Boundary estimates for solutions to linear degenerate parabolic equations

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA-domain and let $\Omega_T = \Omega \times (0,T)$ for some $T > 0$. We study the boundary behaviour of non-negative solutions to the equation

$$Hu = \partial_t u - \partial_{x_i}(a_{ij}(x,t)\partial_{x_j} u) = 0,\ (x,t) \in \Omega_T.$$

We assume that $A(x,t) = \{a_{ij}(x,t)\}$ is measurable, real, symmetric and that

$$\beta^{-1}\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \beta\lambda(x)|\xi|^2 \text{ for all } (x,t) \in \mathbb{R}^{n+1},\ \xi \in \mathbb{R}^n,$$

for some constant $\beta \geq 1$ and for some non-negative and real-valued function $\lambda = \lambda(x)$ belonging to the Muckenhoupt class $A_{1+2/n}(\mathbb{R}^n)$. Our main results include the doubling property of the associated parabolic measure and the Hölder continuity up to the boundary of quotients of non-negative solutions which vanish continuously on a portion of the boundary. Our results generalize previous results of Fabes, Kenig, Jerison, Serapioni, see [18], [19], [20], to a parabolic setting.

Key words: Linear, degenerate, parabolic, boundary Harnack, parabolic measure, NTA.

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1 Introduction and statement of main results

In a sequence of papers, see [18], [19], [20], Fabes, Kenig, Jerison and Serapioni (in the following refered to as Fabes et al.) developed the theory concerning the boundary behaviour of solutions to linear degenerate elliptic equations of the form

$$\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) = 0 \text{ in } \mathbb{R}^n. \tag{1.1}$$

Fabes et al. assume that $A(x) = \{a_{ij}(x)\}$ is measurable, real, symmetric, for every $x \in \mathbb{R}^n$, and that

$$\beta^{-1}\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \beta\lambda(x)|\xi|^2 \tag{1.2}$$

for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and for some constant β, $1 \leq \beta < \infty$. The weight $\lambda = \lambda(x)$ is assumed to belong to the Muckenhoupt class $A_2(\mathbb{R}^n)$. While the results by Fabes et al., to some extent are straight forward generalizations of previous results established in the uniformly elliptic case, that is when $\lambda(x) \equiv 1$, see [5], [12], [2], [32], [43] and the references in these papers, the results by Fabes et al. have recently proved important in several fields within the area of partial differential equations. In particular, firstly in [4], [3], [47], the results are used in the study of the boundary behaviour of non-local operators exemplified by the fractional Laplacian. Secondly, in [34]-[41], a theory concerning the boundary behaviour for solutions to operators of p-Laplace type is developed. Part of the technical toolbox developed in [34]-[41], consists of techniques for establishing boundary Harnack inequalities for p-harmonic functions vanishing on a portion of the boundary of a domain which is 'flat' in the sense that its boundary is well-approximated by hyperplanes. In this case, at the final stage of the analysis, results are derived in the non-linear case by a reduction to linear degenerate elliptic equations of the form considered by Fabes et al.

Based on the above it is natural to attempt to develop a parabolic counterpart of the elliptic theory developed by Fabes et al., and in this case the operators of interest are second order parabolic partial differential operators of the form

$$H = \partial_t - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}), \ (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.3}$$

where again $A = A(x,t) = \{a_{ij}(x,t)\} = \{a_{ij}\}$ is assumed measurable, real and symmetric, for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. To allow for degeneracy we assume that there exists a real valued function $\lambda : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$\beta^{-1}\lambda(x,t)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \beta\lambda(x,t)|\xi|^2 \tag{1.4}$$

for all $(x,t) \in \mathbb{R}^{n+1}$, $\xi \in \mathbb{R}^n$, and for some constant β, $1 \leq \beta < \infty$. In fact, for this type of equations results of interior character were established in [6], [7], [8], [9], [10], [11], under various integrability conditions on the weight $\lambda = \lambda(x,t)$. For example, in [8] the authors establish a Harnack inequality for non-negative solutions to $Hu = 0$ assuming that $\lambda(x,t) = \lambda(x)$, i.e.,

λ is time-independent, and $\lambda \in A_{1+2/n}(\mathbb{R}^n)$. Furthermore, in the same paper the authors also show, by way of examples, that when $n \geq 3$ and λ is time-independent, the condition $\lambda \in A_{1+2/n}(\mathbb{R}^n)$ is sharp among the Muckenhoupt A_p-conditions for the continuity of weak solutions. This is in contrast to the elliptic case, where the condition $\lambda \in A_2(\mathbb{R}^n)$ is sufficient for the same conclusion. See also [9] for many interesting examples concerning the difference between the elliptic and parabolic case in the context of degenerate operators, and some results in the context of degenerate parabolic operators with time-dependent weights.

This paper is the first in a sequence of two papers devoted to the study of the boundary behaviour of non-negative solutions to linear degenerate parabolic operators satisfying (1.4). In this paper we consider operators as in (1.3), satisfying (1.4) for some $\lambda(x,t) = \lambda(x)$, and we assume

$$\lambda \in A_{1+2/n}(\mathbb{R}^n) \text{ and we will denote the } A_{1+2/n}(\mathbb{R}^n)\text{-constant of } \lambda \text{ by } \Lambda. \tag{1.5}$$

In a subsequent paper we intend to consider the case of time-dependent weights as part of an ambition to understand the boundary behaviour of non-negative solutions to non-linear parabolic equations of p-parabolic type somehow along the lines of the elliptic theory developed in [34], [35], [38], [36]. However, already the case of time-independent weights $\lambda(x,t) = \lambda(x) \in A_{1+2/n}(\mathbb{R}^n)$ forces us to revisit essentially all the relevant arguments used in the corresponding context of uniformly parabolic equations. The contribution of the paper is a generalization of results previously established for uniformly parabolic equation in divergence form in the celebrated papers of Fabes, Safonov and Yuan, see [21], [22], [46], to operators as in (1.3) satisfying (1.4) for some $\lambda(x,t) = \lambda(x)$ as in (1.5).

1.1 Statement of main results

Let the operator H be as in (1.3), satisfying (1.4), for some $\lambda(x,t) = \lambda(x)$ as in (1.5). We will work in cylinders $\Omega_T = \Omega \times (0,T)$, $T > 0$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, i.e., a bounded, connected and open set in $\mathbb{R}^n$. Let the parabolic boundary of the cylinder Ω_T, $\partial_p\Omega_T$, be defined as

$$\partial_p\Omega_T = S_T \cup (\bar{\Omega} \times \{0\}), \quad S_T = \partial\Omega \times [0,T].$$

Some restriction on Ω will be needed. We will assume that Ω is an non-tangentially accessible domain, or NTA-domain for short, as introduced in [32]. If Ω is a NTA-domain, with parameters M and R_0, then for any $x_0 \in \partial\Omega$, $0 < R < R_0$, there exists a non-tangential corkscrew point, that is a point $A_R(x_0) \in \Omega$, such that

$$M^{-1}R \leq d(x_0, A_R(x_0)) \leq R, \text{ and } d(A_R(x_0), \partial\Omega) \geq M^{-1}R,$$

where d is the Euclidean distance $d(x,y) = |x-y|$. In Section 4 we prove that if Ω is a bounded NTA-domain then there exists, for each $f \in C(\partial_p\Omega_T)$, a unique (weak) solution $u \in C(\overline{\Omega}_T)$ to the continuous Dirichlet problem

$$Hu = 0 \text{ in } \Omega_T, \quad u = f \text{ on } \partial_p\Omega_T. \tag{1.6}$$

As a consequence there also exists a unique probability measure $\omega(x,t,\cdot)$ on $\partial_p\Omega_T$ such that

$$u(x,t) = \int_{\partial_p\Omega_T} f(y,s)d\omega(x,t,y,s) \tag{1.7}$$

whenever u is the unique solution to the continuous Dirichlet problem (1.6). We will refer to $\omega(x,t,\cdot)$ as the H-parabolic measure, or simply parabolic measure, relative to (x,t) and Ω_T.

The Harnack inequality proven in [8] does not hold in standard parabolic cylinders but rather in cylinders associated with the weight λ. The same is true for our results. Given $R>0$ and $x\in\mathbb{R}^n$, we let

$$r_x(R)=\left(\int_{B(x,R)}\lambda^{-n/2}(\xi)d\xi\right)^{1/n},$$

and

$$\begin{aligned}C_R^*(x,t)&=B(x_0,R)\times\left(t_0-r_{x_0}(R)^2,t_0+r_{x_0}(R)^2\right),\\ \Delta_R^*(x,t)&=S_T\cap C_R^*(x,t).\end{aligned}$$

Note that by construction the cylinders $\{C_R^*(x_0,t_0)\}$ take the degeneracy of H into account and that the use of these intrinsic cylinder allows us to state our main theorems with constants which do not depend on the weight λ directly. We let $\operatorname{diam}(\Omega)=\sup\{|x-y|\mid x,y\in\Omega\}$ denote the Euclidean diameter of Ω and we let $\operatorname{diam}_\lambda(\Omega)=\sup\{r_x(|x-y|)\,|\,x,y\in\Omega\}$. When we in the following write that a constant c depends on the operator H, $c=c(H)$, we mean that c depends on the dimension n, the constant β in (1.4) and the constant Λ in (1.5). The following theorems are the main results proved in this paper. For notation and definitions, we refer to Section 2 and Section 3.

Theorem 1.1. *Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega\subset\mathbb{R}^n$ be a bounded NTA-domain with parameters M, R_0 and let $\Omega_T=\Omega\times(0,T)$ for some $T>0$. Let u be a non-negative solution of $Hu=0$ in Ω_T vanishing continuously on S_T. Then there is an $\bar r_0=\bar r_0(H,M,R_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega))$, $\bar r_0>0$ such that the following holds. Let δ, $0<\delta<\bar r_0$, be a fixed constant, let $(x_0,t_0)\in S_T$, $\delta^2\le t_0$, and assume that R satisfies $r_{x_0}(R)<\delta/2$. Then there exists $c=c(H,M,\operatorname{diam}_\lambda(\Omega),T,\delta)$, $1\le c<\infty$, such that*

$$u(x,t)\le cu\big(A_R(x_0),t_0\big)$$

whenever $(x,t)\in\Omega_T\cap C_R^*(x_0,t_0)$.

