ON THE RELATIONSHIPS BETWEEN TYPES OF L-CONVERGENCE SPACES

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Abstract. This paper focuses on the relationships between stratified L-convergence spaces, stratified strong L-convergence spaces and stratified levelwise L-convergence spaces. It has been known that: (1) a stratified L-convergence space is precisely a left-continuous stratified levelwise L-convergence space; and (2) a stratified strong L-convergence space is naturally a stratified L-convergence space, but the converse is not true generally. In this paper, a strong left-continuity condition for stratified levelwise L-convergence space is given. It is proved that a stratified strong L-convergence space is precisely a strongly left-continuous stratified levelwise L-convergence space. Then a sufficient and necessary condition for a stratified L-convergence space to be a stratified strong L-convergence space is presented.

1. Introduction

Stratified L-convergence spaces were first defined in [9] and then developed in a series of papers [6, 10, 11] for the case that the lattice L is a complete Heyting algebra (or a frame). Later, the theory of these spaces was generalized to the lattice context of complete residuated lattices [25]. In [22], the lattice situation was further extended to enriched cl-premonoids. Recently, under different lattice contexts, a type of L-convergence spaces is proposed from the viewpoint of fuzzy orders [2, 3, 12, 15, 16, 17, 18, 19, 20, 21]. It is proved that these spaces are slightly stronger than stratified L-convergence spaces, and thus are generally called stratified strong L-convergence spaces. In [2, 16], it is proved that a stratified L-convergence space need not to be strong. This leads to a natural question: Does there exist a sufficient and necessary condition that guarantees that a stratified L-convergence space is a stratified strong L-convergence space?

In the case of L being a frame, Flores and his co-workers [6] introduced the concept of stratified levelwise L-convergence space which can be regarded as an extension of the concept of probabilistic convergence space in [24]. They also proposed a so-called left-continuity condition and proved that the left-continuous stratified levelwise L-convergence spaces are precisely the stratified L-convergence spaces. This leads to another question: Can stratified strong L-convergence spaces be characterized by levelwise lattice-valued convergence spaces via some left-continuity condition?

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This article will focus on the above two questions. The contents are arranged as follows. Section 2 recalls some basic notions and notations used in this paper. Section 3 presents the levelwise characterization for the stratified strong L-convergence spaces via some left-continuity condition. Section 4 presents a sufficient and necessary condition for stratified L-convergence spaces to be stratified strong L-convergence spaces. Section 5 makes a conclusion.

2. Preliminaries

In this paper, if not otherwise specified, $(L,*,1)$ is always a complete residuated lattice. That is, L is a complete lattice with a top element 1 and a bottom element 0; $*$ is a binary operation on L such that (i) $(L,*,1)$ is a commutative monoid; and (ii) $*$ distributes over arbitrary joins. Since the binary operation $*$ distributes over arbitrary joins, the mapping $\alpha*(-):L\longrightarrow L$ has a right adjoint $\alpha\to(-):L\longrightarrow L$ given by

$$\alpha\to\beta=\bigvee\{\gamma\in L:\alpha*\gamma\leq\beta\}.$$

The binary operation $\to$ is called the residuation with respect to $*$. We collect some basic properties of the binary operations $*$ and $\to$.

Proposition 2.1. [1, 7] *Let $(L,*,1)$ be a complete residuated lattice. Then*

- (I1) $0*\alpha=0$ and $1\to\alpha=\alpha$;
- (I2) $\alpha\to\beta=1\Leftrightarrow\alpha\leq\beta$;
- (I3) $\alpha*(\alpha\to\beta)\leq\beta$ and $(\alpha\to\beta)*(\beta\to r)\leq\alpha\to r$;
- (I4) $\alpha\to(\beta\to r)=(\alpha*\beta)\to r=\beta\to(\alpha\to r)$;
- (I5) $\alpha\leq(\alpha\to\beta)\to\beta$;
- (I6) $(\bigvee_{j\in J}\alpha_j)\to\beta=\bigwedge_{j\in J}(\alpha_j\to\beta)$;
- (I7) $\alpha\to(\bigwedge_{j\in J}\beta_j)=\bigwedge_{j\in J}(\alpha\to\beta_j)$;
- (I8) $\alpha\leq\beta\Rightarrow\alpha\to\gamma\geq\beta\to\gamma$ and $\gamma\to\alpha\leq\gamma\to\beta$.

