropriate, because it holds that

$$P\big(g_{ji}(x,\xi) \le 0,\ i = 1, \ldots, k_j\big) \ge 1 - \varepsilon_j \Longleftrightarrow P\big(p_j(x,\xi) > 0\big) \le \varepsilon_j. \tag{3}$$

There are two possible stochastic programming formulations of the problem with a random factor using the penalty functions. We can define the problem with generalized integrated chance constraints as follows

$$\varphi_L^{ICC} = \min_{x \in X} \Big\{ f(x) : \ \text{s.t. } \mathbb{E}[p_j(x,\xi)] \le L_j, j = 1, \ldots, m \Big\} \tag{4}$$

for some prescribed bounds $L_j \ge 0$, $L = (L_1, \ldots, L_m)'$. with an optimal solution x_L^{ICC}. The integrated chance constraints were originally defined using the penalty function ϑ^2, cf. [11,12], but any penalty function can be used in their definition.

The expectations of the penalized constraints can also be incorporated into the objective function as a penalty term leading to the problem:

$$\varphi_N = \min_{x \in X} \left[f(x) + N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x,\xi)] \right] \tag{5}$$

with N being a positive parameter. We denote x_N an optimal solution of (5). The approach for solving non-linear deterministic programs with several constraints using the penalty functions is well studied in literature. Algorithms and basic theory based on continuity and Karush–Kuhn–Tucker conditions are explained in [2,15].

A rigorous proof of the relationship between the optimal values of (2) and those of (5) for a special additive penalty function and one chance constraint was given by [8]. The approach was recently extended to a whole class of penalty functions with desirable properties and to the problems with several joint chance constraints, which was done in [3,5]. The reformulation of chance-constrained problems using the penalties has been applied in water management, insurance and engineering, cf. [6,8,24].

We can compare the formulation (4) with the following alternative expression of the penalty function problem

$$\min_{(x,u)\in X\times\mathbb{R}^m}\left\{f(x)+N\sum_{j=1}^{m}u_j:\ \text{s.t. } \mathbb{E}[p_j(x,\xi)]\le u_j, u_j\ge 0,\ j=1,\dots,m\right\}.$$

The problems differ by incorporating the upper bounds u_j into the penalty term of the objective function. However, this term converges to zero as N goes to infinity under quite mild conditions. We can conclude that both problems are equivalent as L_j goes to zero and the penalty parameter N to infinity. We will prove this in the following theorem, including explicit bounds on the optimal values.

THEOREM 2.1 *Consider the two problems* (4) *and* (5), *and assume that* $X\neq\emptyset$ *is a compact set*, $f(x)$ *is a continuous function*, $\vartheta_j:\mathbb{R}^{k_j}\to\mathbb{R}_+$, $j=1,\dots,m$, *are continuous functions*, *nondecreasing in their components*, *which are equal to* 0 *on* $\mathbb{R}^{k_j}_-$ *and positive otherwize*, *denote*

$$p_j(x,\xi)=\vartheta_j(g_{j1}(x,\xi),\dots,g_{jk_j}(x,\xi)),\quad j=1,\dots,m,$$

and assume that

(i) $g_{ji}(\cdot,\xi)$, $i=1,\dots,k_j$, $j=1,\dots,m$, *are almost surely continuous*;
(ii) *there exists a nonnegative random variable* $C(\xi)$ *with* $\mathbb{E}[C(\xi)]<\infty$, *such that* $|p_j(x,\xi)|\le C(\xi)$, $j=1,\dots,m$ *for all* $x\in X$;
(iii) $\mathbb{E}[p_j(x',\xi)]=0$, $j=1,\dots,m$ *for some* $x'\in X$.

For arbitrary $\gamma\in(0,1)$, $N>0$ *and* $L_j\ge 0$ *put*

$$L_j(x)=\mathbb{E}[p_j(x,\xi)],\quad j=1,\dots,m,$$
$$\alpha_N(x)=N\cdot\sum_{j=1}^{m}\mathbb{E}[p_j(x,\xi)],$$
$$\beta_L(x)=\left(\sum_{j=1}^{m}L_j\right)^{\gamma-1}\sum_{j=1}^{m}\mathbb{E}[p_j(x,\xi)],$$

and let $[N^{1/(\gamma-1)}/m]=(N^{1/(\gamma-1)}/m,\dots,N^{1/(\gamma-1)}/m)'$ *be the vector of length m*.

Then for any prescribed $L_j\ge 0$ *there always exists N large enough so that minimization* (5) *generates the optimal solutions* x_N, *which also satisfy the integrated chance constraints* (4) *with the given* $L=(L_1,\dots,L_m)'$.

Moreover, *bounds on the optimal value* φ_L^{ICC} *of* (4) *based on the optimal value* φ_N *of* (5) *and vice versa can be constructed*

$$\varphi_{\left(\sum_{j=1}^{m}L_j(x_N)\right)^{\gamma-1}}-\beta_{L(x_N)}(x_{L(x_N)})\le\varphi^{ICC}_{L(x_N)}\le\varphi_N-\alpha_N(x_N),$$
$$\varphi^{ICC}_{L(x_N)}+\alpha_N(x_N)\le\varphi_N\le\varphi^{ICC}_{[N^{1/(\gamma-1)}/m]}+\beta_{[N^{1/(\gamma-1)}/m]}(x^{ICC}_{[N^{1/(\gamma-1)}/m]}),$$

with

$$\lim_{N\to+\infty}\alpha_N(x_N)=\lim_{N\to+\infty}L_j(x_N)=\lim_{L_{\max}\to 0_+}\beta_L(x_L^{ICC})=0$$

for any sequences of the optimal solutions x_N and x_L^{ICC} where $L_{\max}$ denotes the maximal component of the vector L.

Proof We denote

$$\delta_N=\sum_{j=1}^m \mathbb{E}[p_j(x_N,\xi)]$$

for a sequence x_N of the optimal solutions of the problem (5). Our assumptions and general properties of the penalty function method [2, Theorem 9.2.2] ensure that, for any sequence x_N of the optimal solutions, $\delta_N\to 0_+$ and also $\alpha_N(x_N)=N\delta_N\to 0$ as $N\to\infty$. Then

$$\mathbb{E}[p_j(x_N,\xi)]\to 0,\quad \text{as } N\to\infty,\ j=1,\ldots,m.$$

Based on this result, it is obvious that for N large enough the terms $\mathbb{E}[p_j(x_N,\xi)]$ are arbitrarily small, which implies that the optimal solution x_N is feasible for the integrated chance-constrained problem with L_j equal or greater than $\mathbb{E}[p_j(x_N,\xi)]$.