Theorem 1.2. *Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega\subset\mathbb{R}^n$ be a bounded NTA-domain with parameters M, r_0 and let $\Omega_T=\Omega\times(0,T)$ for some $T>0$. There then is an $\bar r_0=\bar r_0(H,M,r_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega))$, $\bar r_0>0$ such that the following is true. Let $0<\delta<\bar r_0$ be a fixed constant. Let $(x_0,t_0)\in S_T$ be such that $16\delta^2\le t_0$ and $\delta^2\le T-t_0$, and suppose that $r_{x_0}(R)<\delta/2$. Then there exists a constant $c=c(H,M,\operatorname{diam}_\lambda(\Omega),T,\delta)$, $1\le c<\infty$, such that*

$$\omega(x,t,\Delta_{2R}^*(x_0,t_0))\le c\omega(x,t,\Delta_R^*(x_0,t_0)),$$

whenever $(x,t)\in\Omega_T$ is such that $t\ge t_0+16r_{x_0}(R)^2$.

Theorem 1.3. *Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega\subset\mathbb{R}^n$ be a bounded NTA-domain with parameters M, r_0 and let $\Omega_T=\Omega\times(0,T)$ for some $T>0$. There then is an $\bar r_0=\bar r_0(H,M,R_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega))$, $\bar r_0>0$ such that the following is true. Let u,v be non-negative solutions of $Hu=0$ in Ω_T vanishing continuously on S_T. Let δ, $0<\delta<\bar r_0$, be a fixed* *constant. Then u/v is Hölder continuous on the closure of $\Omega\times(\delta^2,T]$. Furthermore, let $(x_0,t_0)\in S_T$, $\delta^2\leq t_0$, and assume that $r_{x_0}(R)<\delta/2$. Then there exist $c=c(H,M,\operatorname{diam}_\lambda(\Omega),T,\delta)$, $1\leq c<\infty$, and $\alpha=\alpha(H,M,\operatorname{diam}_\lambda(\Omega),T,\delta)$, $\alpha\in(0,1)$, such that*

$$\left|\frac{u(x,t)}{v(x,t)}-\frac{u(y,s)}{v(y,s)}\right|\leq c\left(\frac{r_x(|x-y|)+|s-t|^{1/2}}{r_{x_0}(R)}\right)^\alpha\frac{u\big(A_R(x_0),t_0\big)}{v\big(A_R(x_0),t_0\big)}$$

whenever $(x,t),(y,s)\in\Omega_T\cap C^*_{R/c}(x_0,t_0)$.

Note that by a covering argument, versions of Theorems 1.1-1.3 could also be stated using the standard parabolic cylinders $\{C_R(x_0,t_0)=B(x_0,R)\times(t_0-R^2,t_0+R^2)\}$. However, in this case the constants would depend on the quotient of $r_{x_0}(R)$ and R.

Note that Theorem 1.1 and Theorem 1.3 have a global flavor in the sense that we assume that u and v are non-negative solutions of $Hu=0$ in all of Ω_T, vanishing continuously on the entire lateral boundary S_T. Naturally, also local versions can be formulated but we here omit further details.

To put Theorem 1.1-Theorem 1.3 in perspective we note, as briefly mentioned at the beginning of the introduction, that for uniformly parabolic equations, the case $\lambda\equiv1$, the study of the type of problems considered in this paper, and in particular Theorem 1.1-Theorem 1.3, have a long and rich history which culminated with the celebrated papers of Fabes, Safonov and Yuan [21], [22] and [46]. In these works the authors proved Theorem 1.1-Theorem 1.3 for linear uniformly parabolic equations, both in divergence and non-divergence form. We remark that, while these authors work in Lipschitz cylinders, one can easily see that their proofs can be generalized to the setting of bounded NTA-cylinders. While the works Fabes, Safonov and Yuan completed, for linear uniformly parabolic equations, the line of research considered in this paper, contributions by other researchers are contained in [17], [23], [25], [33], [16], [21], [44]. For the elliptic versions of Theorem 1.1-Theorem 1.3 we refer to [18], [19], [20], and we emphasize that in the elliptic case the assumption $\lambda\in A_2(\mathbb{R}^n)$ on the weight is sufficient for the validity of the corresponding versions of Theorem 1.1-Theorem 1.3.

1.2 Organization of the paper

In Section 2, which mainly is of preliminary nature, we introduce our main technical tool which is a weighted distance function d_λ related to the function $r_x(R)$. In this section we also define weak solutions and we state fundamental principles like Cacciopoli estimates, the Harnack inequality, interior Hölder continuity estimates and the weak maximum principle. Section 3 is devoted to geometry. We here introduce the notion of λ-NTA-domains, NTA-domains with respect to d_λ, and we prove, see Lemma 3.4 and Lemma 3.3 below, that the λ-NTA-domains are exactly the classic NTA-domains introduced in [32]. However, the setting of λ-NTA-domains facilitates the use of the Harnack inequality of [8] and is used in the remainder of the paper. In section 4 we establish existence and uniqueness for solutions to the continuous Dirichlet problem stated in (1.6) and the existence of the parabolic measure, assuming that Ω is a λ-NTA-domain. This is done by approximating H with a sequence of uniformly parabolic operators. In Section 5 we establish some technical lemmas and prove Theorem 1.1. In Section 6 and Section 7 we prove Theorem 1.2 and Theorem 1.3, respectively, by first proving the theorems for approximating uniformly parabolic operators and then passing to the limit.

2 Preliminaries

In this section we introduce notation, definitions and preliminary results that will be used throughout the paper.

2.1 Notations and conventions

Points in Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ will usually be denoted by $(x,t) = (x_1, \ldots, x_n, t)$. The notation (y,s) or (ξ, τ) will also be used when needed. Given a set $E \subset \mathbb{R}^n$, let $\bar{E}, \partial E$, be the closure and boundary of E. Let $x \cdot y$ denote the standard inner product on $\mathbb{R}^n$, and $|x| = \sqrt{x \cdot x}$ the Euclidean norm of x. Let dx be Lebesgue n-measure on $\mathbb{R}^n$ and for any measurable $E \in \mathbb{R}^n$ we let $|E| = \int_E dx$. Given $E \in \mathbb{R}^n$ let $\mathrm{diam}(E) = \sup\{|x-y| : x, y \in E\}$. Given $(x,t) \in \mathbb{R}^{n+1}$ and $R > 0$, let $B(x,R)$ denote the standard Euclidean ball

$$B(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\},$$

and let $C_R(x,t)$ denote the standard parabolic cylinder

$$C_R(x,t) = B(x,R) \times (t - R^2, t - R^2).$$

Note that the Euclidean radii of balls will always be denoted by R. Throughout the paper c will denote a positive constant $c \geq 1$, not necessarily the same at each occurrence. In general $c = c(a_1, \ldots, a_m)$ denotes a positive constant $c \geq 1$ may depend only on $a_1, \ldots, a_m$ and which is not necessarily the same at each occurrence. Let H be as in (1.3) and assume (1.4) and (1.5). That c depends on the operator H, $c = c(H)$, means that $c = c(n, \Lambda, \beta)$ where n is the dimension, Λ is the $A_{1+2/n}$-constant of λ and β is as in (1.4). Two quantities A and B are said to be comparable, or $A \approx B$, if $c^{-1} \leq A/B \leq c$ for some $c = c(H)$, $c \geq 1$.

2.2 Weights and distances

Recall that a function λ is said to belong to the Muckenhoupt class $A_p = A_p(\mathbb{R}^n)$, for some p, $1 < p < \infty$, if λ is non-negative, measurable and satisfies

$$\left(\frac{1}{|B(y,R)|} \int_{B(y,R)} \lambda(x) dx\right) \left(\frac{1}{|B(y,R)|} \int_{B(y,R)} \lambda(x)^{-1/(p-1)} dx\right)^{p-1} \leq \Lambda_{\lambda,p} < \infty \tag{2.1}$$

for all $y \in \mathbb{R}^n$, $R > 0$. The constant $\Lambda_{\lambda,p} = \Lambda_p$ is called the A_p constant of λ. Let $\lambda(E) = \int_E \lambda(x) dx$, for all measurable sets E. Then, in particular, every $\lambda \in A_p(\mathbb{R}^n)$ gives a doubling measure with doubling constants depending only on n, p and Λ_p. In the following we let λ be as in (1.5), that is $\lambda \in A_{1+2/n}(\mathbb{R}^n)$ with constant $\Lambda_{1+2/n} = \Lambda$. Let

$$d_\lambda(x,y) = \left(\int_{B(x,|x-y|)} \lambda^{-n/2}(\xi) d\xi\right)^{1/n} \tag{2.2}$$

whenever $x, y \in \mathbb{R}^n$. The function d_λ will be used to measure distances weighted by λ. Note that, up to normalisation, d_λ coincides with the Euclidean metric when $\lambda \equiv 1$. The function d_λ has some of the characteristics of a metric. To start with, for $x \in \mathbb{R}^n$ fixed, $d_\lambda(x,y)$ increases as $|x-y|$ increases. Furthermore, $d_\lambda(x,y)=0$ if and only if $x=y$. However, in general $d_\lambda(x,y)\neq d_\lambda(y,x)$ and d_λ only satisfies a relaxed triangle inequality making d_λ only a quasi-semi-metric on $\mathbb{R}^n$, see Lemma 2.2 below. Having introduced d_λ we let

$$\mathrm{diam}_\lambda(E)=\sup\{d_\lambda(x,y):x,y\in E\}.$$

for any set $E\subset\mathbb{R}^n$. For each $x\in\mathbb{R}^n$ and $r>0$, we let

$$B_\lambda(x,r)=\big\{y\in\mathbb{R}^n:d_\lambda(x,y)<r\big\}, \tag{2.3}$$

denote the corresponding open ball with center at x and with radius r, measured with d_λ. Furthermore, for $(x,t)\in\mathbb{R}^{n+1}$ and $r,r_1,r_2>0$, let

$$\begin{aligned} C_r^\lambda(x,t)&=B_\lambda(x,r)\times(t-r^2,t+r^2), & C_{r_1,r_2}^\lambda(x,t)&=B_\lambda(x,r_1)\times(t-r_2^2,t+r_2^2),\\ C_r^{\lambda,+}(x,t)&=B_\lambda(x,r)\times(t,t+r^2), & C_r^{\lambda,-}(x,t)&=B_\lambda(x,r)\times(t-r^2,t),\\ C_{r_1,r_2}^{\lambda,+}(x,t)&=B_\lambda(x,r_1)\times(t,t+r_2^2), & C_{r_1,r_2}^{\lambda,-}(x,t)&=B_\lambda(x,r_1)\times(t-r_2^2,t). \end{aligned} \tag{2.4}$$