For a set X, the set L^X of functions from X to L with the pointwise order becomes a complete lattice. Each element of L^X is called an L-subset (or a fuzzy subset) of X. For any $\lambda\in L^X$, $\mathscr{K}\subseteq L^X$ and $\alpha\in L$, we define the L-subsets $\alpha*\lambda$, $\alpha\to\lambda$, $\bigvee\mathscr{K}$ and $\bigwedge\mathscr{K}$ by $(\alpha*\lambda)(x)=\alpha*\lambda(x)$, $(\alpha\to\lambda)(x)=\alpha\to\lambda(x)$, $(\bigvee\mathscr{K})(x)=\bigvee_{\mu\in\mathscr{K}}\mu(x)$ and $(\bigwedge\mathscr{K})(x)=\bigwedge_{\mu\in\mathscr{K}}\mu(x)$. Let ϕ be a function, we define $\phi_L^{\rightarrow}:L^X\longrightarrow L^Y$ and $\phi_L^{\leftarrow}:L^Y\longrightarrow L^X$ [8] by $\phi_L^{\rightarrow}(A)(y)=\bigvee_{\phi(x)=y}A(x)$ for $A\in L^X$ and $y\in Y$, and $\phi_L^{\leftarrow}(B)=B\circ\phi$ for $B\in L^Y$.

Let X be a set. A fuzzy partial order (or, an L-partial order) ([1, 26, 27, 28, 29]) on X is a function $R:X\times X\longrightarrow L$ such that

1. (1) $R(a,a)=1$ for every $a\in X$ (reflexivity);
2. (2) $R(a,b)=R(b,a)=1$ implies that $a=b$ for all $a,b\in X$ (antisymmetry);
3. (3) $R(a,b)*R(b,c)\leq R(a,c)$ for all $a,b,c\in X$ (transitivity).

The pair (X,R) is called an L-partially ordered set. An *L-order-preserving function* $f:(X,R)\longrightarrow(Y,S)$ between L-partially ordered sets is a function $f:X\longrightarrow Y$ such that $R(a,b)\leq S(f(a),f(b))$ for all $a,b\in X$.

Let $R : L^X \times L^X \longrightarrow L$ be a function defined by

$$R(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)),$$

then R is an L-partial order on L^X. The value $R(\lambda, \mu) \in L$ is the degree that λ is contained in μ, so the fuzzy partially order R is called *fuzzy inclusion order* ([1]). The L-partially ordered set (L^X, R) is called the fuzzy powerset of X, denoted by $[L^X]$ for short.

Definition 2.2. (Höhle and Šostak [8]) A stratified L-filter on a set X is a function $\mathcal{F} : L^X \longrightarrow L$ such that for each $\lambda, \mu \in L^X$ and each $\alpha \in L$,
(F1) $\mathcal{F}(0) = 0$, $\mathcal{F}(1) = 1$; (F2) $\mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$;
(Fs) $\mathcal{F}(\alpha * \lambda) \geq \alpha * \mathcal{F}(\lambda)$.

The following examples of L-filters belong to the folklore, we list them here because the notations are needed.

Example 2.3. (1) For each point x in a set X, it is obvious that $[x] : L^X \longrightarrow L, [x](\lambda) = \lambda(x)$ is a stratified L-filter on X, called *the principal L-filter* generated by x.

(2) Let J be a non-empty set and $\{\mathcal{F}_j | j \in J\}$ be a family of stratified L-filters on X, then $\bigwedge_{j \in J} \mathcal{F}_j$ is also a stratified L-filter on X. Let $\mathcal{F}_0$ denote the meet of all stratified L-filters on X, i.e., the smallest stratified L-filter on X.

(3) Let $\phi : X \longrightarrow Y$ be a function and $\mathcal{F}$ be a stratified L-filter on X. Then the function $\phi^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ defined by $\lambda \mapsto \mathcal{F}(\lambda \circ \phi)$ is a stratified L-filter on Y, called the *image* of $\mathcal{F}$ under ϕ.

The set of stratified L-filters on a set X is denoted by $\mathcal{F}_L^s(X)$. There is a natural fuzzy partial order on $\mathcal{F}_L^s(X)$ inherited from $L^{(L^X)}$. Precisely, if we let

$$[\mathcal{F}_L^s(X)](\mathcal{F}, \mathcal{G}) = [L^{L^X}](\mathcal{F}, \mathcal{G}) = \bigwedge_{\lambda \in L^X} (\mathcal{F}(\lambda) \to \mathcal{G}(\lambda))$$

for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, then $[\mathcal{F}_L^s(X)]$ is an L-partially order. For simplicity, we use the symbol $(\mathcal{F}, \mathcal{G})$ to denote the value $[\mathcal{F}_L^s(X)](\mathcal{F}, \mathcal{G})$ below. We have an obvious lemma for the L-partial order $[\mathcal{F}_L^s(X)]$.