The following trivial convergence holds

$$\mathbb{E}[p_j(x_L^{ICC},\xi)]\to 0,\quad \text{as } L_j\to 0_+,\ j=1,\ldots,m.$$

Accordingly, for any $\gamma\in(0,1)$

$$\begin{aligned}\beta_L(x_L^{ICC})&=\left(\sum_{j=1}^m L_j\right)^{\gamma-1}\sum_{j=1}^m\mathbb{E}[p_j(x_L^{ICC},\xi)]\\&\le\left(\sum_{j=1}^m L_j\right)^{\gamma}\to 0,\quad \text{as } L_{\max}\to 0_+.\end{aligned}$$

If we set

$$L_j(x_N)=\mathbb{E}[p_j(x_N,\xi)],\quad j=1,\ldots,m,$$

then the optimal solution x_N of the expected value problem is feasible for the integrated chance-constrained program with $L(x_N)=(L_1(x_N),\ldots,L_m(x_N))$.

Hence, we obtain the inequality

$$\begin{aligned}\varphi_N&=f(x_N)+N\cdot\sum_{j=1}^m\mathbb{E}[p_j(x_N,\xi)]\\&\ge f(x_{L(x_N)})+N\cdot\sum_{j=1}^m\mathbb{E}[p_j(x_N,\xi)]\\&=\varphi_{L(x_N)}^{ICC}+\alpha_N(x_N).\end{aligned}$$

Finally,

$$\varphi_L^{ICC} = \left(\varphi_L^{ICC} + \left(\sum_{j=1}^m L_j\right)^{\gamma-1} \sum_{j=1}^m \mathbb{E}\big[p_j(x_L^{ICC}, \xi)\big]\right) - \left(\sum_{j=1}^m L_j\right)^{\gamma-1} \sum_{j=1}^m \mathbb{E}[p_j(x_L^{ICC}, \xi)]$$
$$\geq \varphi_{(\sum_{j=1}^m L_j)^{\gamma-1}} - \beta_L(x_L^{ICC}).$$

Using the previous inequalities we can obtain the bounds on the optimal values. This completes the proof. ■

Note that the theorem does not make any statement on the convergence of the optimal solutions, but it relates the optimal values for certain values of the levels and the penalty parameter. We will investigate the behaviour of the optimal solutions in the numerical study in Section 4.

Remark 1 No assumptions on convexity of underlying functions and sets were necessary in the proof. Hence, the approach can be used for mixed-integer stochastic programs where convexity is not usually present [4].

Remark 2 Assumption (iii) may be very strong because it requires existence of a point for which the random constraints are fulfilled for almost all realizations of the random vector ξ.

3. Sample approximations using Monte-Carlo techniques

In this section, we will address the sample-approximation technique and derive rates of convergence for the problems with generalized integrated chance constraints. The sample-average approximations were applied to the expected value constrained problems in [23]. We will extend this result to the case with several generalized integrated chance constraints.

Usually, the sample approximation of the chance-constrained problems leads only to a feasible solution of the original problem. Using the results of [21], the problem can be formulated as a large mixed-integer non-linear program. However, due to the increasing number of binary variables, it may be very difficult to solve the resulting problem, even using special solvers for the mixed-integer problems. Hence, it may be interesting to investigate the ability to generate a feasible solution of the chance-constrained problem, i.e. a highly reliable solution, using the penalty function problems where no additional integer variables are necessary. Our approach is summarized in Table 1.

Sample-approximation techniques for chance-constrained problems were investigated in [1,13,17] and were generalized for the case with several joint chance constraints in [3]. We can also refer to [22] for the main results on the sample average approximation techniques for the stochastic programs with expectation type objective, which cover the problems with penalties in the objective.

In all of the problems, it is, in general, not clear as to how to choose the parameters ensuring the reliability levels of the original chance-constrained problem. It is unclear even when solving the sample-approximated chance-constrained problem. We will show that by solving all the problems, we are able to obtain highly reliable solutions of the original problem. It is obvious that the necessary sample size and values of the parameters depend on the particular problem.

Table 1. Formulation and approximation schema..

		1. Stochastic programming formulation		2. Sample approximation (SA)		3. Solution validation
		Integrated chance constrained problem (ICC)	→	SA ICC	→	Reliability
		↑				
Program with a random factor	↗ → ↘	Chance constrained problem (CCP)	→	SA CCP	→	Reliability
		↓				
		Problem with penalty objective (PPO)	→	SA PPO	→	Reliability

First results on parameter settings can be found in [24] where an engineering problem of beam design was solved using chance constraints and penalties in the objective.

Now, we turn our attention to the sample-approximation technique applied to the generalized integrated chance constraints. Let $\xi^1, \ldots, \xi^S$ be an independent Monte-Carlo sample of the underlying distribution of the random vector ξ. We will denote the sets of feasible solutions of the original and the sample-approximated problem as

$$X_L = \left\{x \in X : \ p_j(x) := \mathbb{E}[p_j(x,\xi)] \le L_j, \ j = 1, \ldots, m\right\},$$

$$X_L^S = \left\{x \in X : \ p_j^S(x) := \frac{1}{S}\sum_{s=1}^{S} p_j(x,\xi^s) \le L_j, \ j = 1, \ldots, m\right\}.$$

In [23] the set of feasible solutions of the original problem was relaxed. We will consider the case when the approximated constraints are relaxed and the original constraints remain unchanged. Hence we deal with the probability $P(X_L \subseteq X_{L+\tau}^S)$ that the original set of feasible solutions is contained in the relaxed sample-approximated set and we will show that it increases exponentially with increasing sample size. The same results are valid for the probability $P(X_{L-\tau}^S \subseteq X_L)$. Using the rate of convergence, we are able to estimate sample size necessary to ensure that the feasible solutions of the original problem are feasible for the relaxed sample-approximated problems with a high probability and vice versa. We will direct our attention to the case when the set of feasible solutions is finite or bounded infinite, where additional assumptions are necessary. We will denote by $|X|$ the cardinality of the set X.