Finally, let the (weighted) parabolic distance, be defined as

$$d_{\lambda,p}((x,t),(y,s))=d_{\lambda,p}(x,t,y,s)=((d_\lambda(x,y))^2+|s-t|)^{1/2}, \tag{2.5}$$

whenever $(x,t),(y,s)\in\mathbb{R}^{n+1}$. To gain further intuition concerning our weighted setting it is important to note that the set $B_\lambda(x,r)$ is in fact an Euclidean ball. In particular, for every $x\in\mathbb{R}^n$ and $R>0$ there is an r such that $B_\lambda(x,r)=B(x,R)$ and vice versa. To formalize this we define, if $\lambda\in A_{1+2/n}$, $x\in\mathbb{R}^n$ and $R>0$,

$$r_x(R)=\left(\int_{B(x,R)}\lambda^{-n/2}(\xi)d\xi\right)^{1/n}, \tag{2.6}$$

and we note, using this notation, that $B_\lambda\big(x,r_x(R)\big)=B(x,R)$. Also note that the cylinder used in the statement of Theorems 1.1-1.3 is exactly $C_R^*(x,t)=C_{r_x(R)}^\lambda(x,t)$, however for the sake of brevity we avoided this notation in the introduction of the paper. Since the function r_x is strictly increasing, it has an inverse $R_x(r)=r_x^{-1}(r)$. This means that $B\big(x,R_x(r)\big)=B_\lambda(x,r)$. Note also that R_x is strictly increasing. The connection between d_λ and the Euclidean metric allows us recover some geometrical information from the quasi-semi-metric. In particular, combining the A_p condition and the Hölder inequality we derive the useful comparison

$$r_x(R)^2\approx\frac{R^{n+2}}{\lambda\left(B(x,R)\right)}$$

valid for every $x\in\mathbb{R}^n$ and $R>0$, with comparison constants depending only on n and Λ. Furthermore, when comparing the radii of two balls, the following lemma allows us to switch between the weighted and the Euclidean settings.

Lemma 2.1. *Let λ be as in* (1.5). *Let $x,\hat{x}\in\mathbb{R}^n$ and $R,\hat{R}>0$ be such that $B(\hat{x},\hat{R})\subset B(x,R)$. Then there exist $c=c(n,\Lambda)$, $c\geq 1$, such that*

$$c^{-1}\left(\hat{R}/R\right)^c\leq\frac{r_{\hat{x}}(\hat{R})}{r_x(R)}\leq c\left(\hat{R}/R\right)^{1/c}. \tag{2.7}$$

Proof. Let $\mu(x) = \lambda^{-n/2}(x)$ for all $x \in \mathbb{R}^n$. Then, by (2.1), $\mu \in A_{1+2/n}$ with $A_{1+2/n}$-constant depending only on n and Λ. Noting that $r_x(R) = \mu(B(x,R))^{1/n}$, the lemma follows from Lemma 5 in [12]. $\square$

Lemma 2.2. *Let λ be as in* (1.5) *and let $d_\lambda(x,y)$, $x, y \in \mathbb{R}^n$, be defined as in* (2.2). *Then there exists a constant $c_\Delta = c_\Delta(n,\Lambda)$, $1 \le c_\Delta < \infty$, such that*

$$d_\lambda(x,z) \le c_\Delta(d_\lambda(x,y) + d_\lambda(y,z)) \tag{2.8}$$

and

$$d_\lambda(x,z) \le c_\Delta d_\lambda(z,x) \tag{2.9}$$

whenever $x, y, z \in \mathbb{R}^n$.

Proof. We first note that if $|x-z| \le |x-y|$, then $d_\lambda(x,z) \le d_\lambda(x,y)$ and hence there is nothing more to prove. We may therefore assume that $|x-z| > |x-y|$. Assume now that x, y, and z are collinear and y lies between x and z. Then $B(y,|y-z|) \subset B(x,|x-z|)$ and using Lemma 2.1 it follows that

$$c^{-1} d_\lambda(x,z)\frac{|y-z|^c}{|x-z|^c} \le d_\lambda(y,z),\ \ c^{-1} d_\lambda(x,z)\frac{|x-y|^c}{|x-z|^c} \le d_\lambda(x,y),$$

for some $c = c(n,\Lambda) \ge 1$. Adding the estimates in the last display we see that

$$c^{-1} d_\lambda(x,z)\left(\frac{|x-y|^c + |y-z|^c}{|x-z|^c}\right) \le d_\lambda(x,y) + d_\lambda(y,z).$$

Since $|x-z|^c \le 2^{c-1}\left(|x-y|^c + |y-z|^c\right)$ the proof of (2.8) is complete in this case. In the general case, let z' be the point collinear with x and y such that y lies between x and z' and such that $|y-z'| = |y-z|$, $|x-z| < |x-z'|$ and $|x-y| < |x-z'|$. We can then apply the previous argument to x, y, z' and we deduce that

$$d_\lambda(x,z) \le d_\lambda(x,z') \le c(d(x,y) + d(y,z')) = c(d(x,y) + d(y,z)).$$

This finishes the proof of the triangle inequality in (2.8). The inequality in (2.9) follows from Lemma 2.1, since

$$d_\lambda(x,y) = r_x(|x-y|) \le r_x(2|x-y|) \le c r_y(|x-y|) = c d_\lambda(y,x).$$

Hence the proof is complete. $\square$

Remark 2.2. *If $c_\Delta > 2$ the triangle inequality* (2.8) *does not give any lower bound for $d_\lambda\left(\partial B_\lambda(x,r), \partial B_\lambda(x,2r)\right)$. However, $d_\lambda\left(\partial B_\lambda(x,r), \partial B_\lambda(x,2c_\Delta r)\right) \ge r$. This technical remark is of some importance since you for example can not construct a function that is constant* 1 *on $B_\lambda(x,r)$ and supported $B_\lambda(x,2r)$ and has controlled gradient.*

Repeated use of the triangle inequality gives

$$d(x,w) \le c_\Delta\big(c_\Delta\big(d(x,y) + d(y,z)\big) + d(z,w)\big) \le {c_\Delta}^2(d(x,y) + d(y,z) + d(z,w))$$

and so forth. Adding j distances in this linear fashion makes the constant grow as ${c_\Delta}^{j-1}$. This simple iteration is usually enough. However, when the growth of the constant is important, the following lemma is needed to make effective use of the triangle inequality.

Lemma 2.3. *Suppose that $x_0, x_1, \dots, x_{2^k} \in \mathbb{R}^n$ for some $k > 1$. Then*

$$d_\lambda(x_0, x_{2^k}) \le {c_\Delta}^k \sum_{i=1}^{2^k} d_\lambda(x_{i-1}, x_i).$$

Proof. For $k = 2$ we have by the triangle inequality that

$$\begin{aligned} d_\lambda(x_0, x_4) &\le c_\Delta\big(d_\lambda(x_0, x_2) + d_\lambda(x_2, x_4)\big) \\ &\le c_\Delta\Big(c_\Delta\big(d_\lambda(x_0, x_1) + d_\lambda(x_1, x_2)\big) + c_\Delta\big(d_\lambda(x_2, x_3) + d_\lambda(x_3, x_4)\big)\Big). \end{aligned}$$

The general case follows by induction. □

2.3 Weak solutions

Let in the following $\lambda \in A_{1+2/n}(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $L^2_\lambda(\Omega)$ denote the Hilbert space of functions defined on Ω which are square integrable on Ω with respect to the measure $\lambda(x)dx$. Let $L^2_\lambda(\Omega)$ be equipped with the natural weighted L^2-norm $\|\cdot\|_{L^2_\lambda(\Omega)}$. Furthermore, let $W^{1,2}_\lambda(\Omega)$, be the space of equivalence classes of functions u with distributional gradient $\nabla u = (u_{x_1}, \dots, u_{x_n})$, both of which belong to $L^2_\lambda(\Omega)$. Let

$$\|u\|_{W^{1,2}_\lambda(\Omega)} = \|f\|_{L^2_\lambda(\Omega)} + \|\,|\nabla f|\,\|_{L^2_\lambda(\Omega)}$$

be the norm in $W^{1,2}_\lambda(\Omega)$. Let $C_0^\infty(\Omega)$ denote the set of infinitely differentiable functions with compact support in Ω and let $W^{1,2}_{\lambda,0}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{1,2}_\lambda(\Omega)}$. $W^{1,2}_{\lambda,loc}(\Omega)$ is defined in the standard way. Given $t_1 < t_2$, let $L^2(t_1, t_2, W^{1,2}_\lambda(\Omega))$ denote the space of functions such that for almost every t, $t_1 \le t \le t_2$, the function $x \to u(x,t)$ belongs to $W^{1,2}_\lambda(\Omega)$ and

$$\|u\|_{L^2(t_1,t_2,W^{1,2}_\lambda(\Omega))} := \Big(\int_{t_1}^{t_2}\int_\Omega \Big(|u(x,t)|^2 + |\nabla u(x,t)|^2\Big)\lambda(x)dxdt\Big)^{1/2} < \infty.$$

The space $L^2\left(t_1, t_2, W^{1,2}_{\lambda,loc}(\Omega)\right)$ is defined analogously. Let H be as in (1.3), assume (1.4) and (1.5). Let Ω be a bounded domain and $T > 0$. A function u is said to be a weak solution of $Hu = 0$ in Ω_T if, for all open sets $\Omega' \subseteq \Omega$ and $0 < t_1 < t_2 < T$, we have $u \in L^2(t_1, t_2, W^{1,2}_\lambda(\Omega'))$ and

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\Omega'} a_{ij}(x,t)\partial_{x_i}u\partial_{x_j}\theta dxdt - \int_{t_1}^{t_2}\int_{\Omega'} u\partial_t\theta dxdt \\ &+ \int_{\Omega'} u(x,t_2)\theta(x,t_2)dx - \int_{\Omega'} u(x,t_1)\theta(x,t_1)dx = 0 \end{aligned} \tag{2.10}$$

whenever $\theta \in C_0^\infty(\Omega'_T)$. Furthermore, u is said to be a *weak supersolution* to $Hu = 0$ if the left hand side of (2.10) is non-negative for all $\theta \in C_0^\infty(\Omega'_T)$ with $\theta \ge 0$. If instead the left hand side is non-positive u is said to be a *weak subsolution*. For the existence of weak solutions to $Hu = 0$ we refer to [9].