Lemma 2.4. *Let $\phi : X \longrightarrow Y$ be a function. Then $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $(\mathcal{F}, \mathcal{G}) \leq (\phi^{\Rightarrow}(\mathcal{F}), \phi^{\Rightarrow}(\mathcal{G}))$.*

Definition 2.5. (Flores and Mohapatra and Richardson [6] for L a frame) A collection $\overline{q} = (q_\alpha)_{\alpha \in L}$, where $q_\alpha : \mathcal{F}_L^s(X) \longrightarrow \mathcal{P}(X)$, is called a *stratified levelwise L-convergence structure* on X if it satisfies:

(LL1) $[x] \overset{q_\alpha}{\to} x, \mathcal{F}_0 \overset{q_0}{\to} x$ for each $x \in X$,

(LL2) $\mathcal{G} \geq \mathcal{F} \overset{q_\alpha}{\to} x$ implies $\mathcal{G} \overset{q_\alpha}{\to} x$,

(LL3) $\mathcal{F} \overset{q_\alpha}{\to} x$ implies $\mathcal{F} \overset{q_\beta}{\to} x$ whenever $\beta \leq \alpha$.

The notation, $\mathcal{F} \overset{q_\alpha}{\to} x$, means that $x \in q_\alpha(\mathcal{F})$. The pair $(X, \overline{q})$ is called a *stratified levelwise L-convergence space.*

A function $\phi : X \longrightarrow Y$ between stratified levelwise L-convergence spaces $(X, \overline{q})$ and $(Y, \overline{q'})$ is said to be continuous if $\mathcal{F} \stackrel{q_\alpha}{\rightarrow} x$ implies $\phi^{\Rightarrow}(\mathcal{F}) \stackrel{q'_\alpha}{\rightarrow} \phi(x)$ for each $\mathcal{F} \in \mathcal{F}^s_L(X), \alpha \in L, x \in X$. The category of stratified levelwise L-convergence spaces and continuous functions is denoted by L-**SLC**.

Definition 2.6. (Jäger [9] and Yao [25]) A stratified L-convergence structure on a set X is a function $\lim : \mathcal{F}^s_L(X) \longrightarrow L^X$ satisfying

(LC1) $\lim[x](x) = 1$ for every $x \in X$; and

(LC2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, $\mathcal{F} \leq \mathcal{G} \Longrightarrow \lim\mathcal{F} \leq \lim\mathcal{G}$.

The pair $(X, \lim)$ is called a stratified L-convergence space.

A function $\phi : X \longrightarrow Y$ between stratified L-convergence spaces $(X, \lim)$ and $(Y, \lim')$ is said to be continuous if $\lim \mathcal{F}(x) \leq \lim' \phi^{\Rightarrow}(\mathcal{F})(\phi(x))$ for each $\mathcal{F} \in \mathcal{F}^s_L(X)$ and each $x \in X$. The category of stratified L-convergence spaces and continuous functions is denoted by L-**SC**.

In [16], a notion of stratified strong L-convergence space (called stratified L-ordered convergence space in [2]) was proposed, and it was defined as a function $\lim : \mathcal{F}^s_L(X) \longrightarrow L^X$ satisfying **(LC1)** and

(LC2′) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, $(\mathcal{F}, \mathcal{G}) \leq \lim\mathcal{F} \rightarrow \lim\mathcal{G}$, or equivalent, $(\mathcal{F}, \mathcal{G}) * \lim\mathcal{F} \leq \lim\mathcal{G}$. That is, lim is L-order-preserving.

The full subcategory of L-**SC** consisting of strong objects is denoted by L-**SSC**.

Except for the above three kinds of lattice-valued convergence spaces, there are many other kinds of lattice-valued convergence spaces [4, 5, 13, 14, 23] are discussed by many researchers.

3. A Levelwise Characterization for Stratified Strong L-convergence Space Via Strong Left-continuity Condition

In [6], for L a frame, a stratified levelwise L-convergence space $(X, \overline{q})$ is said to be *left-continuous* if $\mathcal{F} \stackrel{\alpha}{\rightarrow} x$ iff there exists $A \subseteq L$ such that $\vee A = \alpha$ and $\mathcal{F} \stackrel{q_\beta}{\rightarrow} x$ for each $\beta \in A$. The category of left-continuous stratified levelwise L-convergence spaces is proved to be isomorphic to the category of stratified L-convergence spaces. Observing the proof in [6], we find that its left-continuity condition does not work in the general lattice context. But if we make a slight modification, that is, replacing $\vee A = \alpha$ by $\vee A \geq \alpha$, it will do. The proof is similar to that in [6], thus, it is omitted.

In this section, we shall present a strong left-continuity condition for stratified levelwise L-convergence spaces and prove the category of strongly left-continuous stratified levelwise L-convergence spaces is isomorphic to the category of stratified strong L-convergence spaces.

Definition 3.1. A stratified levelwise L-convergence space $(X, \overline{q})$ is said to be *strongly left-continuous* if $\mathcal{F} \stackrel{q_\alpha}{\rightarrow} x$ iff there exists $A \subseteq L$ and $\mathcal{G}_\beta \stackrel{q_\beta}{\rightarrow} x$ for each $\beta \in A$ such that $\bigvee_{\beta \in A}(\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$.