3.1. *Finite* $|X|$

Theorem 3.1 *Let*

(i) *the set of feasible solutions be finite*, *i.e.* $|X| < \infty$,
(ii) *the moment generating function of the difference* $p_j(x,\xi) - p_j(x)$ *be finite.*

Then for small $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$,

(a) *the probability that the set of feasible solutions is contained in the relaxed sample-approximated set of feasible solutions increases exponentially fast with increasing sample size, and it holds that*

$$P(X_L \subseteq X^S_{L+\tau}) \geq 1 - m|X| \exp\left\{ -S \min_{j,x} \frac{\tau_j^2}{2\sigma_{jx}^2} \right\},$$

where $\sigma_{jx}^2 = \mathrm{Var}[p_j(x,\xi) - p_j(x)]$ *and the minimum is taken over* $x \in X$ *and* $j \in \{1, \ldots, m\}$.

(b) *it is possible to estimate the sample size S such that the feasible solutions of the original problem are feasible for the relaxed sample-approximated problems with a high probability* $1-\delta$, *i.e.*

$$S \geq \frac{1}{\min_{j,x} \tau_j^2 / 2\sigma_{jx}^2} \ln \frac{m|X|}{\delta}.$$

Proof For $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$, it holds that

$$\begin{aligned} 1 - P(X_L \subseteq X^S_{L+\tau}) &= P\big(\exists_{\tilde{j} \in \{1,\ldots,m\}} \exists x \in X: \ p_{\tilde{j}}(x) \leq L_{\tilde{j}} \\ &\leq \sum_{j=1}^m \sum_{x \in X} P(p_j^S(x) - p_j(x) > \tau_j) \\ &\leq \sum_{j=1}^m \sum_{x \in X} \exp\{-S I_{jx}(\tau_j)\}, \end{aligned}$$

where I_{jx} denotes the large deviations rate function, which is the Fenchel dual to the logarithm of the moment generating function of the difference $p_j(x,\xi) - p_j(x)$, which is assumed to be finite, i.e.

$$I_{jx}(\tau_j) = \sup_{t \in \mathbb{R}} \left\{ t\tau_j - \ln \mathbb{E}\big[e^{t(p_j(x,\xi) - p_j(x))}\big] \right\}.$$

For a small τ_j we can use the following estimate

$$I_{jx}(\tau_j) \geq \frac{\tau_j^2}{2\sigma_{jx}^2},$$

where $\sigma_{jx}^2 = \mathrm{Var}[p_j(x,\xi) - p_j(x)]$. Using the estimate we get

$$\begin{aligned} 1 - P(X_L \subseteq X^S_{L+\tau}) &\leq \sum_{j=1}^m \sum_{x \in X} \exp\left\{ -S \frac{\tau_j^2}{2\sigma_{jx}^2} \right\} \\ &\leq m|X| \exp\left\{ -S \min_{j,x} \frac{\tau_j^2}{2\sigma_{jx}^2} \right\}, \end{aligned}$$

where the minimum is taken over $x \in X$ and $j \in \{1, \ldots, m\}$. Using the previous upper bound it is possible to estimate the sample size S such that the feasible solutions of the original problem are feasible for the relaxed sample-approximated problems with a high probability. ■

3.2. *Bounded* $|X|$

If the set of feasible solutions is infinite but bounded, we must add an assumption on Lipschitz continuity of the penalized constraints. Then we must differentiate between two cases: when the Lipschitz modulus is fixed or random.

We can use the result valid for the finite case, because for a given $\upsilon > 0$ there exists a finite set X_υ with $|X_\upsilon| \le D^n/\upsilon^n$ such that for any $x \in X$ there is a $x' \in X_\upsilon$ such that $\|x - x'\| \le \upsilon$. The choice of υ will be discussed in the proofs below.

3.2.1. *Fixed Lipschitz modulus*

THEOREM 3.2 *Let*

(i) *the set of feasible solutions X be bounded, not necessarily finite, with diameter $D = \sup_{x,x' \in X} \|x - x'\|$,*

(ii) *the functions $p_j(x, \xi)$ be Lipschitz continuous on X moduli $M_j > 0$ which do not depend on ξ, i.e.*

$$|p_j(x,\xi) - p_j(x',\xi)| \le M_j \|x - x'\| \quad \forall x, x' \in X, \ \forall \xi \in \Xi, \ \forall j,$$

(iii) *the moment generating function of $p_j(x,\xi) - p_j(x)$ be finite.*

Then for small $\tau_j > 0$, $\tau = (\tau_1, \dots, \tau_m)$,

(a) *the probability that the set of feasible solutions is contained in the relaxed sample-approximated set of feasible solutions increases exponentially with increasing sample size, and it holds that*

$$P(X_L \subseteq X^S_{L+\tau}) \ge 1 - m\left(\frac{D}{\upsilon}\right)^n \exp\left\{-S \min_{j,x} \frac{(\tau_j - 2M_j\upsilon)^2}{2\sigma^2_{jx}}\right\},$$

where $\sigma^2_{jx} = \mathrm{Var}[p_j(x,\xi) - p_j(x)]$ and the minimum is taken over $x \in X_\upsilon$ and $j \in \{1, \dots, m\}$. The constant υ is chosen such that $\tau_j - 2M_j\upsilon$ is small $\forall j$.

(b) *we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original feasibility set and vice versa with a high probability, equal to $1 - \delta$*

$$S \ge \frac{1}{\min_{j,x}(\tau_j - 2M_j\upsilon)^2/2\sigma^2_{jx}} \left(\ln\frac{m}{\delta} + n\ln\frac{D}{\upsilon}\right).$$

Proof For $\tau_j > 0$, $\tau = (\tau_1, \dots, \tau_m)$, it holds that

$$\begin{aligned}
1 - P(X_L \subseteq X^S_{L+\tau}) &= P\big(\exists_{\tilde{j} \in \{1,\dots,m\}} \exists x \in X : \ p_{\tilde{j}}(x) \le L_{\tilde{j}} \ \& \ p^S_{\tilde{j}}(x) > L_{\tilde{j}} + \tau_{\tilde{j}}\big) \\
&\le P\big(\exists_{\tilde{j} \in \{1,\dots,m\}} \exists x \in X_\upsilon : \ p_{\tilde{j}}(x) \le L + M_{\tilde{j}}\upsilon \\
&\quad \& \ p^S_{\tilde{j}}(x) > L + \tau_{\tilde{j}} - M_{\tilde{j}}\upsilon\big) \\
&\le \sum_{j=1}^m \sum_{x \in X_\upsilon} P(p^S_j(x) - p_j(x) > \tau_j - 2M_j\upsilon) \\
&\le \sum_{j=1}^m \sum_{x \in X_\upsilon} \exp\{-SI_{jx}(\tau_j - 2M_j\upsilon)\}.
\end{aligned}$$