2.4 Fundamental principles

Lemma 2.4. *(Parabolic Cacciopoli type estimate) Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$, and let $\Omega_T = \Omega \times (0,T)$. Suppose that u is a bounded weak subsolution to $Hu = 0$ in Ω_T. Let $(x_0,t_0) \in \mathbb{R}^{n+1}$, $r_1 > 0$, $r_2 > 0$. There exists a constant $\eta = \eta(H)$, $1 \leq \gamma < \infty$, such that the following holds. Assume that $C^{\lambda,-}_{r_1,r_2}(x_0,t_0) \subset \Omega_T$, and that ϕ is a smooth function defined in $C^{\lambda,-}_{r_1,r_2}(x_0,t_0)$ satisfying $\phi(x,t) = 0$ whenever $x \in \mathbb{R}^n \setminus \overline{B_\lambda(x_0,r_1)}$. Then*

$$
\begin{aligned}
&\sup_{t_0 - r_2^2 < t < t_0} \int_{B_\lambda(x_0,r_1)} u^2\,\phi^2(x,t)\,dx + \eta^{-1} \iint_{C^{\lambda,-}_{r_1,r_2}(x_0,t_0)} |\nabla u|^2 \phi^2\,\lambda(x) dx d\tau \\
\leq\ & \int_{B_\lambda(x_0,r_1)} u^2\,\phi^2(x,t_0 - r_2^2)\,dx + \eta \iint_{C^{\lambda,-}_{r_1,r_2}(x_0,t_0)} u^2\,|\nabla\phi|^2\,\lambda(x) dx d\tau + \eta \iint_{C^{\lambda,-}_{r_1,r_2}(x_0,t_0)} u^2\,\phi\,\phi_t\,dx d\tau.
\end{aligned}
$$

Proof. The proof follows by standard manipulations by formally taking $\theta := u\phi^2$ as the test function in the weak formulation of subsolutions. □

An important tool is the interior Harnack inequality for positive solutions to $Hu = 0$ where H satisfies (1.4). Harnack inequalities have been established for operators satisfying (1.4), for various assumptions on λ, see [8], [9], [10], [29], [28]. As shown in [9], if H is degenerate, the Harnack inequality does not hold in standard Euclidean cylinders with constants independent of the cylinder.

Lemma 2.5. *(Harnack inequality) Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $(x_0,t_0) \in \mathbb{R}^{n+1}$, $r > 0$ and $\gamma > 0$. Let $R = R_{x_0}(r)$. Suppose that u is a bounded weak solution to $Hu = 0$ in $B(x_0,2R) \times (t_0 - \gamma r^2, t_0 + \gamma r^2)$. Then there is a constant $c = c(H,\gamma)$ such that*

$$
\sup_{B_\lambda(x_0,r) \times (t_0 - \frac{3}{4}\gamma r^2, t_0 - \frac{1}{4}\gamma r^2)} u \leq c \inf_{B_\lambda(x_0,r) \times (t_0 + \frac{1}{4}\gamma r^2, t_0 + \gamma r^2)} u. \tag{2.11}
$$

Proof. For $\gamma = 1$ the lemma is just a reformulation of Theorem 1.1 in [8]. For $\gamma > 0$ the lemma follows from either a modification of the proof in [8], or can be derived directly from Theorem A in [28]. □

Lemma 2.6. *(Interior Hölder continuity) Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $(x_0,t_0) \in \mathbb{R}^{n+1}$ and $r > 0$. Let u be a solution to $Hu = 0$ in $C^\lambda_{2r}(x_0,t_0)$. Then, after a redefinition on a set of measure zero, u is continuous on $C^\lambda_{2r}(x_0,t_0)$. Furthermore, there exist constants $c = c(H)$, $1 \leq c < \infty$, $\alpha = \alpha(H)$, $0 < \alpha < 1$, such that*

$$
|u(x,t) - u(y,s)| \leq c \left(\frac{d_{p,\lambda}(x,t,y,s)}{r} \right)^\alpha \sup_{C^\lambda_{2r}(x_0,t_0)} |u|
$$

whenever (x,t), $(y,s) \in C^\lambda_r(x_0,t_0)$.

Proof. The lemma follows from Lemma 2.5 by a standard iteration argument. □

Lemma 2.7. *(Weak maximum principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$, $\Omega_T = \Omega \times (0, T)$. Let u and v be a (weak) supersolution and a subsolution to $Hu = 0$ Ω_T, respectively. Assume that* $\min\{u - v, 0\} \in L^2(0, T, W^{1,2}_{\lambda,0}(\Omega))$. *Then $u \geq v$ a.e. in Ω_T.*

Proof. This follows by standard arguments. See for example [13, pp.160-161], for a similar situation. □

3 Geometry and Harnack chains

The Harnack inequality stated in Lemma 2.5 holds in cylinders given by the weighted distance function introduced in Section 2. We need to be able to compare the values of a non-negative solution u to $Hu = 0$ in Ω_T, where $\Omega \subset \mathbb{R}^n$ is a domain and $T > 0$, by repeatedly applying the Harnack inequality. To do this in a controlled fashion, some restrictions on the domain Ω is needed, especially when considering points close to the boundary. In this section we formulate and analyse such restrictions using the notion of non-tangentially accessible domains with respect to the weighted distance.

3.1 Notion of λ-NTA-domains

In what follows, c_Δ denotes the constant in the triangle inequality, Lemma 2.2.

Definition 3.1. *Let $\lambda \in A_{1+2/n}(\mathbb{R}^n)$. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and $M \geq 2c_\Delta$, we say that $B_\lambda(x, r) \subset \Omega$ is M-non-tangential (with respect to d_λ) if*

$$M^{-1}r < d_\lambda\left(B_\lambda(x, r), \partial\Omega\right) < Mr.$$

Given $x, x' \in \Omega$ a sequence of M-non-tangential balls in Ω, $B_\lambda(x_1, r_1)$,..., $B_\lambda(x_N, r_N)$, is called a M-Harnack chain of length N joining x and x', if $x \in B_\lambda(x_1, r_1)$, $x' \in B_\lambda(x_N, r_N)$, and $B_\lambda(x_i, r_i) \cap B_\lambda(x_{i+1}, r_{i+1}) \neq \varnothing$ for $i \in \{1, ..., N - 1\}$.

Remark 3.1. *Note that since two consecutive balls in a Harnack chain are both M-non-tangential and have non-empty intersection, they have comparable radii.*

Definition 3.2. *Let $\lambda \in A_{1+2/n}(\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that Ω is a non-tangentially accessible domain in $\mathbb{R}^n$, with respect to d_λ, a λ-NTA-domain hereafter, if there exist $M \geq 2c_\Delta$, $r_0 > 0$ such that the following holds.*

- *(i) (Interior corkscrew condition) For any $x_0 \in \partial\Omega$ and $r \leq r_0$ there exists $A^\lambda_r(x_0) \in \Omega$ such that $r/M \leq d_\lambda(A^\lambda_r(x_0), x_0) \leq r$ and $d_\lambda(A^\lambda_r(x_0), \partial\Omega) \geq r/M$.*
- *(ii) (Exterior corkscrew condition) $\Omega^c = \mathbb{R}^n \setminus \Omega$ satisfies property (i).*
- *(iii) (Harnack chain condition) Whenever $\epsilon > 0$ and $x_1, x_2 \in \Omega$ are such that $d_\lambda(x_i, \partial\Omega) > \epsilon$, $i \in \{1, 2\}$, and $d_\lambda(x_1, x_2) \leq \eta\epsilon$, for some constant $\eta \geq 1$, then there exists an M-Harnack chain, of length $N = N(n, \Lambda, M, \eta)$, joining x_1 and x_2.*

Remark 3.2. *The constants M, r_0, will be called the λ-NTA parameters of Ω. When it is clear from the context, will suppress the λ-dependence and write $A_r(x_0) = A^\lambda_r(x_0)$.*

If λ is constant, d_λ is the standard Euclidean metric and $c_\Delta = 1$, then Definition 3.1 and Definition 3.2 coincide with the original definition of NTA-domains formulated in [32]. We refer this classical notion of NTA-domains as NTA-domains defined with respect to the Euclidean metric or simply NTA-domains. For a general weight λ we will always use the label λ-NTA-domains. However, the following two lemmas show that in our setting the two notions are essentially equivalent.

Lemma 3.3. *Suppose that $\lambda \in A_{1+2/n}(\mathbb{R}^n)$ and that $\Omega \subset \mathbb{R}^n$ is a bounded domain. For any $M \geq 2$, there exists $\tilde{M} = \tilde{M}(n, M, \Lambda)$, $\tilde{M} \geq 2c_\Delta$ such that if $B(x, R)$ is M-non-tangential (with respect to the Euclidean metric), then $B_\lambda(x, r_x(R))$ is $\tilde{M}$-non-tangential (with respect to d_λ). Similarly, for any $\tilde{M} \geq 2$, there exists $\bar{M} = \bar{M}(n, \tilde{M}, \Lambda)$, $\bar{M} \geq 2c_\Delta$, such that if $B_\lambda(x, r)$ is $\tilde{M}$-non-tangential (with respect to d_λ), then $B(x, R_x(r))$ is $\bar{M}$-non-tangential (with respect to the Euclidean metric).*

Proof. We only present the proof of the first implication since the second implication is proved similarly. Let $\hat{x} \in \partial B(x, R)$. By Lemma 2.1 we see that

$$d_\lambda(\hat{x}, \partial\Omega) \leq r_{\hat{x}}(MR) \leq cM^c r_{\hat{x}}(R),$$

and

$$d_\lambda(\hat{x}, \partial\Omega) \geq r_{\hat{x}}(M^{-1}R) \geq c^{-1}M^{-c} r_{\hat{x}}(R),$$

for some $c = c(n, \Lambda)$. Similarly, since $B(\hat{x}, R) \subset B(x, 2R)$ we have that

$$c^{-1} r_{\hat{x}}(R) \leq r_x(R) \leq c r_{\hat{x}}(R),$$

for $c = c(n, \Lambda)$. Letting $\tilde{M} = c^2 M^c$ the proof is complete. □

Lemma 3.4. *Suppose that $\lambda \in A_{1+2/n}(\mathbb{R}^n)$ and that $\Omega \subset \mathbb{R}^n$ is a bounded domain. For every $M \geq 2$ and $R_0 > 0$ there exist $\tilde{M}$ and $\tilde{r}_0$, with $\tilde{M} = \tilde{M}(n, \Lambda, M)$, $2c_\Delta \leq \tilde{M} < \infty$ and $\tilde{r}_0 = \tilde{r}_0(n, \Lambda, M, R_0, \mathrm{diam}(\Omega), \mathrm{diam}_\lambda(\Omega))$, $\tilde{r}_0 > 0$, such that if Ω is an NTA-domain, with respect to the Euclidean metric, with parameters M and R_0, then Ω is a λ-NTA-domain with parameters $\tilde{M}$ and $\tilde{r}_0$. Conversely, for every $\tilde{M} \geq 2c_\Delta$ and $\tilde{r}_0 > 0$ there exist $\bar{M} = \bar{M}(n, \Lambda, \tilde{M})$, $2 \leq \bar{M} < \infty$, and $\bar{R}_0 = \bar{R}_0(\Lambda, n, \tilde{M}, \tilde{r}_0, \mathrm{diam}(\Omega), \mathrm{diam}_\lambda(\Omega))$, $\bar{R}_0 > 0$, such that if Ω is a λ-NTA-domain, with parameters $\tilde{M}$ and $\tilde{r}_0$, then Ω is an NTA-domain, with respect to the Euclidean metric, with parameters $\bar{M}$ and $\bar{R}_0$.*