Remark 3.2. Note that the if part of strong left-continuity is obvious. Thus, when examining it, we only need to check its only if part. In addition, it is easily seen that strong left-continuity implies left-continuity.

The full subcategory of L-**SLC** consisting of strongly left-continuous objects is denoted by L-**SLSLC**. We prove below that L-**SLSLC** is isomorphic to L-**SSC**.

Proposition 3.3. *Let $(X,\overline{q})$ be a stratified levelwise L-convergence space. Then the pair $(X,\lim_{\overline{q}})$ is a stratified strong L-convergence space, where*

$$\lim_{\overline{q}}:\mathcal{F}_L^s(X)\longrightarrow L^X,\quad \lim_{\overline{q}}\mathcal{F}(x)=\bigvee\{\alpha*(\mathcal{G},\mathcal{F})\big|\mathcal{G}\overset{q_\alpha}{\to}x\}.$$

Proof. **(LC1)**: By **(LL1)**, $[x]\overset{q_\alpha}{\to}x$ for each $\alpha\in L, x\in X$. Thus

$$\lim_{\overline{q}}[x](x)\geq\bigvee\{\alpha\big|\alpha\in L\}=1.$$

(LC2′): For each $\mathcal{F},\mathcal{G}\in\mathcal{F}_L^s(X), x\in X$, we have

$$\begin{aligned}(\mathcal{F},\mathcal{G})*\lim_{\overline{q}}\mathcal{F}(x) &= (\mathcal{F},\mathcal{G})*\bigvee\{\alpha*(\mathcal{H},\mathcal{F})|\mathcal{H}\overset{q_\alpha}{\to}x\}\\ &= \bigvee\{\alpha*(\mathcal{F},\mathcal{G})*(\mathcal{H},\mathcal{F})|\mathcal{H}\overset{q_\alpha}{\to}x\}\\ &\leq \bigvee\{\alpha*(\mathcal{H},\mathcal{G})|\mathcal{H}\overset{q_\alpha}{\to}x\}=\lim_{\overline{q}}(\mathcal{G})(x).\end{aligned}$$

□

Remark 3.4. Taking $\mathcal{G}\equiv\mathcal{F}$, $\lim_{\overline{q}}$ appeared in [6]. In this case, $\lim_{\overline{q}}$ is only a stratified L-convergence space.

Proposition 3.5. *Let $(X,\overline{q})$ and $(Y,\overline{q'})$ be stratified levelwise L-convergence spaces. If $\phi:(X,\overline{q})\longrightarrow(Y,\overline{q'})$ is continuous, then so is $\phi:(X,\lim_{\overline{q}})\longrightarrow(Y,\lim_{\overline{q'}})$.*

Proof. Let $\mathcal{F}\in\mathcal{F}_L^s(X), x\in X$. Then

$$\begin{aligned}\lim_{\overline{q'}}\phi^{\Rightarrow}(\mathcal{F})(\phi(x)) &= \bigvee\{\alpha*(\mathcal{H},\phi^{\Rightarrow}(\mathcal{F}))|\mathcal{H}\overset{q'_\alpha}{\to}\phi(x)\}\\ &\geq \bigvee\{\alpha*(\phi^{\Rightarrow}(\mathcal{G}),\phi^{\Rightarrow}(\mathcal{F}))|\mathcal{G}\overset{q_\alpha}{\to}x\}\\ &\geq \bigvee\{\alpha*(\mathcal{G},\mathcal{F})|\mathcal{G}\overset{q_\alpha}{\to}x\}=\lim_{\overline{q}}\mathcal{F}(x).\end{aligned}$$

Where the first inequality holds for the continuity of ϕ and the second inequality follows from Lemma 2.4. □

Proposition 3.6. *Let $(X,\lim)$ be a stratified strong L-convergence space, then the pair $(X,\overline{q^{\lim}})$ is a strongly left-continuous stratified levelwise L-convergence space, where*

$$\forall\alpha\in L,\mathcal{F}\overset{(q^{\lim})_\alpha}{\longrightarrow}x\Longleftrightarrow\lim\mathcal{F}(x)\geq\alpha.$$

Proof. In the case of L being a frame, for a stratified L-convergence space $(X,\lim)$, it is proved in [6] that $(X,\overline{q^{\lim}})$ is a stratified levelwise L-convergence space. So, it suffices to prove the strong left-continuity of $\overline{q^{\lim}}$.