If υ is chosen such that $\tau_j - 2M_j\upsilon$ are small, we can use the following bound

$$I_{jx}(\tau_j - 2M_j\upsilon) \geq \frac{(\tau_j - 2M_j\upsilon)^2}{2\sigma_{jx}^2},$$

where $\sigma_{jx}^2 = \mathrm{Var}[p_j(x,\xi) - p_j(x)]$. Then we obtain

$$\begin{aligned} 1 - P(X_L \subseteq X_{L+\tau}^S) &\leq |X_\upsilon| \sum_{j=1}^{m} \exp\left\{ - S\frac{(\tau_j - 2M_j\upsilon)^2}{2\sigma_{jx}^2} \right\} \\ &\leq m\left(\frac{D}{\upsilon}\right)^n \exp\left\{ - S\min_{j,x} \frac{(\tau_j - 2M_j\upsilon)^2}{2\sigma_{jx}^2} \right\}. \end{aligned}$$

Finally, we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original feasibility set with a high probability. ■

3.2.2. *Random Lipschitz modulus*

Theorem 3.3 *Let*

(i) *the set of feasible solutions X be bounded with a diameter $D = \sup_{x,x' \in X} \|x - x'\|$,*
(ii) *$p_j(x,\xi)$ be Lipschitz continuous on X moduli $M_j(\xi) > 0$ which depend on ξ, i.e.*

$$|p_j(x,\xi) - p_j(x',\xi)| \leq M_j(\xi)\|x - x'\| \quad \forall x, x' \in X,\ \forall \xi \in \Xi,\ \forall j,$$

(iii) *$M_j = \mathbb{E}[M_j(\xi)] < \infty\ \forall j$,*
(iv) *the moment generating functions of $p_j(x,\xi) - p_j(x)$ and $M_j(\xi) - M_j$ be finite.*

Then for $\tau_j > 0$ small, $\tau = (\tau_1, \ldots, \tau_m)$,

(a) *the probability that the set of feasible solutions is contained in the relaxed sample-approximated set of feasible solutions increases exponentially with increasing sample size, and it holds that*

$$P(X_L \subseteq X_{L+\tau}^S) \geq 1 - m\left(1 + \frac{D^n}{\upsilon^n}\right)\exp\{-Sd(\tau)\},$$

where

$$\begin{aligned} \sigma_{jx}^2 &= \mathrm{Var}[p_j(x,\xi) - p_j(x)], \\ \sigma_{M_j}^2 &= \mathrm{Var}[M_j(\xi) - M_j], \\ \mathrm{d}(\tau) &= \min\left\{ \min_{j,x} \frac{\tau_j^2}{8\sigma_{jx}^2},\ \min_{j} \frac{\tau_j^2}{8\sigma_{M_j}^2} \right\}, \end{aligned}$$

the minimum is taken over $x \in X_\upsilon$ and $j \in \{1, \ldots, m\}$ and the constant $\upsilon = \max_j \tau_j/(4M_j + \tau_j)$.

(b) *we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original* *feasibility set and vice versa with a high probability, equal to* $1-\delta$

$$S \geq \frac{1}{\mathrm{d}(\tau)}\left(\ln\frac{m}{\delta}+\ln\left(1+\frac{D^n}{\upsilon^n}\right)\right).$$

Proof Denote $M_j^S = 1/S\sum_{s=1}^S M_j(\xi^s)$ the sample-approximated Lipschitz modulus. For $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$, it holds that

$$\begin{aligned}
1 - P(X_L \subseteq X_{L+\tau}^S) &= P\big(\exists_{\tilde{j}\in\{1,\ldots,m\}}\exists x\in X:\ p_{\tilde{j}}(x) \leq L_{\tilde{j}}\ \&\ p_{\tilde{j}}^S(x) > L_{\tilde{j}} + \tau_{\tilde{j}}\big)\\
&\leq P\big(\exists_{\tilde{j}\in\{1,\ldots,m\}}\exists x\in X_\upsilon:\ p_{\tilde{j}}(x) \leq L + M_{\tilde{j}}\upsilon\\
&\quad\ \&\ p_{\tilde{j}}^S(x) > L + \tau_{\tilde{j}} - M_{\tilde{j}}^S\upsilon\big)\\
&= P(\exists_{\tilde{j}\in\{1,\ldots,m\}}\exists x\in X_\upsilon : p_{\tilde{j}}^S(x) - p_{\tilde{j}}(x) > \tau_{\tilde{j}} - (M_{\tilde{j}}^S + M_{\tilde{j}})\upsilon).
\end{aligned}$$

In this case, such υ should be chosen that the following implications hold for any j

$$M_j^S - M_j > \frac{\tau_j}{2} \ \Rightarrow\ (M_j^S + M_j)\upsilon > \frac{\tau_j}{2}.$$

It is enough to set

$$\upsilon_j = \frac{\tau_j}{4M_j + \tau_j}$$

and $\upsilon = \max_j \upsilon_j$. Then, if the moment generating functions of $p_j(x,\xi) - p_j(x)$ and $M_j(\xi) - M_j$ are finite, we can use the following estimates

$$\begin{aligned}
&P(\exists_{\tilde{j}\in\{1,\ldots,m\}}\exists x\in X_\upsilon : p_{\tilde{j}}^S(x) - p_{\tilde{j}}(x) + (M_{\tilde{j}}^S + M_{\tilde{j}})\upsilon > \tau_{\tilde{j}})\\
&\quad\leq \sum_{j=1}^m P(M_j^S > M_j + \tau_j/2) + \sum_{j=1}^m P(\exists x\in X_\upsilon : p_j^S(x) - p_j(x) > \tau_j/2)\\
&\quad\leq \sum_{j=1}^m P(M_j^S - M_j > \tau_j/2) + \sum_{j=1}^m\sum_{x\in X_\upsilon} P(p_j^S(x) - p_j(x) > \tau_j/2)\\
&\quad\leq \sum_{j=1}^m \exp\{-SI_{M_j}(\tau_j/2)\} + \sum_{j=1}^m\sum_{x\in X_\upsilon}\exp\{-SI_{jx}(\tau_j/2)\},
\end{aligned}$$

where I_{M_j}, I_{jx} are the corresponding large deviation functions. If τ_j are small, we can use the estimates

$$I_{jx}(\tau_j/2) \geq \frac{\tau_j^2}{8\sigma_{jx}^2}, \quad I_{M_j}(\tau_j/2) \geq \frac{\tau_j^2}{8\sigma_{M_j}^2},$$

where $\sigma_{M_j}^2 = \mathrm{Var}[M_j^S - M_j]$. We set

$$d(\tau) = \min\left\{\min_{j,x}\frac{\tau_j^2}{8\sigma_{jx}^2},\ \min_j\frac{\tau_j^2}{8\sigma_{M_j}^2}\right\}.$$