Proof. It is enough to prove the first implication, the second implication is proved similarly. Assume that Ω is an NTA-domain with parameters M and r_0. To establish the Harnack chain condition, let $x, x' \in \Omega$, $\epsilon > 1$ and $\eta \geq 1$ be such that

$$d_\lambda(x, \partial\Omega), d_\lambda(x', \partial\Omega) > \epsilon, \tag{3.1}$$

and

$$d_\lambda(x, x') \leq \eta\epsilon. \tag{3.2}$$

Let $\hat{R} = \max\{R_x(\epsilon), R_{x'}(\epsilon)\}$, so that $d(x, \partial\Omega), d(x', \partial\Omega) > \hat{R}$. Then, using Lemma 2.1, we see that

$$|x - x'| = R_x\big(d_\lambda(x, x')\big) \leq R_x(\eta\epsilon) \leq c\eta^c R_x(\epsilon) \leq c\eta^c \hat{R}.$$

Since Ω is an NTA-domain, x and x' can be joined by an Euclidean M-Harnack chain of length $N(n, M, c\eta^c)$. By the proof of Lemma 3.3, this chain is $\tilde{M}_1$-non-tangential with respect to λ for $\tilde{M}_1 = c^2 M^c$ for some $c = c(\Lambda, n)$. Noting that $N = N(n, M, c\eta^c) = N(n, \Lambda, \tilde{M}_1, \eta)$, the proof of the Harnack chain condition is complete. To verify the corkscrew conditions, an appropriate scale parameter r_0 has to be found. Let $y, y' \in \partial\Omega$ be such that $\mathrm{diam}(\Omega) = |y - y'|$. By the triangle inequality $\mathrm{diam}_\lambda(\Omega) \leq c_\Delta r_y(\mathrm{diam}(\Omega))$. Since $B(x_0, R_0) \subset B(x_0, \mathrm{diam}(\Omega) + R_0)$ for all $x_0 \in \partial\Omega$ it follows, using Lemma 2.1, that

$$\begin{aligned} r_{x_0}(R_0) &\geq c^{-1}\left(\frac{R_0}{\mathrm{diam}(\Omega) + R_0}\right)^c r_y\left(\mathrm{diam}(\Omega) + R_0\right) \\ &\geq c^{-2}\left(\frac{R_0}{\mathrm{diam}(\Omega) + R_0}\right)^c \left(\frac{\mathrm{diam}(\Omega) + R_0}{\mathrm{diam}(\Omega)}\right)^{1/c} r_y(\mathrm{diam}(\Omega)) \\ &\geq c^{-2}\left(\frac{R_0}{\mathrm{diam}(\Omega) + R_0}\right)^c \left(\frac{\mathrm{diam}(\Omega) + R_0}{\mathrm{diam}(\Omega)}\right)^{1/c} {c_\Delta}^{-1}\,\mathrm{diam}_\lambda(\Omega), \end{aligned}$$

whenever $x_0 \in \partial\Omega$. Let

$$\tilde{r}_0 := c^{-2}\left(\frac{R_0}{\mathrm{diam}(\Omega) + R_0}\right)^c \left(\frac{\mathrm{diam}(\Omega) + R_0}{\mathrm{diam}(\Omega)}\right)^{1/c} \mathrm{diam}_\lambda(\Omega).$$

Consider $x_0 \in \partial\Omega$ and $r \leq \tilde{r}_0$. Then $R_{x_0}({c_\Delta}^{-1} r) \leq R_0$ and hence, since Ω is an NTA-domain, there is a point $A_{R_{x_0}({c_\Delta}^{-1} r)}(x_0)$ satisfying the interior corkscrew condition with respect to the Euclidean distance. Let $A_r^\lambda(x_0) := A_{R_{x_0}({c_\Delta}^{-1} r)}(x_0)$, then by definition

$$d_\lambda\big(A_r^\lambda(x_0), x_0\big) \leq c_\Delta r_{x_0}\left(\big|A_r^\lambda(x_0) - x_0\big|\right) \leq c_\Delta r_{x_0}(R_{x_0}({c_\Delta}^{-1} r)) = r,$$

and, by Lemma 2.1,

$$d_\lambda\big(A_r^\lambda(x_0), \partial\Omega\big) \geq {c_\Delta}^{-1} r_{x_0}\big(d(A_R, \partial\Omega)\big) \geq r_{x_0}\big(M^{-1} R_{x_0}({c_\Delta}^{-1} r)\big) \geq {c_\Delta}^{-2} c^{-1} M^{-c} r =: \tilde{M}_2 r.$$

The exterior corkscrew condition is verified in the same way. Setting $\tilde{M} = \max\{\tilde{M}_1, \tilde{M}_2\}$ completes the proof. □

Remark 3.4. *By Lemma 3.4 we see that Ω is an NTA-domain if and only if it is a λ-NTA-domain. However, the construction of $\tilde{M}$ and $\tilde{r}_0$ in the proof of Lemma 3.4 is not reversible. In other words, the mappings $\tilde{M} \mapsto \bar{M}$ and $\tilde{r}_0 \mapsto \bar{R}_0$ are not the inverses of $M \mapsto \tilde{M}$ and $R_0 \mapsto \tilde{r}_0$.*

3.2 The Harnack inequality in λ-NTA-cylinders

Lemma 3.5. *Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega \subset \mathbb{R}^n$ be a bounded λ-NTA-domain with parameters M and r_0, let $T > 0$, and let u be a non-negative solution to $Hu = 0$*

in Ω_T. Assume that $(x,t) \in \Omega_T$ and $(x',t') \in \Omega_T$ are such that $t' > t$, $(t'-t)^{1/2} \geq \sigma^{-1} d_\lambda(x,x')$ for some $\sigma > 1$, and that $d_\lambda(x,d\Omega) > \epsilon$, $d_\lambda(x',d\Omega) > \epsilon$, $t > \epsilon^2$, $d_{p,\lambda}(x,t,x',t') < \eta\epsilon$, for $\epsilon > 0$ and $\eta \geq 1$. Then there exists $c = c(H,M,\sigma,\eta)$, $1 \leq c < \infty$, such that

$$u(x,t) \leq cu(x',t').$$

Proof. Without loss of generality u can be assumed to be a solution to $Hu = 0$ in $\Omega \times (0,\infty)$. Consider first the case when $x = x'$. Let $r = r_x(R_x(\epsilon)/2)$. By Lemma 2.1 there is some c_0 such that $r < c_0^{-1}\epsilon$. Let k be the largest integer such that $k < \eta^2 c_0^2$. Let $t_1 = t + \frac{1}{2}r^2$ and for $1 \leq i < k$ let $t_{i+1} = t_i + r^2$ so that $t_k < t + kc_0^{-2}\epsilon^2 < t'$. Then $C^\lambda_{\epsilon,r}(x,t_i) \subset \Omega_T$ and repeated application of the Harnack inequality gives $u(x,t) \leq c^k u(x,t_k)$. Finally, letting $r' := \sqrt{t'-t_k} < c_0^{-1}\epsilon$, applying the Harnack inequality to the cylinder $B(x,2R_x(r')) \times (t' - \frac{3}{2}r'^2, t' + \frac{1}{2}r'^2)$, gives $u(x,t_k) \leq c_0 u(x,t')$. Now, assume that $x \neq x'$. Since Ω is λ-NTA there exists an M-Harnack chain $\{B_\lambda(x_i,r_i)\}$ of length $N = N(n,\Lambda,M,\eta)$ joining x and x' . As noted in Remark 3.1 there is a constant $c_1 = c_1(n,\Lambda,M)$ such that $r_i < c_1 r_{i+1}$ and hence $r_i < c_1^N r_N$ for every $1 \leq i \leq N$. Without loss of generality it can be assumed that $d_\lambda(x',\partial\Omega) = \epsilon$ and hence that $r_i \leq c_1^N M\epsilon$. Note that if $B(x_i, 2R_{x_i}(r_i)) \subset \Omega$ for each i, the lemma would follow by applying the Harnack inequality to a sequence of cylinders with the balls from the Harnack chain as bases. This is not the case in general, but the original Harnack chain can be refined in such a way that this is true. By Lemma 3.3 the chain $\{B_\lambda(x_i,r_i)\}$ is also an Euclidean $\overline{M}$-Harnack chain for some $\overline{M}$. Without loss of generality, assume that $\overline{M}$ is an integer. Refine the original Harnack chain by replacing each original ball $B_\lambda(x_i,r_i)$ with $\overline{M}$ balls of Euclidean radii $R_{x_i}(r_i)/\overline{M}$ to get a new (not necessarily non-tangential) chain $\{B_\lambda(\tilde{x}_j,\tilde{r}_j)\}$, of length $\tilde{N} = \overline{M}N$, joining x and x'. Then by construction $B(\tilde{x}_i, 2R_{x_i}(\tilde{r}_i)) \subset \Omega$. Also note that if $B_\lambda(\tilde{x}_j,\tilde{r}_j) \subset B_\lambda(x_i,r_i)$ then $\tilde{r}_j < r_i$. By Lemma 2.1 and Lemma 2.2 there is a constant $c_2 = c_2(n,\Lambda,M)$ such that if $d_\lambda(x,x') < c_2 r$ and x' lies in some M-non-tangential ball $B_\lambda(x_0,r)$ then $d_\lambda\big(B_\lambda\big(x', d_\lambda(x',x)\big),\partial\Omega\big) > M^{-1}d_\lambda(x',x)$. Hence in this case the original Harnack chain could be replaced with just the ball $B_\lambda\big(x', d_\lambda(x',x)\big)$. Let $c_3 := \max\{c_1,c_2,\sigma,M\}$ then, by the argument above, we may assume that $d_\lambda(x,x') \geq c_3 r_N$. Also note that by construction $\tilde{r}_j \leq c_3^N r_N \leq c_3^{N+1}\epsilon$. Let $\gamma > 0$ be a number to be chosen later, let $s_1 = t + \frac{1}{2}\gamma r_1^2$ and for $1 < i \leq \tilde{N}$ let

$$s_i := t + \frac{1}{2}\gamma \sum_{\ell=2}^{i} \left(\tilde{r}_\ell^2 + \tilde{r}_{\ell-1}^2\right).$$

Now let γ be so small that

$$s_i - \gamma\tilde{r}_i^2 \geq t - \gamma c^{2N+2}\epsilon^2 > \left(1 - \gamma c^{2N+2}\right)\epsilon^2 > 0$$

and

$$\begin{aligned} t' - s_{\tilde{N}} &= t' - t - \frac{1}{2}\gamma \sum_{i=2}^{\tilde{N}} \left(\tilde{r}_i^2 + \tilde{r}_{i-1}^2\right) \geq \frac{d_\lambda(x,x')^2}{\sigma^2} - \gamma \sum_{i=1}^{\tilde{N}} \tilde{r}_i^2 \\ &\geq \left(1 - \gamma\tilde{N}c^{2N}\right) r_N^2 \geq \left(1 - \gamma\tilde{N}c^{2N}\right)\tilde{r}_{\tilde{N}}^2 \geq \frac{3}{4}\tilde{r}_{\tilde{N}}^2. \end{aligned}$$

Note that this choice of γ does not depend on ϵ. By repeated use of Lemma 2.5 we have that $u(x,t) \leq c^{\tilde{N}} u(x', s_{\tilde{N}} + \frac{3}{4}\tilde{r}_{\tilde{N}}^2)$. The lemma now follows from the case $x = x'$. □

The proof of Lemma 3.5 does not use that u is a solution to $Hu = 0$ in the whole of Ω_T. In particular, Lemma 3.5 can be refined as follows.

