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Research Article

On $(N, p, q)(E, 1)$ Summability of Fourier Series

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A new theorem on $(N, p, q)(E, 1)$ summability of Fourier series has been established.

1. Introduction

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex, such that

$$\begin{aligned} P_n &= p_0 + p_1 + p_2 + \cdots + p_n = \sum_{\nu=0}^{n} p_\nu \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \\ Q_n &= q_0 + q_1 + q_2 + \cdots + q_n = \sum_{\nu=0}^{n} q_\nu \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \\ R_n &= p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 = \sum_{\nu=0}^{n} p_\nu q_{n-\nu} \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{1.1}$$

Given two sequences $\{p_n\}$ and $\{q_n\}$ convolution $(p * q)$ is defined as

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k. \tag{1.2}$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its nth partial sums $\{s_n\}$.

We write

$$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k s_k. \tag{1.3}$$

If $R_n \neq 0$, for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

If $t_n^{p,q} \to S$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is summable to S by generalized Nörlund method (Borwein [1]) and is denoted by

$$S_n \longrightarrow \mathrm{S}(N, p, q). \tag{1.4}$$

The necessary and sufficient conditions for (N, p, q) method to be regular are

$$\sum_{k=0}^{n} \left| p_{n-k} q_k \right| = O(|R_n|), \tag{1.5}$$

and $p_{n-k} = o(|R_n|)$, as $n \to \infty$ for every fixed $k \geq 0$, for which $q_k \neq 0$.

Now

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k} s_k. \tag{1.6}$$

If $E_n^1 \to s$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be $(E, 1)$ summable to s (Hardy [2]):

$$\begin{aligned} t_n^{p,q,E} &= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k E_k^1 \\ &= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \cdot \frac{1}{2^k}\sum_{\nu=0}^{k} \binom{k}{\nu} s_\nu. \end{aligned} \tag{1.7}$$

If $t_n^{p,q,E} \to \infty$, as $n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is summable to S by $(N, p, q)(E, 1)$ summability method.

Particular Cases

(1) $(N, p, q)(E, 1)$ mean reduces to $(N, p_n)(E, 1)$ summability mean if $q_n = 1$, $\forall n$.

(2) $(N, p, q)(E, 1)$ mean reduces to $(N, 1/(n+1))(E, 1)$ mean if $p_n = 1/(n+1)$ and $q_n = 1$, $\forall n$.

(3) $(N, p, q)(E, 1)$ method reduces to $(\overline{N}, q_n)(E, 1)$ if $p_n = 1$, $\forall n$.

(4) $(N, p, q)(E, 1)$ method reduces to $(C, \alpha)(E, 1)$ if $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, and $q_n = 1$, $\forall n$.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$.

Let its Fourier series be given by

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{1.8}$$

We use the following notations:

$$\begin{gathered}
\phi(t) = f(x+t) - f(x-t) - 2f(x), \\
\Phi(t) = \int_0^t |\phi(u)|\, du, \\
\tau = \left[\frac{1}{t}\right] = \text{the integral part of } \frac{1}{t}, \\
R\left(\frac{1}{t}\right) = R_\tau, \qquad R_n = R(n), \\
K_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{\cos^k(t/2)\cos(k+1)(t/2)}{\sin(t/2)}.
\end{gathered} \tag{1.9}$$

2. Theorem

A quite good amount of work is known for Fourier series by ordinary summability method. The purpose of this paper is to study Fourier series by $(N, p, q)(E, 1)$ summability method in the following form.