Then

$$1 - P(X_L \subseteq X^S_{L+\tau}) \le m(1 + |X_\upsilon|)\exp\{-Sd(\tau)\}.$$

Finally, we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original feasibility set with a high probability. ■

4. Numerical comparison

In this section, we deal with blending problem which was originally defined as a joint chance-constrained problem. The costs of the fertilizers are minimized subject to the constraints on minimal nutrients necessary to increase the production of a crop. However, the first fertilizer has an uncertain nutrient content.

The problem was solved using the sample-approximation method in [17]. We will show that by solving the sample-approximated problems with expectation of the penalized random constraints we can also obtain reliable solutions of the underlying problem.

The blending problem is formulated as jointly chance-constrained program

$$\begin{aligned} \varphi_\varepsilon^{CCP} = \min\, & x_1 + x_2 \\ \text{s.t.}\ & \\ & P(\xi_1 x_1 + x_2 \ge 7, \xi_2 x_1 + x_2 \ge 4) \ge 1 - \varepsilon, \\ & x_1 \ge 0, \quad x_2 \ge 0, \end{aligned}$$

where $\varepsilon \in (0, 1)$ and the random components (ξ_1, ξ_2) are independent and have uniform distributions on the intervals $[1, 4]$ and $[1/3, 1]$. The explicit solution can be obtained, cf. [17], depending on the choice of the level $\varepsilon \in [0.5, 1]$:

$$x_1^* = \frac{9}{11 - 9(1-\varepsilon)}, \quad x_2^* = \frac{41 - 36(1-\varepsilon)}{11 - 9(1-\varepsilon)}, \quad \varphi_\varepsilon^{CCP} = \frac{50 - 36(1-\varepsilon)}{11 - 9(1-\varepsilon)}.$$

Optimal solutions for particular choices of the level ε are given in Table 2.

Using a penalty function $\vartheta : \mathbb{R}^2 \to \mathbb{R}_+$ we can obtain the problem with a generalized integrated chance constraint

$$\begin{aligned} & \min x_1 + x_2 \\ & \text{s.t.} \\ & \mathbb{E}\Big[\vartheta(7 - \xi_1 x_1 - x_2, 4 - \xi_2 x_1 - x_2)\Big] \le L, \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

Table 2. Blending problem – optimal solutions.

ε	x_1^*	x_2^*	$\varphi_\varepsilon^{CCP}$
0.1	3.1034	2.9655	6.0690
0.05	3.6735	2.7755	6.4490
0.01	4.3062	2.5646	6.8708
0.005	4.4010	2.5330	6.9340
0.001	4.4798	2.5067	6.9866

for some level $L \geq 0$. If we incorporate the expectation of the penalized constraints into the objective, we get

$$\begin{aligned}
&\min x_1 + x_2 + N \cdot \mathbb{E}\Big[\vartheta(7 - \xi_1 x_1 - x_2, 4 - \xi_2 x_1 - x_2)\Big] \\
&\text{s.t.} \\
&x_1 \geq 0,\ x_2 \geq 0,
\end{aligned}$$

for some $N > 0$. The reformulation using penalty objective was also considered in [5] to study stability of the optimal value with respect to the changes of the underlying probability measure using contamination technique. In a case where the penalty term is small enough, the problem with penalty objective can serve as a good approximation of the chance-constrained problem. This was confirmed by the proposed numerical experiment.

We generated 100 samples for each sample size (100, 150, 200, 500) and solved the resulting problems using GAMS 23.2 and IBM ILOG CPlex 12.1. The sample sizes were chosen according to the recommendations given in [17] (100–200) and one larger sample size (500) was taken into consideration. We used two penalty functions in both problems: $\vartheta^{1,1}(u_1, u_2) = [u_1]^+ + [u_2]^+$ and $\vartheta^2(u_1, u_2) = \max\{[u_1]^+, [u_2]^+\}$.

To solve the sample-approximated chance constrained problem, additional binary variables y_s are necessary

$$\begin{aligned}
&\min x_1 + x_2 \\
&\text{s.t.} \\
&7 - \xi_1^s x_1 + x_2 \leq C(1 - y_s), \\
&4 - \xi_2^s x_1 + x_2 \leq C(1 - y_s), \\
&\frac{1}{S}\sum_{s=1}^{S} y_s \geq 1 - \gamma, \\
&x_1 \geq 0,\ x_2 \geq 0,\ y_s \in \{0, 1\}, s = 1, \ldots, S,
\end{aligned}$$

where C is a large enough constant. On the other hand, in the penalty problems no additional binary variables are necessary and only continuous variables are needed to model positive parts in the penalty functions. We obtained a sample-approximated problem with generalized integrated chance constraint using the penalty function $\vartheta^{1,1}$:

$$\begin{aligned}
&\min x_1 + x_2 \\
&\text{s.t.} \\
&7 - \xi_1^s x_1 + x_2 \leq u_{1s}, \\
&4 - \xi_2^s x_1 + x_2 \leq u_{2s}, \\
&\frac{1}{S}\sum_{s=1}^{S} u_{1s} + u_{2s} \leq L, \\
&x_1 \geq 0,\ x_2 \geq 0,\ u_{1s}, u_{2s} \geq 0, s = 1, \ldots, S,
\end{aligned}$$

Table 3. Blending problem – CCP.