Assume that there exists $A\subseteq L$ and $\mathcal{G}_\beta\overset{(q^{\lim})_\beta}{\longrightarrow}x$ for each $\beta\in A$ such that $\bigvee_{\beta\in A}(\beta*(\mathcal{G}_\beta,\mathcal{F}))\geq\alpha$. Then by $\mathcal{G}_\beta\overset{(q^{\lim})_\beta}{\longrightarrow}x\Longleftrightarrow\lim\mathcal{G}_\beta(x)\geq\beta$ and **(LC2′)** we have

$$\lim\mathcal{F}(x)\geq\lim\mathcal{G}_\beta(x)*(\mathcal{G}_\beta,\mathcal{F}),\forall\beta\in A.$$

So,

$$\lim \mathcal{F}(x) \geq \bigvee_{\beta\in A} (\lim \mathcal{G}_\beta(x) * (\mathcal{G}_\beta, \mathcal{F})) \geq \bigvee_{\beta\in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha.$$

This means $\mathcal{F} \overset{(q^{\lim})_\alpha}{\longrightarrow} x$ as desired. □

Proposition 3.7. *Let* $(X, \lim)$ *and* $(Y, \lim')$ *be stratified strong* L*-convergence spaces. If* $\phi : (X, \lim) \longrightarrow (Y, \lim')$ *is continuous, then so is* $\phi : (X, \overline{q^{\lim}}) \longrightarrow (Y, \overline{q^{\lim'}})$.

Proof. Suppose that $\mathcal{F} \overset{(q^{\lim})_\alpha}{\longrightarrow} x$, i.e., $\lim \mathcal{F}(x) \geq \alpha$. By the continuity of ϕ, we get $\lim' \phi^{\Rightarrow}(\mathcal{F})(\phi(x)) \geq \lim \mathcal{F}(x) \geq \alpha$. So, $\phi^{\Rightarrow}(\mathcal{F}) \overset{(q^{\lim'})_\alpha}{\longrightarrow} \phi(x)$. □

We define $\begin{cases} \Psi : L\text{-}\mathbf{SLSLC} \longrightarrow L\text{-}\mathbf{SSC}, \\ \Psi(\phi) = \phi, \\ \Psi(X, \overline{q}) = (X, \lim_{\overline{q}}). \end{cases}$ $\begin{cases} \Delta : L\text{-}\mathbf{SSC} \longrightarrow L\text{-}\mathbf{SLSLC}, \\ \Delta(\phi) = \phi, \\ \Delta(X, \lim) = (X, \overline{q_{\lim}}). \end{cases}$

Then Ψ and Δ are all concrete functors. The following theorem shows that Δ is an isomorphism, i.e., L-**SLSLC** is isomorphic to L-**SSC**.

Theorem 3.8. $\Psi \circ \Delta = \mathrm{id}_{L\text{-}\mathbf{SSC}}$*,* $\Delta \circ \Psi = \mathrm{id}_{L\text{-}\mathbf{SLSLC}}$*.*

Proof. (1) For $(X, \lim) \in L$-**SSC**, we check that $\Psi \circ \Delta(\lim) = \lim$.

Indeed, $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\forall x \in X$.

$$\begin{aligned} \lim_{\overline{q^{\lim}}} \mathcal{F}(x) &= \bigvee \{\alpha * (\mathcal{G}, \mathcal{F}) | \mathcal{G} \overset{(q^{\lim})_\alpha}{\longrightarrow} x\} \geq \bigvee \{\alpha * (\mathcal{F}, \mathcal{F}) | \mathcal{F} \overset{(q^{\lim})_\alpha}{\longrightarrow} x\} \\ &= \bigvee \{\alpha | \lim \mathcal{F}(x) \geq \alpha\} = \lim \mathcal{F}(x) \end{aligned}$$

Conversely, let

$$\lim_{\overline{q^{\lim}}} \mathcal{F}(x) = \bigvee \{\alpha * (\mathcal{G}, \mathcal{F}) | \mathcal{G} \overset{(q^{\lim})_\alpha}{\longrightarrow} x\} = \beta.$$

By the strong left-continuity of $\overline{q^{\lim}}$, we have $\mathcal{F} \overset{(q^{\lim})_\beta}{\longrightarrow} x$, i.e., $\lim \mathcal{F}(x) \geq \beta = \lim_{\overline{q^{\lim}}} \mathcal{F}(x)$. Thus $\lim \mathcal{F}(x) = \lim_{\overline{q^{\lim}}} \mathcal{F}(x)$ as desired.

(2) For $(X, \overline{q}) \in L$-**SLSLC**, we check that $\Delta \circ \Psi(\overline{q}) = \overline{q}$.

In fact,

$$\mathcal{F} \overset{(q^{\lim_{\overline{q}}})_\alpha}{\longrightarrow} x \Longleftrightarrow \lim_{\overline{q}} \mathcal{F}(x) = \bigvee \{\beta * (\mathcal{G}, \mathcal{F}) | \mathcal{G} \overset{q_\beta}{\to} x\} \geq \alpha \Longleftrightarrow \mathcal{F} \overset{q_\alpha}{\to} x.$$

Where the second equivalence holds by the strong left-continuity of $(X, \overline{q})$. □

Theorem 3.9. L-**SLSLC** *(and thus* L-**SSC***) is reflective in* L-**SLC**.