Lemma 3.6. *There is a $K = K(H, M)$ such that the following is true. Let H be as in* (1.3), *assume* (1.4) *and* (1.5). *Let $\Omega \subset \mathbb{R}^n$ be a bounded λ-NTA-domain with parameters M, r_0, and let $T > 0$. Furthermore, let $(x_0, t_0) \in S_T$ and $r < \sqrt{t_0/4}$. Assume that u is a non-negative solution of $Hu = 0$ in $C^\lambda_{Kr,2r}(x_0, t_0) \cap \Omega_T$. Assume that $(x, t) \in C^\lambda_r(x_0, t_0) \cap \Omega_T$ and $(x', t') \in C^\lambda_r(x_0, t_0) \cap \Omega_T$ are such that $t' > t$, $(t' - t)^{1/2} \geq \sigma^{-1} r$, for some $\sigma > 1$, and that $d_\lambda(x, d\Omega) > \eta^{-1} r$, $d_\lambda(x', d\Omega) > \eta^{-1} r$, for some $\eta \geq 1$. Then there exists $c = c(H, M, \sigma, \eta)$, $1 \leq c < \infty$, such that*

$$u(x, t) \leq cu(x', t')$$

Proof. Note that $d_\lambda(x, x') \leq 2c_\Delta r \leq Mr$ so x and x' can be connected using a Harnack chain $\{B_\lambda(x_i, r_i)\}$ of length $N = N(M)$. By the triangle inequality there is a $K = K(n, \Lambda, M)$ such that $\bigcup_{i=1}^N \bigcup_{y \in B_\lambda(x_i, r_i)} B_\lambda(y, M^{-1} r_i) \subset B_\lambda(x_0, Kr)$. The conclusion now follows from the proof of Lemma 3.5. □

4 Approximation Results and the Dirichlet Problem

The purpose of this section is to solve the continuous Dirichlet problem for H, where H is as in (1.3) assuming (1.4) and (1.5). Throughout the section, $\Omega \subset \mathbb{R}^n$ is a bounded λ-NTA-domain with parameters M, r_0, and $\Omega_T = \Omega \times (0, T)$ for some $T > 0$ and λ. An important tool to solve the Dirichlet problem is the following lemma proved in [24].

Lemma 4.1. *Consider p, $1 < p < \infty$, fixed and let $\lambda \in A_p(\mathbb{R}^n)$ with $A_p(\mathbb{R}^n)$-constant bounded by $\Lambda_{p,\lambda}$. Assume that $\{a_{ij}(x, t)\} = \{a_{ij}\}$ is measurable, real, symmetric, for every $(x, t) \in \mathbb{R}^{n+1}$, and that*

$$\beta^{-1} \lambda(x) |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \leq \beta \lambda(x) |\xi|^2 \tag{4.1}$$

for all $\xi \in \mathbb{R}^n$ and for almost all $(x, t) \in \Omega \times [0, T]$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then there exist $\tilde{\lambda}_1, \tilde{\lambda}_2 \in A_p(\mathbb{R}^n)$, with $\Lambda_{p,\tilde{\lambda}_1}$ and $\Lambda_{p,\tilde{\lambda}_2}$ depending only on $\Lambda_{p,\lambda}$, such that the following is true for all $\ell \in \mathbb{N}$. Given $\ell \in \mathbb{N}$ there exists a measurable function λ^ℓ and a matrix $\{\tilde{a}^\ell_{ij}(x, t)\} = \{\tilde{a}^\ell_{ij}\}$ which is measurable, real, symmetric, for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, such that the following holds.

- *(i)* $\tilde{\lambda}_1 \leq \lambda^\ell \leq \tilde{\lambda}_2$,
- *(ii)* $\ell^{-1} c_1 \leq \lambda^\ell \leq c_2 \ell$ *in* Ω, *where* $c_1 = c_1(n, \Lambda_{p,\lambda}, \Omega)$ *and* $c_2 = c_2(n, \Lambda_{p,\lambda}, \Omega)$,
- *(iii)* $\lambda^\ell \in A_p(\mathbb{R}^n)$ *with* $\Lambda_{p,\lambda^\ell} = \Lambda_{p,\lambda^\ell}(\Lambda_{p,\lambda})$,
- *(iv)* *There exists a closed set F^ℓ such that $\tilde{a}^\ell_{ij} = a_{ij}$, $\lambda^\ell = \lambda$,*
  and $c^{-1} \tilde{\lambda}_2 \leq \lambda \leq c \tilde{\lambda}_1$ in F^ℓ where $c = c(\ell)$,
- *(v)* *The set F^ℓ is increasing in ℓ and the complement of $\bigcup_{\ell=1}^\infty F^\ell$*
  has Lebesgue measure zero,
- *(vi)* $\lambda^\ell \to \lambda$ *almost everywhere in* $\mathbb{R}^n$.

Furthermore,

$$\beta^{-1}\lambda^\ell(x)|\xi|^2 \le \sum_{i,j=1}^n \tilde a^\ell_{ij}(x,t)\xi_i\xi_j \le \beta\lambda^\ell(x)|\xi|^2 \tag{4.2}$$

for all $\xi\in\mathbb{R}^n$ and almost all $(x,t)\in\Omega\times[0,T]$.

Proof. This is Lemma 2.1 from [24]. □

We will also need the following lemma concerning weak solutions with zero boundary values.

Lemma 4.2. *Let $(x_0,t_0)\in\partial_p\Omega_T$ and $r<r_0$. Let u be a solution of $Hu=0$ in $\Omega_T\cap C^\lambda_r(x_0,t_0)$ vanishing continuously on $\partial_p\Omega_T\cap C^\lambda_r(x_0,t_0)$. Then there exist $c=c(H,M)$, $1\le c<\infty$, and $\gamma_1=\gamma_1(H,M)$, $\gamma_1\in(0,1)$, such that*

$$u(x,t)\le c\left(\frac{d_{p,\lambda}(x,t,x_0,t_0)}{r}\right)^{\gamma_1}\sup_{\Omega_T\cap C^\lambda_r(x_0,t_0)}|u|$$

for all $(x,t)\in\Omega_T\cap C^\lambda_{r/c}(x_0,t_0)$.

Proof. To prove the lemma one has to consider three cases depending on the location of $(x_0,t_0)\in\partial_p\Omega_T$. However, using versions of Lemma 2.4 applied to $(u-k)_\pm$, for appropriate choices of k, weighted Sobolev inequalities, properties of the weight λ, and the uniform (in measure) outer density condition satisfied at each point $(x_0,t_0)\in\partial_p\Omega_T$, Lemma 4.2 can be proved by a fairly standard iterative argument. See for example [13]. Further details are omitted. □

Lemma 4.3. *Let H be as in (1.3), assume (1.4) and (1.5). Then for each $f\in C(\partial_p\Omega_T)$ there exists a unique weak solution $u\in C(\overline{\Omega}_T)$ to the problem*

$$Hu=0 \text{ in } \Omega_T,\quad u=f \text{ on } \partial_p\Omega_T. \tag{4.3}$$

Proof. Given $\{a_{ij}(x,t)\}=\{a_{ij}\}$ and $\ell\in\mathbb{N}$, let $\tilde a^\ell_{ij}(x,t)$ be as in Lemma 4.1 and let H_ℓ be the operator corresponding to $\tilde a^\ell_{ij}(x,t)$. Then by the corresponding result for uniformly parabolic equations, see [14], [26], there exists a unique weak solution $u_\ell\in C(\overline{\Omega}_T)$ to the problem

$$H_\ell u_\ell=0 \text{ in } \Omega_T,\quad u_\ell=f \text{ on } \partial_p\Omega_T. \tag{4.4}$$

Furthermore, it follows from Lemma 2.4 that $\{u_\ell\}_{\ell=1}^\infty$ is bounded in $L^2\big(t_1,t_2,W^{1,2}_{\lambda,loc}\big)$ whenever $0<t_1<t_2<T$, and hence that there is a subsequence $\{w_\ell\}_{\ell=1}^\infty$ of $\{u_\ell\}_{\ell=1}^\infty$ such that $\{w_\ell\}$ converges weakly in $L^2\big(t_1,t_2,W^{1,2}_{\lambda,loc}(\Omega)\big)$ to some function $u\in L^2\big(t_1,t_2,W^{1,2}_{\lambda,loc}(\Omega)\big)$. Furthermore it follows from Lemma 2.6, the Arzelà-Ascoli theorem and a diagonal argument that there is another subsequence of $\{u_\ell\}_{\ell=1}^\infty$, say $\{v_\ell\}_{\ell=1}^\infty$, such that $v_\ell\to u$ locally uniformly, and hence u is continuous on Ω_T. To see that u is indeed a solution to $Hu=0$ in Ω_T, let $\Omega'\subset\Omega$ be open and let $0<t_1<t_2<T$. Since u_ℓ is a solution to $H_\ell u_\ell=0$ in Ω_T it follows from (iv) of Lemma 4.1 that

$$\int_{t_1}^{t_2}\int_{\Omega'} a_{ij}(x,t)\partial_{x_i}u_\ell\partial_{x_j}\theta\,dxdt-\int_{t_1}^{t_2}\int_{\Omega'}u_\ell\partial_t\theta\,dxdt+\int_{\Omega'}u_\ell(x,t_2)\theta(x,t_2)dx-\int_{\Omega'}u_\ell(x,t_1)\theta(x,t_1)dx$$
$$=\int_{t_1}^{t_2}\int_{\Omega'\setminus F^\ell}a_{ij}(x,t)\partial_{x_i}u_\ell\partial_{x_j}\theta\,dxdt-\int_{t_1}^{t_2}\int_{\Omega'\setminus F^\ell}\tilde a^\ell_{ij}(x,t)\partial_{x_i}u_\ell\partial_{x_j}\theta\,dxdt,$$

for every $\theta \in C_0^\infty(\Omega'_T)$. Using this relation we can now use (1.4), Lemma 2.4, the maximum principle and the fact that the Lebesgue measure of $\Omega' \setminus F^\ell$ tends to zero, as $\ell \to \infty$, to conclude that u is a weak solution to $Hu = 0$ in Ω_T. Finally, to conclude that u is a solution to the continuous Dirichlet problem in (4.3) it remains to prove that u is continuous up to $\partial_p \Omega_T$. To do this, consider $(x_0, t_0) \in \partial_p \Omega_T$ and let $\epsilon > 0$. Now choose $\delta > 0$ so small that

$$|f(x,t) - f(x_0,t_0)| \leq \epsilon \text{ whenever } (x,t) \in \partial_p \Omega_T \cap C_\delta^\lambda(x_0,t_0). \tag{4.5}$$