S	γ	Minimal reliability	Mean reliability	Mean optimal solution x_1	x_2	Mean optimal value
100	0.1	0.7383	0.8739	3.0255	2.9660	5.9916
100	0.05	0.8183	0.9308	3.6090	2.7717	6.3807
100	0.01	0.9225	0.9700	4.1738	2.5814	6.7552
100	0.005	0.9330	0.9788	4.3266	2.5294	6.8560
100	0.001	0.9330	0.9788	4.3266	2.5294	6.8560
150	0.1	0.8108	0.8876	3.0854	2.9563	6.0418
150	0.05	0.8816	0.9433	3.6892	2.7563	6.4456
150	0.01	0.9404	0.9812	4.2654	2.5638	6.8291
150	0.005	0.9488	0.9885	4.3976	2.5194	6.9170
150	0.001	0.9488	0.9885	4.3976	2.5194	6.9170
200	0.1	0.8321	0.8921	3.1193	2.9448	6.0641
200	0.05	0.8854	0.9435	3.6936	2.7539	6.4475
200	0.01	0.9611	0.9869	4.3786	2.5243	6.9029
200	0.005	0.9600	0.9858	4.3563	2.5318	6.8881
200	0.001	0.9611	0.9903	4.4423	2.5028	6.9451
500	0.1	0.8587	0.8952	3.0928	2.9634	6.0561
500	0.05	0.9113	0.9470	3.6682	2.7728	6.4409
500	0.01	0.9674	0.9909	4.3635	2.5406	6.9041
500	0.005	0.9817	0.9958	4.4577	2.5091	6.9668
500	0.001	0.9817	0.9962	4.4654	2.5066	6.9719

for some level $L > 0$, sample-approximated problem with integrated chance constraint using the penalty function ϑ^2

$$
\begin{aligned}
&\min x_1 + x_2 \\
&\text{s.t.} \\
&7 - \xi_1^s x_1 + x_2 \le u_s, \\
&4 - \xi_2^s x_1 + x_2 \le u_s, \\
&\frac{1}{S}\sum_{s=1}^{S} u_s \le L, \\
&x_1 \ge 0,\ x_2 \ge 0,\ u_s \ge 0, s = 1, \ldots, S,
\end{aligned}
$$

for some level $L > 0$, a sample-approximated problem with penalty objective using the penalty function $\vartheta^{1,1}$

$$
\begin{aligned}
&\min x_1 + x_2 + \frac{N}{S}\sum_{s=1}^{S} u_{1s} + u_{2s} \\
&\text{s.t.} \\
&7 - \xi_1^s x_1 + x_2 \le u_{1s}, \\
&4 - \xi_2^s x_1 + x_2 \le u_{2s}, \\
&x_1 \ge 0,\ x_2 \ge 0,\ u_{1s}, u_{2s} \ge 0,\ s = 1, \ldots, S,
\end{aligned}
$$

Table 4. Blending problem – penalty constraint $\vartheta^{1,1}$.

S	L	Minimal reliability	Mean reliability	Mean optimal solution x_1	x_2	Mean optimal value
100	0.1	0.6570	0.7808	2.7301	2.9644	5.6944
100	0.05	0.7652	0.8588	3.0962	2.8954	5.9916
100	0.01	0.8769	0.9418	3.7534	2.7206	6.4740
100	0.005	0.9090	0.9562	3.9439	2.6591	6.6031
100	0.001	0.9250	0.9723	4.2082	2.5696	6.7778
100	0.0005	0.9291	0.9753	4.2618	2.5514	6.8132
100	0.0001	0.9322	0.9780	4.3124	2.5342	6.8466
150	0.1	0.7029	0.7927	2.7457	2.9744	5.7201
150	0.05	0.7829	0.8697	3.1207	2.9016	6.0224
150	0.01	0.8964	0.9485	3.7725	2.7252	6.4977
150	0.005	0.9224	0.9631	3.9603	2.6662	6.6265
150	0.001	0.9392	0.9801	4.2402	2.5724	6.8126
150	0.0005	0.9431	0.9837	4.3063	2.5501	6.8565
150	0.0001	0.9477	0.9874	4.3763	2.5266	6.9029
200	0.1	0.7276	0.7986	2.7721	2.9685	5.7406
200	0.05	0.8160	0.8721	3.1560	2.8866	6.0426
200	0.01	0.9207	0.9511	3.8128	2.7105	6.5234
200	0.005	0.9389	0.9654	3.9999	2.6519	6.6519
200	0.001	0.9551	0.9813	4.2709	2.5606	6.8315
200	0.0005	0.9576	0.9848	4.3360	2.5386	6.8747
200	0.0001	0.9598	0.9889	4.4154	2.5119	6.9273
500	0.1	0.7526	0.7972	2.7398	2.9845	5.7243
500	0.05	0.8348	0.8739	3.1166	2.9107	6.0273
500	0.01	0.9161	0.9524	3.7863	2.7259	6.5123
500	0.005	0.9466	0.9674	3.9738	2.6686	6.6424
500	0.001	0.9692	0.9851	4.2535	2.5774	6.8309
500	0.0005	0.9732	0.9888	4.3227	2.5543	6.8770
500	0.0001	0.9784	0.9935	4.4126	2.5242	6.9368

for a penalty parameter $N>0$, and sample-approximated problem with penalty objective using the penalty function ϑ^2

$$
\begin{aligned}
&\min x_1 + x_2 + \frac{N}{S}\sum_{s=1}^{S} u_s \\
&\text{s.t.} \\
&7 - \xi_1^s x_1 + x_2 \le u_s, \\
&4 - \xi_2^s x_1 + x_2 \le u_s, \\
&x_1 \ge 0,\ x_2 \ge 0,\ u_s \ge 0,\ s = 1, \ldots, S
\end{aligned}
$$

for a penalty parameter $N>0$.

Due to the simple structure of the problem we can compute the exact probability of fulfilling the constraints for obtained solutions of the sample-approximated problems. Mean and minimal probabilities are contained in Tables 3–7. The tables

Table 5. Blending problem – penalty constraint ϑ^2.