Proof. Given $(X, \overline{q}) \in L$-**SLC**, define $(X, \overline{Lq})$ as follows: $\mathcal{F} \overset{(Lq)_\alpha}{\longrightarrow} x$ iff there exists $A \subseteq L$ and $\mathcal{G}_\beta \overset{q_\beta}{\to} x$ for each $\beta \in A$ such that $\bigvee_{\beta\in A}(\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$. It is not difficult to check that $(X, L\overline{q}) \in L$-**SLC**. Next, we prove that it is strongly left-continuous.

Suppose that there exists $A \subseteq L$ and $\mathcal{G}_\beta \overset{(Lq)_\beta}{\longrightarrow} x$ for each $\beta \in A$ such that $\bigvee_{\beta\in A}(\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$. We fix $\beta \in A$. Because $\mathcal{G}_\beta \overset{(Lq)_\beta}{\rightarrow} x$, then there exists $A_\beta \subseteq L$ and $\mathcal{H}_\gamma \overset{q_\gamma}{\rightarrow} x$ for each $\gamma \in A_\beta$ such that $\bigvee_{\gamma\in A_\beta}(\gamma * (\mathcal{H}_\gamma, \mathcal{G}_\beta)) \geq \beta$. Denoting $B = \bigcup\{A_\beta | \beta \in A\}$, then

$$\begin{aligned}
\bigvee_{\gamma\in B}\big(\gamma * (\mathcal{H}_\gamma, \mathcal{F})\big) &= \bigvee_{\beta\in A}\bigvee_{\gamma\in A_\beta}\big(\gamma * (\mathcal{H}_\gamma, \mathcal{F})\big)\\
&\geq \bigvee_{\beta\in A}\bigvee_{\gamma\in A_\beta}\big(\gamma * (\mathcal{H}_\gamma, \mathcal{G}_\beta) * (\mathcal{G}_\beta, \mathcal{F})\big)\\
&= \bigvee_{\beta\in A}\big((\mathcal{G}_\beta, \mathcal{F}) * (\bigvee_{\gamma\in A_\beta}(\gamma * (\mathcal{H}_\gamma, \mathcal{G}_\beta)))\big)\\
&\geq \bigvee_{\beta\in A}\big(\beta * (\mathcal{G}_\beta, \mathcal{F})\big) \geq \alpha.
\end{aligned}$$

Thus $\mathcal{F} \overset{(Lq)_\alpha}{\longrightarrow} x$ as desired.

We check below that $\mathrm{id}_X \colon (X, \overline{q}) \longrightarrow (X, L\overline{q})$ is the L-**SLSLC**-reflection.

Since $\mathcal{F} \overset{q_\beta}{\rightarrow} x$ implies $\mathcal{F} \overset{(Lq)_\beta}{\rightarrow} x$, thus id_X is obvious continuous. Assume that $(Y, \overline{p}) \in L$-**SLSLC**, and $\phi : (X, \overline{q}) \longrightarrow (Y, \overline{p})$ is continuous. Next, we check that $\phi : (X, \overline{Lq}) \longrightarrow (Y, \overline{p})$ is also continuous. Let $\mathcal{F} \overset{(Lq)_\alpha}{\rightarrow} x$. There exists $A \subseteq L$ and $\mathcal{G}_\beta \overset{q_\beta}{\rightarrow} x$ for each $\beta \in A$ such that $\bigvee_{\beta\in A}(\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$. Because $\phi : (X, \overline{q}) \longrightarrow (Y, \overline{p})$ is continuous, thus $\forall \beta \in A, \phi^{\Rightarrow}(\mathcal{G}_\beta) \overset{p_\beta}{\rightarrow} \phi(x)$. So, by Lemma 2.4

$$\bigvee_{\beta\in A}\big(\beta * (\phi^{\Rightarrow}(\mathcal{G}_\beta), \phi^{\Rightarrow}(\mathcal{F}))\big) \geq \bigvee_{\beta\in A}\big(\beta * (\mathcal{G}_\beta, \mathcal{F})\big) \geq \alpha.$$

Therefore, $\phi^{\Rightarrow}(\mathcal{F}) \overset{p_\alpha}{\rightarrow} x$ by the strong left-continuity of $(Y, \overline{p})$. □

4. A Sufficient and Necessary Condition Such That Stratified L-convergence Spaces to be Stratified Strong L-convergence Spaces

It is known that a stratified L-convergence structure needs not to be strong [2, 16]. In the following, we shall give a sufficient and necessary condition such that stratified L-convergence spaces to be strong. Our idea comes from the following equivalence [28]: A function $f : L^X \longrightarrow L^Y$ is L-order-preserving iff

(1) f is order-preserving function, i.e., $\lambda \leq \mu \Longrightarrow f(\lambda) \leq f(\mu)$;

(2) For each $\alpha \in L, \lambda \in L^X$, $f(\alpha * \lambda) \geq \alpha * f(\lambda)$.