Let ϕ be a test function satisfying $0 \leq \phi \leq 1$, with support in $C_\delta^\lambda(x_0,t_0)$, such that $\phi \equiv 1$ on $C_{\delta/2}^\lambda(x_0,t_0)$. Let $\hat{f}(x,t) = \phi(x,t)\big(f(x,t) - f(x_0,t_0)\big)$ and $\tilde{f}(x,t) = \big(1 - \phi(x,t)\big)\big(f(x,t) - f(x_0,t_0)\big)$. Now let $\hat{w}_\ell$ and $\tilde{w}_\ell$ be the unique solutions to the problem in (4.4) with f replaced by $\hat{f}$ and $\tilde{f}$ respectively. It then follows from the maximum principle that $w_\ell(x,t) - f(x_0,t_0) = \hat{w}_\ell(x,t) + \tilde{w}_\ell(x,t)$ whenever (x,t) is in the closure of Ω_T and that

$$||\hat{w}_\ell||_{L^\infty(\Omega_T)} \leq \epsilon. \tag{4.6}$$

Arguing as above we conclude that $u(x,t) - f(x_0,t_0) = \hat{u}(x,t) + \tilde{u}(x,t)$ on Ω_T where $\hat{u}$, $\tilde{u}$, are the uniform limits on compact subsets of appropriate subsequences of $\{\hat{w}_\ell\}_{\ell=1}^\infty$ and $\{\tilde{w}_\ell\}_{\ell=1}^\infty$, respectively. Using (4.6) and the maximum principle, we see that

$$||\hat{u}||_{L^\infty(\Omega_T)} \leq \epsilon. \tag{4.7}$$

By Lemma 4.2, each element in the sequence $\{\tilde{w}_l\}_{\ell=1}^\infty$ is Hölder continuous up to the parabolic boundary with constants independent of ℓ. In particular this means that

$$\lim_{(x,t)\to(x_0,t_0),\ (x,t)\in\Omega_T} \tilde{u}(x,t) = 0. \tag{4.8}$$

Combining (4.7) and (4.8), it follows that

$$\limsup_{(x,t)\to(x_0,t_0),\ (x,t)\in\Omega_T} |u(x,t) - f(x_0,t_0)| \leq \epsilon.$$

Since $\epsilon > 0$ and $(x_0,t_0) \in \partial_p \Omega_T$ are arbitrary it follows that u is a solution to the continuous Dirichlet problem in (4.3). Uniqueness follows from the maximum principle. □

It follows from the proof of Lemma 4.3 that the solution u to the problem in (4.3) is given as

$$u(x,t) = \lim_{\ell\to\infty} u_\ell(x,t) \tag{4.9}$$

where u_ℓ solves (4.4) and where H_ℓ is an operator approximating H in the sense of Lemma 4.1. It also follows from the maximum principle and the Riesz representation theorem that there exists a probability measure $\omega(x,t,y,s)$ on $\partial_p \Omega_T$ such that

$$u(x,t) = \int_{\partial_p \Omega_T} f(y,s) d\omega(x,t,y,s), \tag{4.10}$$

for each $f \in C(\partial_p \Omega_T)$. The measure ω is called the parabolic measure associated to the operator H. In the same way one defines the probability measures $\omega_\ell(x,t,y,s)$ associated to the operators H_ℓ. In fact, $\omega_\ell \to \omega$ in the weak-* topology of Radon measures as we here note the following lemma.

Lemma 4.4. *Suppose that ω and ω_ℓ are as above. Then*

$$\omega_\ell \to \omega \quad \textit{in the weak-* topology of Radon measures.}$$

Proof. It follows directly from (4.9) that

$$\int_{\partial_p\Omega_T} f(y,s)d\omega(x,t,y,s) = \lim_{\ell\to\infty}\int_{\partial_p\Omega_T} f(y,s)d\omega_\ell(x,t,y,s), \quad \forall f \in C(\partial_p\Omega_T), \tag{4.11}$$

or equivalently that ω_ℓ converges to ω in the weak-* topology. □

5 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. Throughout this section, let H be as in (1.3) and assume (1.4) and (1.5). Let also $\Omega \subset \mathbb{R}^n$ be a bounded λ-NTA-domain with parameters M, r_0 and let $\Omega_T = \Omega \times (0,T)$ for some $T > 0$. Let

$$A_r^+(x_0,t_0) = (A_r(x_0), t_0 + 2r^2) \text{ and } A_r^-(x_0,t_0) = (A_r(x_0), t_0 - 2r^2).$$

Furthermore, if $x \in \Omega$ let $d_\lambda(x,\partial\Omega)$ denote the distance from x to $\partial\Omega$ measured by the weighted distance function d_λ.

Lemma 5.1. *There exists $K = K(H,M) \geq 1$ such that the following is true. Let $(x_0,t_0) \in S_T$ and $r < \min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Let u be a non-negative solution to $Hu = 0$ in $\Omega_T \cap C^\lambda_{Kr,2r}(x_0,t_0)$. Then there exist $c = c(H,M)$, $1 \leq c < \infty$, and $\gamma_2 = \gamma_2(H,M)$, $\gamma_2 > 0$, such that*

$$(i) \qquad u(x,t) \leq c\left(\frac{r}{d_\lambda(x,\partial\Omega)}\right)^{\gamma_2} u(A_r^+(x_0,t_0))$$

and

$$(ii) \qquad u(A_r^-(x_0,t_0)) \leq c\left(\frac{r}{d_\lambda(x,\partial\Omega)}\right)^{\gamma_2} u(x,t)$$

for all $(x,t) \in \Omega_T \cap C^\lambda_r(x_0,t_0)$.

Proof. We only prove (i) since (ii) can be proven analogously. Let $\hat K$ be as in Lemma 3.6 and set $K = M^4\hat K$. Let (x,t) be a fixed, arbitrary point in $\Omega_T \cap C^\lambda_r(x_0,t_0)$. If $d_\lambda(x,\partial\Omega) > r/(2M)$ the conclusion follows directly from Lemma 3.6 and thus we may assume that $d_\lambda(x,\partial\Omega) \leq r/(2M)$. Let k denote the largest integer such that

$$(2M)^k \leq \frac{r}{d_\lambda(x,\partial\Omega)}. \tag{5.1}$$

Let $\overline{x}_0 \in \partial\Omega$ denote any point such that $d_\lambda(x,\overline{x}_0) = d_\lambda(x,\partial\Omega)$. For each integer $1 \leq i \leq k$ let $r_i = (2M)^i d_\lambda(x,\partial\Omega)$, $\hat x_i = A^+_{r_i}(\overline{x}_0)$ and $\hat t_i = t + r_i^2/2$. By definition $\hat x_{i-1}, \hat x_i \in \Omega^{r_i/M^2} \cap$

$B_\lambda(\overline{x}_0, Mr_i)$ and $\hat{t}_i - \hat{t}_{i-1} > r_i^2/3$ for all $1 < i \leq k$. Furthermore, by the triangle inequality and the choice of K we have that

$$C^\lambda_{\hat{K}Mr_i,2Mr_i}(\overline{x}_0, t) \subset C^\lambda_{Kr,2r}(x_0, t_0)$$

for all $1 \leq i \leq k-1$. Thus u is a solution to $Hu = 0$ on $C^\lambda_{\hat{K}Mr_i,2Mr_i}(\overline{x}_0, t) \cap \Omega_T$ and applying Lemma 3.5 we see that

$$u(x,t) \leq cu(\hat{x}_1, \hat{t}_1) \tag{5.2}$$

and

$$u(\hat{x}_i, \hat{t}_i) \leq cu(\hat{x}_{i+1}, \hat{t}_{i+1}) \tag{5.3}$$

for all $1 \leq i \leq k-1$. It follows from (5.2), and (5.3), that

$$u(x,t) \leq c^k u(\hat{x}_k, \hat{t}_k).$$

Finally, since $\hat{x}_k, A_r(x_0) \in \Omega^{r/M^2} \cap B_\lambda(x_0, Mr)$ and $t_0 + 2r^2 - \hat{t}_k > r^2/2$, we can again apply Lemma 3.6 to conclude that

$$u(x,t) \leq c^k u(\hat{x}_k, \hat{t}_k) \leq c^{k+1} u(A_r(x_0), t_0 + 2r^2). \tag{5.4}$$

By combining (5.1) and (5.4) the proof of (i) is complete since (x,t) is an arbitrary point in $C^\lambda_r(x_0, t_0)$. □

Lemma 5.2. *There exists $K = K(H, M) \geq 1$ such that the following is true. Let $(x_0, t_0) \in S_T$ and $r < \min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Let u be a non-negative solution of $Hu = 0$ in $\Omega_T \cap C^\lambda_{Kr,2r}(x_0, t_0)$ vanishing continuously on $S_T \cap C^\lambda_r(x_0, t_0)$. Then there exists $c = c(H, M)$, $1 \leq c < \infty$, such that*

$$u(x,t) \leq cu(A_r^+(x_0, t_0)) \tag{5.5}$$

for all $(x,t) \in \Omega_T \cap C^\lambda_{r/c,r}(x_0, t_0)$.