S	L	Minimal reliability	Mean reliability	Mean optimal solution x_1	x_2	Mean optimal value
100	0.1	0.6455	0.7724	2.7587	2.9325	5.6912
100	0.05	0.7645	0.8564	3.1039	2.8868	5.9907
100	0.01	0.8769	0.9413	3.7553	2.7185	6.4739
100	0.005	0.9090	0.9561	3.9444	2.6586	6.6030
100	0.001	0.9250	0.9723	4.2084	2.5694	6.7778
100	0.0005	0.9291	0.9753	4.2618	2.5514	6.8132
100	0.0001	0.9322	0.9780	4.3124	2.5342	6.8466
150	0.1	0.6914	0.7859	2.7673	2.9496	5.7170
150	0.05	0.7777	0.8665	3.1318	2.8897	6.0215
150	0.01	0.8964	0.9484	3.7728	2.7248	6.4976
150	0.005	0.9224	0.9631	3.9605	2.6660	6.6265
150	0.001	0.9392	0.9801	4.2405	2.5721	6.8126
150	0.0005	0.9431	0.9837	4.3066	2.5498	6.8564
150	0.0001	0.9477	0.9874	4.3765	2.5264	6.9029
200	0.1	0.7185	0.7918	2.7949	2.9431	5.7380
200	0.05	0.8143	0.8699	3.1641	2.8779	6.0420
200	0.01	0.9207	0.9510	3.8132	2.7102	6.5233
200	0.005	0.9389	0.9654	4.0002	2.6517	6.6519
200	0.001	0.9551	0.9813	4.2709	2.5606	6.8315
200	0.0005	0.9576	0.9848	4.3360	2.5386	6.8747
200	0.0001	0.9598	0.9889	4.4154	2.5119	6.9273
500	0.1	0.7417	0.7910	2.7600	2.9618	5.7218
500	0.05	0.8284	0.8715	3.1246	2.9019	6.0266
500	0.01	0.9161	0.9523	3.7867	2.7255	6.5122
500	0.005	0.9466	0.9673	3.9742	2.6682	6.6424
500	0.001	0.9692	0.9851	4.2535	2.5774	6.8309
500	0.0005	0.9732	0.9888	4.3227	2.5543	6.8770
500	0.0001	0.9784	0.9935	4.4126	2.5242	6.9368

also contain mean optimal solutions and mean optimal values. In Tables 6 and 7, there are also means of the penalty terms. As we can see, the penalty terms actually decrease with increasing N and penalize the violations of the random constraints.

For instance, the obtained solutions for sample size $S=200$ and choice of parameters $\gamma=0.001$, $L=0.0001$, $N=100$ are reliable for at least 95% and, on average, for almost 99%. Of course, the setting of parameters for a particular sample size depends on the concrete problem. In our blending problem, there is no significant difference in results caused by choosing any of the penalty functions.

5. Conclusion

In this article, we dealt with several possible stochastic programming formulations of a problem with constraints depending on a random factor. Chance-constrained

Table 6. Blending problem – penalty objective $\vartheta^{1,1}$.

S	N	Minimal reliability	Mean reliability	Mean optimal solution x_1	Mean optimal solution x_2	Mean optimal value	Penalty term
100	5	0.6920	0.7936	2.7707	2.9643	6.1859	0.4509
100	10	0.7783	0.8857	3.2526	2.8637	6.4629	0.3466
100	50	0.9225	0.9700	4.1738	2.5814	6.8084	0.0532
100	100	0.9330	0.9788	4.3266	2.5294	6.8560	0.0000
100	200	0.9330	0.9788	4.3266	2.5294	6.8560	0.0000
100	500	0.9330	0.9788	4.3266	2.5294	6.8560	0.0000
150	5	0.7409	0.8013	2.7633	2.9794	6.2152	0.4725
150	10	0.7991	0.8957	3.3002	2.8573	6.4966	0.3391
150	50	0.9393	0.9747	4.1517	2.6023	6.8364	0.0824
150	100	0.9404	0.9813	4.2651	2.5640	6.8893	0.0602
150	200	0.9488	0.9885	4.3976	2.5194	6.9170	0.0000
150	500	0.9488	0.9885	4.3976	2.5194	6.9170	0.0000
200	5	0.7211	0.8060	2.7996	2.9658	6.2365	0.4710
200	10	0.8467	0.8990	3.3202	2.8520	6.5207	0.3484
200	50	0.9467	0.9761	4.1806	2.5911	6.8631	0.0913
200	100	0.9600	0.9858	4.3563	2.5318	6.9171	0.0290
200	200	0.9611	0.9903	4.4423	2.5028	6.9451	0.0000
200	500	0.9611	0.9903	4.4423	2.5028	6.9451	0.0000
500	5	0.7587	0.8059	2.7733	2.9800	6.2220	0.4687
500	10	0.8589	0.8992	3.2852	2.8699	6.5093	0.3542
500	50	0.9527	0.9774	4.1212	2.6217	6.8603	0.1174
500	100	0.9674	0.9881	4.3096	2.5586	6.9228	0.0545
500	200	0.9774	0.9925	4.3934	2.5306	6.9528	0.0287
500	500	0.9817	0.9962	4.4654	2.5066	6.9719	0.0000

problems and problems with penalties in the constraints or in the objective were considered. We focused on the problems with expectation-type constraints which generalize the so-called integrated chance constraints. We showed that these problems are, under quite mild conditions, asymptotically equivalent to those with penalties in the objective. This is a result complementary to [3,5,8] where similar equivalence was proven between the chance-constrained problems and the problems with a penalty objective.

We generalized results on the exponential rates of convergence for sample-approximated integrated chance-constrained problems. Based on these rates, it is possible to estimate sample sizes which are necessary to ensure that the feasible solutions of the original problem are also feasible for a relaxed sample-approximated problem. Moreover, no additional binary variables were necessary to be introduced in order to solve the penalty problems.

In the numerical study, we obtained promising results that the sample-approximated penalty problems can provide highly reliable solutions. This is possible without using mixed-integer formulation, avoiding to the binary variables by employing continuous variables to model the penalty terms.

Table 7. Blending problem – penalty objective ϑ^2.