The axiom **(LC2′)** means that lim is L-order-preserving. Analogizing the above equivalence, it seems **(LC2′)** can be split into **(LC2)** and the following condition:

For each $\alpha \in L, \mathcal{F} \in \mathcal{F}_L^s(X)$, $\lim(\alpha * \mathcal{F}) \geq \alpha * \lim \mathcal{F}$.

Unfortunately, $\alpha * \mathcal{F}$ is not a stratified L-filter in general. We use

$$[\alpha * \mathcal{F}] = \bigwedge\{\mathcal{G} \in \mathcal{F}_L^s(X) | \mathcal{G} \geq \alpha * \mathcal{F}\} = \bigwedge\{\mathcal{G} \in \mathcal{F}_L^s(X) | \alpha \leq (\mathcal{F}, \mathcal{G})\}$$

to replace it. Note that the set $\{\mathcal{G} \in \mathcal{F}_L^s(X) | \mathcal{G} \geq \alpha * \mathcal{F}\}$ is not empty since $\mathcal{F} \geq \alpha * \mathcal{F}$. Intuitively, $[\alpha * \mathcal{F}]$ is the coarsest stratified L-filter finer than the function $\lambda \mapsto \alpha * \mathcal{F}(\lambda)$. In this way, we obtain the condition

(LC2″) For all $\alpha \in L, \mathcal{F} \in \mathcal{F}_L^s(X), \lim[\alpha * \mathcal{F}] \geq \alpha * \lim \mathcal{F}$.

Theorem 4.1. *Let* $(X, \lim)$ *be a stratified* L*-convergence space, then*

$$(\mathbf{LC2'}) \Longleftrightarrow (\mathbf{LC2}) + (\mathbf{LC2''}).$$

Proof. **(LC2′)⟹(LC2″).** For each $\alpha \in L, \mathcal{F} \in \mathcal{F}_L^s(X)$, by $\alpha * \mathcal{F} \leq [\alpha * \mathcal{F}]$ we have $(\mathcal{F}, [\alpha * \mathcal{F}]) \geq \alpha$. So by **(LC2′)**

$$\alpha * \lim \mathcal{F} \leq (\mathcal{F}, [\alpha * \mathcal{F}]) * \lim \mathcal{F} \leq \lim[\alpha * \mathcal{F}].$$

Conversely, $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, let $\alpha = (\mathcal{F}, \mathcal{G})$. Then $\alpha * \mathcal{F} \leq \mathcal{G}$. Thus by **(LC2)** and **(LC2″)**

$$(\mathcal{F}, \mathcal{G}) * \lim \mathcal{F} = \alpha * \lim \mathcal{F} \leq \lim[\alpha * \mathcal{F}] \leq \lim \mathcal{G}. \quad \square$$

Note that for $L = \{0, 1\}$, the axiom **(LC2″)** simply states that

$$\lim[0 * \mathcal{F}] \geq 0 * \lim \mathcal{F} = 0, \quad \lim \mathcal{F} \geq \lim \mathcal{F}.$$

Therefore, in this case, we have the equivalence **(LC2)⟺(LC2′)**.

The next two examples show that this equivalence is not always true for $L \neq \{0, 1\}$.

Example 4.2. [2, 17] Let L be the linearly ordered frame $(\{0, a, 1\}, \wedge, 1)$ with $0 < a < 1$. Assume $X = \{x, y\}$.

For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $z \in X$, let $\lim \mathcal{F}(z) = \begin{cases} 1, & \mathcal{F} \geq [z]; \\ 0, & \text{others}. \end{cases}$ Then $(X, \lim)$ satisfies the axiom **(LC2)**. Let $\mathcal{F}_x(\lambda) = \begin{cases} 1, & \lambda = 1_X; \\ a, & \lambda(x) = 1, \lambda(y) \neq 1; \\ a, & \lambda(x) = a; \\ 0, & \lambda(x) = 0. \end{cases}, \forall \lambda \in L^X.$

Then $\mathcal{F}_x$ is a stratified L-filter on X. It is easily seen that $\mathcal{F}_x = [a \wedge [x]]$ and $[x] \not\leq \mathcal{F}_x$. So,

$$a \wedge \lim[x](x) = a \not\leq 0 = \lim[a \wedge [x]](x).$$

That means $(X, \lim)$ does not satisfy **(LC2″)**.