Theorem 2.1. *Let $\{p_n\}$ and $\{q_n\}$ be positive monotonic, nonincreasing sequences of real numbers such that*

$$R_n = \sum_{k=0}^{n} p_k q_{n-k} \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \tag{2.1}$$

Let $\alpha(t)$ be a positive, nondecreasing function of t. If

$$\Phi(t) = \int_0^t |\phi(u)| du = o\left(\frac{t}{\alpha(1/t)}\right), \quad \text{as } t \longrightarrow +0, \tag{2.2}$$

$$\alpha(n) \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \tag{2.3}$$

then a sufficient condition that the Fourier Series (1.8) be summable $(N, p, q)(E, 1)$ to $f(x)$ at the point $t = x$ is

$$\int_1^n \frac{R(u)}{u\alpha(u)}\, du = O(R_n), \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

3. Lemmas

Proof of the theorem needs some lemmas.

Lemma 3.1. *For* $0 \le t \le 1/n$,

$$|K_n(t)| = O(n). \tag{3.1}$$

Proof.

$$\begin{aligned} |K_n(t)| &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k \frac{\cos^k(t/2)\sin(k+1)(t/2)}{\sin(t/2)}\right| \\ &\le \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k \frac{(k+1)|\sin(t/2)|}{|\sin(t/2)|} = O(n)\frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k = O(n). \end{aligned} \tag{3.2}$$

□

Lemma 3.2. *If* $\{p_n\}$ *and* $\{q_n\}$ *are nonnegative and nonincreasing, then for* $0 \le a \le b < \infty$, $0 \le t \le \pi$, *and any* n *we have*

$$\frac{1}{2\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \frac{\cos^k(t/2)\sin(k+1)(t/2)}{\sin(t/2)}\right| = O\left(\frac{R_\tau}{tR_n}\right). \tag{3.3}$$

Proof.

$$\begin{aligned} &\frac{1}{2\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \frac{\cos^k(t/2)\ \sin(k+1)(t/2)}{\sin(t/2)}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)\sin(k+1)\frac{t}{2}\right| \\ &\quad= \frac{1}{t\pi R_n}\left|\operatorname{Im}\left\{\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{i(k+1)(t/2)}\right\}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{ikt/2}\right|\left|e^{it/2}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{ikt/2}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left\{\left|\sum_{k=a}^{\tau-1} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{\frac{ikt}{2}}\right| + \left|\sum_{k=\tau}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{ik\left(\frac{t}{2}\right)}\right|\right\}. \end{aligned} \tag{3.4}$$

Now considering first term of (3.4), we have

$$\begin{aligned}\frac{1}{t\pi R_n}\left|\sum_{k=a}^{\tau-1} p_{n-k}q_k\cos^k\left(\frac{t}{2}\right)e^{ik(t/2)}\right| &\le \frac{1}{t\pi R_n}\sum_{k=a}^{\tau-1} p_{n-k}q_k\left|e^{ik(t/2)}\right| \le \frac{1}{t\pi R_n}\sum_{k=a}^{\tau-1} p_{n-k}q_k \\ &\le \frac{1}{t\pi R_n}\sum_{k=a}^{\tau-1} p_{\tau-k}q_k \le \frac{1}{t\pi R_n}(R_\tau) = O\left(\frac{R_\tau}{tR_n}\right).\end{aligned} \tag{3.5}$$

Now considering second term of (3.4) and using Abel's lemma, we have

$$\begin{aligned}\frac{1}{t\pi R_n}\left|\sum_{k=\tau}^{b} p_{n-k}q_k\cos^k\left(\frac{t}{2}\right)e^{ik(t/2)}\right| &\le \frac{1}{t\pi R_n}\left|\sum_{k=\tau}^{b} p_{n-k}q_k e^{ik(t/2)}\right| \\ &\le \frac{2p_{n-b}q_\tau}{t\pi R_n}\max_{\tau+1\le k\le b}\left|\frac{1-e^{i(k+1)(t/2)}}{1-e^{it/2}}\right| \\ &\le \frac{4p_{n-b}q_\tau}{t\pi R_n}\left|\frac{e^{-it/4}}{e^{it/4}-e^{-it/4}}\right| \\ &\le \frac{2q_\tau}{t\pi R_n}\left(\frac{p_{n-b}}{P_\tau}\right)P_\tau\left|\frac{1}{\sin(t/4)}\right| \quad \left(\text{where } P_\tau = \sum_{k=0}^{\tau} p_{\tau-k}\right) \\ &\le \frac{8q_\tau}{t\pi R_n}\left(\frac{p_{n-b}}{P_\tau}\right)P_\tau\left|\frac{1}{t}\right| \\ &\le \frac{8q_\tau P_\tau}{t\pi R_n} \\ &\le \frac{8R_\tau}{t\pi R_n} \quad \left(\text{since } R_\tau = \sum_{k=0}^{\tau} p_{\tau-k}q_k \ge P_\tau q_\tau\right) \\ &= O\left(\frac{R_\tau}{tR_n}\right).\end{aligned} \tag{3.6}$$