S	N	Minimal reliability	Mean reliability	Mean optimal solution x_1	x_2	Mean optimal value	Penalty term
100	5	0.6877	0.7859	2.7855	2.9422	6.1832	0.4556
100	10	0.7783	0.8838	3.2580	2.8571	6.4625	0.3474
100	50	0.9225	0.9700	4.1738	2.5814	6.8084	0.0532
100	100	0.9330	0.9788	4.3266	2.5294	6.8560	0.0000
100	200	0.9330	0.9788	4.3266	2.5294	6.8560	0.0000
100	500	0.9330	0.9788	4.3266	2.5294	6.8560	0.0000
150	5	0.7324	0.7928	2.7801	2.9545	6.2124	0.4778
150	10	0.7991	0.8942	3.3038	2.8525	6.4963	0.3400
150	50	0.9393	0.9747	4.1520	2.6020	6.8364	0.0824
150	100	0.9404	0.9812	4.2653	2.5638	6.8893	0.0602
150	200	0.9488	0.9885	4.3976	2.5194	6.9170	0.0000
150	500	0.9488	0.9885	4.3976	2.5194	6.9170	0.0000
200	5	0.7090	0.7982	2.8125	2.9445	6.2343	0.4773
200	10	0.8462	0.8976	3.3256	2.8466	6.5204	0.3482
200	50	0.9467	0.9761	4.1806	2.5911	6.8631	0.0913
200	100	0.9600	0.9858	4.3563	2.5318	6.9171	0.0290
200	200	0.9611	0.9903	4.4423	2.5028	6.9451	0.0000
200	500	0.9611	0.9903	4.4423	2.5028	6.9451	0.0000
500	5	0.7533	0.7987	2.7859	2.9600	6.2197	0.4738
500	10	0.8545	0.8976	3.2898	2.8644	6.5090	0.3547
500	50	0.9527	0.9773	4.1213	2.6216	6.8603	0.1174
500	100	0.9674	0.9881	4.3096	2.5586	6.9228	0.0545
500	200	0.9774	0.9925	4.3934	2.5306	6.9528	0.0287
500	500	0.9817	0.9962	4.4654	2.5066	6.9719	0.0000

Acknolwedgements

This work was supported by the grants GA CR P402/10/1610 and SVV 261315/2010.

References

[1] S. Ahmed and A. Shapiro, *Solving chance-constrained stochastic programs via sampling and integer programming*, in *Tutorials in Operations Research*, Z.L. Chen and S. Raghavan, eds., INFORMS, Hanover, 2008, pp. 261–269.

[2] M.S. Bazara, H.D. Sherali, and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*, Wiley, Singapore, 1993.

[3] M. Branda, *Reformulation of General Chance Constrained Problems Using the Penalty Functions*, Stochastic Programming E-Print Series (SPEPS), 2010.

[4] M. Branda, *Solving real-life portfolio problem using stochastic programming and Monte-Carlo techniques*, in *Proceedings of Mathematical Methods in Economics*, M. Houda and J. Friebelová, eds., University of South Bohemia, České Budějovice, 2010, pp. 67–72.

[5] M. Branda and J. Dupačová, *Approximations and contamination bounds for probabilistic programs*, Ann. Oper. Res. (online first; See also SPEPS 2008-13). [Available at http://www.springerlink.com/content/b52u4p2520285781].

[6] J. Dupačová, A. Gaivoronski, Z. Kos, and T. Szantai, *Stochastic programming in water management: A case study and a comparison of solution techniques*, Eur. J. Oper. Res. 52 (1991), pp. 28–44.
[7] J. Dupačová and M. Kopa, *Robustness in stochastic programs with risk constraints*, Ann. Oper. Res. (online first). [Available at http://www.springerlink.com/content/f0051tw68v370580/].
[8] Y.M. Ermoliev, T.Y. Ermolieva, G.J. Macdonald, and V.I. Norkin, *Stochastic optimization of insurance portfolios for managing exposure to catastrophic risks*, Ann. Oper. Res. 99 (2000), pp. 207–225.
[9] A. Fügenschuh and A. Martin, *Computational integer programming and cutting planes*, in *Handbook on Discrete Optimization, Handbooks in Operations Research and Management Science*, Vol. 12, K. Aardal, G. Nemhauser, and R. Weismantel, eds., Elsevier, Amsterdam, 2005, pp. 69–122.
[10] A. Genz and F. Bretz, *Computation of Multivariate Normal and t Probabilities*, Springer Series: Lecture Notes in Statistics, Vol. 195, Springer-Verlag, Berling, Heidelberg, 2009.
[11] W.K. Klein Haneveld, *Duality in Stochastic Linear and Dynamic Programming*, *Lecture,* Notes in Economics and Mathematical Systems, Vol. 274, Springer, Berlin, 1986.
[12] W.K. Klein Haneveld and M. van der Vlerk, *Integrated chance constraints: Reduced forms and an algorithm*, Comput. Manag. Sci. 3 (2006), pp. 245–269.
[13] J. Luedtke and S. Ahmed, *A sample approximation approach for optimization with probabilistic constraints*, SIAM J. Optim. 19 (2008), pp. 674–699.
[14] J. Luedtke, S. Ahmed, and G.L. Nemhauser, *An integer programming approach for linear programs with probabilistic constraints*, Math. Program. 122 (2010), pp. 247–272.
[15] J. Nocedal and S.J. Wright, *Numerical Optimization*, 2nd ed., Springer, New York, 2006.
[16] B. Pagnoncelli, S. Ahmed, and A. Shapiro, *Computational study of a chance constrained portfolio selection problem*, Optimization (online), 2008 [Available at http://www.optimization-online.org/DB_HTML/2008/02/1899.html].
[17] B. Pagnoncelli, S. Ahmed, and A. Shapiro, *Sample average approximation method for chance constrained programming: Theory and applications*, J. Optim. Theory Appl. 142 (2009), pp. 399–416.
[18] A. Prékopa, *Dual method for the solution of a one-stage stochastic programming problem with random RHS obeying a discrete probability distribution*, Math. Methods Oper. Res. 34 (1990), pp. 441–461.
[19] A. Prékopa, *Stochastic Programming*, Kluwer, Dordrecht and Acadmiai Kiad, Budapest, 1995.
[20] A. Prékopa, *Probabilistic programming*, in *Stochastic Programming, Handbook in Operations Research and Management Science*, Vol. 10, A. Ruszczynski and A. Shapiro, eds., Elsevier, Amsterdam, 2003, pp. 483–554.
[21] W.M. Raike, *Dissection methods for solutions in chance constrained programming problems under discrete distributions*, Manag. Sci. 16 (1970), pp. 708–715.
[22] A. Shapiro, *Monte Carlo sampling methods*, in *Stochastic Programming, Handbook in Operations Research and Management Science*, Vol. 10, A. Ruszczynski and A. Shapiro, eds., Elsevier, Amsterdam, 2003, pp. 483–554.
[23] W. Wang and S. Ahmed, *Sample average approximation of expected value constrained stochastic programs*, Oper. Res. Lett. 36 (2008), pp. 515–519.
[24] E. Žampachová, and M. Mrázek, *Stochastic optimization in beam design and its reliability check*, in *MENDEL 2010 – 16th International Conference on Soft Computing*, R. Matousek, ed., Mendel Journal series, FME BUT, Brno, 2010, pp. 405–410.