Example 4.3. Let L, X and $\mathcal{F}_z$ be defined as the above example. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $z \in X$, let

$$\lim \mathcal{F}(z) = \begin{cases} 1, & \mathcal{F} = \mathcal{F}_z \text{ or } \mathcal{F} = [z]; \\ 0, & \text{others}. \end{cases}$$

When $\alpha = 0, 1$, the axiom **(LC2″)** holds obviously. For $\alpha = a$, it suffices to consider the case of $\mathcal{F} = \mathcal{F}_z, [z]$. It is easily seen that $[a \wedge [z]] = [a \wedge \mathcal{F}_z] = \mathcal{F}_z$. So,

$$\lim[a \wedge \mathcal{F}](z) = 1 \geq a = a \wedge \lim \mathcal{F}(z).$$

This means that lim satisfies the axioms **(LC2″)**. Let

$$\mathcal{F}_1(\lambda) = \begin{cases} 0, & \lambda(x) = 0; \\ 1, & \text{others}. \end{cases}, \forall \lambda \in L^X.$$

Then $\mathcal{F}_1$ a stratified L-filter and $\mathcal{F}_1 \geq \mathcal{F}_x$. However, $\lim \mathcal{F}_1(x) = 0 < 1 = \lim \mathcal{F}_x(x)$. That means lim does not satisfy the axiom **(LC2)**.

In [2], for L being a frame, it is proved that L-**SSC** is a reflective subcategory of L-**SC**. Indeed, we observe easily that this result also holds in the general lattice context.

Let $E_{\lim}(X)$ be the set of all stratified strong L-convergence structures on X. For $\lim, \lim' \in E_{\lim}(X)$, we define $\lim \leq \lim'$ iff $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \lim \mathcal{F} \leq \lim' \mathcal{F}$. Then it is easily seen that for each non-void subset $\{\lim_i | i \in I\}$ of $E_{\lim}(X)$, we have $\bigwedge_{i\in I} \lim_i \in E_{\lim}(X)$. The least element of $(E_{\lim}(X), \leq)$ is defined as $\lim \mathcal{F}(x) = ([x], \mathcal{F})$, and the largest element of $(E_{\lim}(X), \leq)$ is defined as $\lim \mathcal{F}(x) \equiv 1$. Thus $(E_{\lim}(X), \leq)$ forms a complete lattice.

For each $(X, \lim) \in L$-**SC**, the L-**SSC**-reflection [2] is given by $\mathrm{id}_X : (X, \lim) \longrightarrow (X, \lim^*)$, where $\lim^* = \bigwedge\{\lim' \in E_{\lim}(X) | \lim' \geq \lim\}$. The following proposition shows that $\lim_*$ can be constructed by lim itself.

Proposition 4.4. *Let $(X, \lim)$ be a stratified L-convergence space. Then the function $\overline{\lim} : \mathcal{F}_L^s(X) \longrightarrow L^X$ defined by*

$$\overline{\lim}\mathcal{F} = \bigvee_{\mathcal{G}\in\mathcal{F}_L^s(X)} ((\mathcal{G}, \mathcal{F}) * \lim \mathcal{G})$$

is the least stratified strong L-convergence structure on X larger than or equal to lim. *So,* $\lim^* = \overline{\lim}$.

Proof. **(LC1)** For each $x \in X$, $\overline{\lim}[x](x) \geq \lim[x](x) * ([x], [x]) = 1$.

(LC2′) For all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$,

$$\begin{aligned} \overline{\lim}\mathcal{F} * (\mathcal{F}, \mathcal{G}) &= \bigvee_{\mathcal{H}\in\mathcal{F}_L^s(X)} (\lim \mathcal{H} * (\mathcal{H}, \mathcal{F}) * (\mathcal{F}, \mathcal{G})) \\ &\leq \bigvee_{\mathcal{H}\in\mathcal{F}_L^s(X)} (\lim \mathcal{H} * (\mathcal{H}, \mathcal{G})) = \overline{\lim}\mathcal{G}. \end{aligned}$$

That $\overline{\lim} \geq \lim$ is obvious. For all $\lim' \in E_{\lim}(X)$ with $\lim' \geq \lim$, we have

$$\overline{\lim}\mathcal{F} = \bigvee_{\mathcal{H}\in\mathcal{F}_L^s(X)} (\lim \mathcal{H} * (\mathcal{H}, \mathcal{F})) \leq \bigvee_{\mathcal{H}\in\mathcal{F}_L^s(X)} (\lim' \mathcal{H} * (\mathcal{H}, \mathcal{F})) \overset{\mathbf{(LC2')}}{\leq} \lim' \mathcal{F}. \quad \square$$

In the proof of $\overline{\lim} \in L$-**SSC**, **(LC2)** is not used. Thus one can construct a stratified strong L-convergence structure from a function lim only with **(LC1)**.

5. **Conclusions**

This paper focuses on the relationships between three types of L-convergence spaces: stratified L-convergence spaces, stratified strong L-convergence spaces and stratified levelwise L-convergence spaces. The main results are summarized as the following two: (1) a levelwise characterization for stratified strong L-convergence spaces via strong left-continuity condition is obtained; (2) a sufficient and necessary condition such that a stratified L-convergence space is a stratified strong L-convergence space is obtained.

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