Using (3.5) and (3.6), we get the required result of Lemma 3.2. □

4. Proof of Theorem

Following Zygmund [3], the nth partial sum $s_n(x)$ of the series (1.8) at $t = x$ is given by

$$s_n(x) = f(x) + \frac{1}{2\pi}\int_0^\pi \phi_x(t)\frac{\sin(n+1/2)t}{\sin(t/2)}dt. \tag{4.1}$$

So the $(E,1)$ mean of the series (1.8) at $t = x$ is given by

$$\begin{aligned}
E_n^1(x) &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k(x) \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n}\binom{n}{k}\ \sin\left(k+\frac{1}{2}\right)t\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}\left(1+e^{it}\right)^n\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}(1+\cos\ t+i\sin t)^n\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}2^n\cos^n\left(\frac{t}{2}\right)\left(\cos\frac{t}{2}+i\sin\frac{t}{2}\right)^n\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}2^n\cos^n\left(\frac{t}{2}\right)\left\{\left(\cos\frac{nt}{2}+i\sin\frac{nt}{2}\right)\right\}\right\}dt \\
&= f(x) + \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\ \frac{\cos^n(t/2)\sin(n+1)(t/2)}{\sin(t/2)}\ dt.
\end{aligned} \tag{4.2}$$

Therefore (N,p,q) transform of $\{E_n^1(x)\}$ is given by

$$\begin{aligned}
t_n^{p,q,E}(x) &= \frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_kE_k^1(x) = f(x) + \frac{1}{2\pi}\int_0^{\pi}\frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k\phi_x(t)\frac{\cos^k(t)\sin(k+1)(t/2)}{\sin(t/2)} \\
&= f(x) + \int_0^{\pi}K_n(t)\phi_x(t)dt, \\
t_n^{p,q,E}(x) - f(x) &= \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right]K_n(t)\phi_x(t)dt = I_1 + I_2 + I_3 \quad (\text{say}).
\end{aligned} \tag{4.3}$$

We have

$$\begin{aligned}
|I_1| &\le \int_0^{1/n}|K_n(t)|\ \left|\phi_x(t)\right|dt \\
&= O(n)\int_0^{1/n}\left|\phi_x(t)\right|dt \quad (\text{using Lemma 3.1}) \\
&= O(n)o\left(\frac{1}{n\alpha(n)}\right) \quad (\text{by (2.2)}) \\
&= o\left(\frac{1}{\alpha(n)}\right) = o(1) \quad \text{as } n \longrightarrow \infty \quad (\text{by (2.3)}).
\end{aligned} \tag{4.4}$$

Now we consider

$$
\begin{aligned}
|I_2| &\le \int_{1/n}^{\delta} |K_n(t)|\left|\phi_x(t)\right| dt \quad (\text{where } 0<\delta<1)\\
&= \int_{1/n}^{\delta} O\left(\frac{R(1/t)}{tR_n}\right)\left|\phi_x(t)\right| dt \quad (\text{using Lemma 3.2})\\
&= O\left(\frac{1}{R_n}\right)\int_{1/n}^{\delta}\left(\frac{R(1/t)}{t}\right)\left|\phi_x(t)\right| dt\\
&= O\left(\frac{1}{R_n}\right)\left[\left\{\left(\frac{R(1/t)}{t}\right)\phi_x(t)\right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} d\left(\frac{R(1/t)}{t}\right)\phi_x(t)\right]\\
&= O\left(\frac{1}{R(n)}\right)\left[\left\{o\left(\frac{R(1/t)}{\alpha(1/t)}\right)\right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} \phi_x(t) d\left(\frac{R(1/t)}{t}\right)\right] \quad (\text{by } (2.1))\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o\left(\frac{1}{R(n)}\right)\left[\int_{1/n}^{\delta}\phi_x(t)\left\{d\left(\frac{R(1/t)\alpha(1/t)}{t\alpha(1/t)}\right)\right\}\right]\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o\left(\frac{1}{R(n)}\right)\\
&\quad\times\left[\int_{1/n}^{\delta} o\left(\frac{t}{\alpha(1/t)}\right)\left\{d\alpha(1/t)\left(\frac{R(1/t)}{t\alpha(1/t)}\right)\right\} + \alpha\left(\frac{1}{t}\right)d\left(\frac{R(1/t)}{t\alpha(1/t)}\right)\right]\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o(1)\left[\int_{1/n}^{\delta}\frac{d\alpha(1/t)}{\{\alpha(1/t)\}^2}+o\left(\frac{1}{R(n)}\right)\int_{1/n}^{\delta} t\, d\left(\frac{R(1/t)}{t\alpha(1/t)}\right)\right]\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o(1)\left\{\frac{1}{\alpha(1/t)}\right\}_{1/n}^{\delta}\\
&\quad+o\left(\frac{1}{R(n)}\right)\left[\left\{\frac{tR(1/t)}{t\alpha(1/t)}\right\}_{1/n}^{\delta}-\int_{1/n}^{\delta}\left(\frac{R(1/t)}{t\alpha(1/t)}\right)dt\right]\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o\left(\frac{1}{R(n)}\right)\int_{1/n}^{1}\left(\frac{R(1/t)}{t\alpha(1/t)}\right)dt\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o\left(\frac{1}{R(n)}\right)\int_{1}^{n}\left(\frac{R(u)}{u\alpha(u)}\right)du \quad \left\{\because \frac{1}{t}=u\right\}\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o\left(\frac{1}{R(n)}\right)O(R_n) \quad (\text{by } (2.4))\\
&= o\left(\frac{1}{R(n)}\right)+o\left(\frac{1}{\alpha(n)}\right)+o(1)\\
&= o(1), \quad \text{as } n\to\infty \ (\text{by virtue of } (2.1) \text{ and } (2.2)).
\end{aligned} \tag{4.5}
$$

Now by Riemann-Lebesgue theorem and by regularity of the method of summability we have

$$I_3 = \int_\delta^\pi |k_n(t)| \left|\phi_x(t)\right| dt \\ = o(1), \quad \text{as } n \longrightarrow \infty. \tag{4.6}$$

This completes the proof of the theorem.

5. Corollaries

Following corollaries can be derived from our main theorem.

Corollary 5.1. *If*

$$\Phi(t) = o\left[\frac{t}{\log(1/t)}\right], \quad \text{as } t \longrightarrow +0, \tag{5.1}$$

then the Fourier series (1.8) *is* $(C,1)(E,1)$ *summable to* $f(x)$ *at the point* $t = x$*.*

Corollary 5.2. *If*

$$\Phi(t) = o(t), \quad \text{as } t \longrightarrow +0, \tag{5.2}$$

then the Fourier series (1.8) *is* $(N,p_n)(E,1)$ *summable to* $f(x)$ *at the point* $t = x$*, provided that* $\{p_n\}$ *be a positive, monotonic, and nonincreasing sequence of real numbers such that*

$$p_n = p_0 + p_1 + \cdots + p_n \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \tag{5.3}$$

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