Proof. The argument follows along the lines of [2] and [45]. Let K be the constant from Lemma 5.1 and let $c := \max(c_1, c_2, 2c_\Delta)$ where c_1 is the constant from Lemma 4.2, c_2 is the constant from Lemma 5.1, and c_Δ is the constant in the triangle inequality, see Lemma 2.2. Let γ_2 denote the exponent from Lemma 5.1 and γ_1 the exponent from Lemma 4.2. By rescaling, we may assume that $r = 1$ and $u(A_r^+(x_0, t_0)) = 1$. The proof is by contradiction and we assume that there exists $(\hat{y}_0, \hat{t}_0) \in \Omega_T \cap C^\lambda_{1/c}(x_0, t_0)$ such that

$$u(\hat{y}_0, \hat{t}_0) > c^{N+1} \tag{5.6}$$

for some N to be determined later. Denote $d_0 = d_\lambda(\hat{y}_0, \partial\Omega)$. Then, using Lemma 5.1 we see that

$$u(\hat{y}_0, \hat{t}_0) \leq cd_0^{-\gamma_2}. \tag{5.7}$$

Combining (5.6) and (5.7) we have that $d_0 \le c^{-N/\gamma_2}$. Let $\hat{x}_0 \in \partial\Omega$ be any point such that $d_\lambda(\hat{y}_0, \hat{x}_0) = d_0$ and let $\rho_0 = c^{-N/\gamma_2 + L/\gamma_1}$ for some $L > \gamma_1$ to be determined later. Then by Lemma 4.2

$$u(\hat{y}_0, \hat{t}_0) \le c\left(\frac{d_0}{\rho_0}\right)^{\gamma_1} \sup_{\Omega_T \cap C^\lambda_{\rho_1}(\hat{x}_0, \hat{t}_0)} u < c^{1-L} \sup_{\Omega_T \cap C^\lambda_{\rho_1}(\hat{x}_0, \hat{t}_0)} u.$$

Hence, there must exist $(\hat{y}_1, \hat{t}_1) \in \Omega_T \cap C^\lambda_{\rho_1}(\hat{x}_0, \hat{t}_0)$ such that

$$u(\hat{y}_0, \hat{t}_0) \le c^{1-L} u(\hat{x}_1, \hat{t}_1), \tag{5.8}$$

and by (5.6) and (5.8) it follows that $u(\hat{y}_1, \hat{t}_1) > c^{N+L}$. Note that if $N = N(L)$ is large enough, then $C^\lambda_{\rho_0}(\hat{x}_0, \hat{t}_0) \subset C^\lambda_1(x_0, t_0)$ so the argument above can be repeated for $(\hat{x}_1, \hat{t}_1)$. Indeed for any $k \ge 2$ there is some $N = N(k, L)$ such if we define $\rho_i = c^{(i(1-L)-N)/\gamma_2 + L/\gamma_1}$ and if $\hat{y}_i, \hat{x}_i$ and d_i are defined for $1 \le i \le k$ by the same procedure as above, then $u(\hat{y}_i, \hat{t}_i) \ge c^{i(L-1)+N+1}$ and $(\hat{y}_i, \hat{t}_i) \in C^\lambda_{\rho_{i-1}}(\hat{x}_{i-1}, \hat{t}_{i-1}) \subset C^\lambda_1(x_0, t_0)$, and hence $d_i \le c^{(i(1-L)-N)/\gamma_2}$. To find L and N, not depending on k, such that $C^\lambda_{\rho_k}(\hat{x}_k, \hat{t}_k) \subset C^\lambda_1(x_0, t_0)$ for all $k \ge 0$, note that it follows from the triangle inequality that

$$\begin{aligned} d(\hat{y}_0, \hat{y}_k) &\le \sum_{i=0}^{k-1} {c_\Delta}^{i+1}\big(d_\lambda(\hat{y}_i, \hat{x}_i) + d_\lambda(\hat{x}_i, \hat{y}_{i+1})\big) \\ &\le \sum_{i=0}^{k-1} c^{i+1}\left(d_i + \rho_i\right) \le c^{2+L/\gamma_1 - N/\gamma_2} \sum_{i=0}^{\infty} c^{i(1+(1-L)/\gamma_2)}, \end{aligned} \tag{5.9}$$

and likewise that

$$|\hat{t}_k - \hat{t}_0| \le \sum_{i=0}^{k} \rho_i^2 \le c^{2(L/\gamma_1 - N/\gamma_2)} \sum_{i=0}^{\infty} c^{2i(1-L)/\gamma_2}. \tag{5.10}$$

Now choose L so large that $c^{2(1-L)/\gamma_2} < c^{1+(1-L)/\gamma_2} < 1/2$ and choose N so large that

$$c^{2(L/\gamma_1 - N/\gamma_2)} < c^{2-N/\gamma_2 - L\gamma_1} < 1/2c.$$

Then, using the triangle inequality and (5.9), we see that

$$d(x_0, \hat{y}_k) \le c_\Delta(d(x_0, \hat{y}_0) + d(\hat{y}_0, \hat{y}_k)) \le c_\Delta\left(1/c + 1/c\right) \le 1,$$

and, using (5.10), that

$$|\hat{t}_k - t_0| < 1/c^2 + 1/c < 1.$$

Thus there exists a sequence of points $(\hat{y}_i, \hat{t}_i) \in C^\lambda_1(x_0, t_0)$ such that $d(\hat{y}_i, \partial\Omega) \to 0$ and $u(\hat{x}_i, \hat{t}_i) \to \infty$ as $i \to \infty$, contradicting that u vanishes continuously on $S_T \cap C^\lambda_1(x_0, t_0)$. □

Remark 5.2. *If u is a non-negative solution to $Hu = 0$ in all of Ω_T which, in addition to the assumptions of Lemma 5.2, vanishes continuously on $S_T \cap C_{2c_\Delta r, 2r}(x_0, t_0)$, then, by a covering argument and Lemma 3.5, the estimate (5.5) holds for all $(x, t) \in C^\lambda_r(x_0, t_0)$.*

Lemma 5.3. *Given $0<\delta<\sqrt{T/4}$, let $\Omega^\delta=\{x\in\Omega : d_\lambda(x,\partial\Omega)>\delta\}$. Let u be a non-negative solution to $Hu=0$ in Ω_T and assume that u vanishes continuously on S_T. Then there is a constant $c=c(H,M,\mathrm{diam}_\lambda(\Omega),T,\delta)$, $1\le c<\infty$ such that*

$$\sup_{\Omega^\delta\times(\delta^2,T)} u \le c \inf_{\Omega^\delta\times(\delta^2,T)} u$$

Proof. The proof of the corresponding lemma in [44] can easily be adapted to prove Lemma 5.3. We omit further details. □

Lemma 5.4. *There exists a $\widehat{K}\gg 1$, $\widehat{K}=\widehat{K}(H)$, such that the following is true whenever $(x_0,t_0)\in\mathbb{R}^{n+1}$, $r>0$, $K\ge\hat{K}$. Let u be a non-negative solution to $Hu=0$ in $C^{\lambda,-}_{Kr,r}(x_0,t_0)$ vanishing continuously on $B_\lambda(x_0,Kr)\times\{t_0-r^2\}$. Then, there exists a constant $c=c(H)$, $1\le c<\infty$, and $\nu=\nu(H)$, $\nu\in(0,1)$ such that*

$$\sup_{C^{\lambda,-}_{r}(x_0,t_0)} u \le ce^{-K^\nu/c} \sup_{C^{\lambda,-}_{Kr,r}(x_0,t_0)} u.$$

Proof. Let $\widehat{K}=(2c_\Delta c_1^{2/\gamma_1})$ where c_1 and γ_1 are as in Lemma 4.2. We may assume, without loss of generality, that $c_1\ge c_\Delta$. Given $K\ge\widehat{K}$ let k be the largest integer such that $2^k {c_\Delta}^k\widehat{K}\le K$. Let $\hat{r}_0=\widehat{K}r$ and let, for each integer $0\le j<2^k$,

$$\hat{r}_{j+1}=\sup\big\{d(x_0,y) : y\in B_\lambda(x,\widehat{K}r), x\in\partial B_\lambda(x_0,\hat{r}_j)\big\}$$

and let x_j be an arbitrary, fixed, point on $\partial B_\lambda(x_0,\hat{r}_j)$. Note that by construction $B_\lambda(x_j,\widehat{K}r)\subset B_\lambda(x_0,\hat{r}_{j+1})$, and by Lemma 2.3 we have that $\hat{r}_{2^k}\le 2^k {c_\Delta}^k\hat{K}r\le Kr$. Now let v be a solution to $Hv=0$ in $C^{\lambda,+}_{\widehat{K}r}(x_j,t_0-r^2)$ such that $v=u$ on $\partial_p C^{\lambda,+}_{\widehat{K}r,r}(x_j,t_0-r^2)$ and $v(x,t)=u(x,t_0)$ whenever $(x,t)\in\partial_p C^{\lambda,+}_{\widehat{K}r}(x_j,t_0-r^2)\setminus\partial_p C^{\lambda,+}_{\widehat{K}r,r}(x_j,t_0-r^2)$. Then, by the maximum principle and the construction of v,

$$\sup_{C^{\lambda,+}_{\widehat{K}r}(x_1,t_0-r^2)} v \le \sup_{C^{\lambda,+}_{\widehat{K}r,r}(x_1,t_0-r^2)} u \le \sup_{C^{\lambda,-}_{\hat{r}_{j+1},r}(x_0,t_0)} u \tag{5.11}$$

Furthermore, it follows from Lemma 4.2 and the construction of $\widehat{K}$ that

$$v(x,t)\le c_1\left(\frac{d_p(x,t,\partial_p\Omega_T)}{\hat{K}r}\right)^{\gamma_1}\sup_{C^{\lambda,+}_{\widehat{K}r}(x_1,t_0-r^2)} v \le c^{-1}\sup_{C^{\lambda,+}_{\widehat{K}r}(x_1,t_0-r^2)} v \tag{5.12}$$

whenever $(x,t)\in C^{\lambda,+}_r(x_j,t_0-r^2)$. Noting that $v(x,t)=u(x,t)$ for all $(x,t)\in C^{\lambda,+}_{\widehat{K}r,r}(x_j,t_0-r^2)$, and in particular for all $(x,t)\in C^{\lambda,+}_r(x_j,t_0-r^2)$, we have by combining (5.11) and (5.12) that

$$u(x,t)\le c^{-1}\sup_{C^{\lambda,-}_{\hat{r}_{j+1},r}(x_0,t_0)} u \tag{5.13}$$

for all $(x,t)\in C^{\lambda,+}_r(x_j,t_0-r^2)$. Since x_j was taken arbitrarily on $\partial_p B_\lambda(x_0,\hat{r}_j)$ we have that (5.13) holds for all $(x,t)\in\partial B_\lambda(x_0,\hat{r}_j)\times(t_0-r^2,t_0)$ and, remembering that $u=0$ on $B_\lambda(x_0,K)\times$

$\{t_0 - r^2\}$, we can use the maximum principle to conclude that (5.13) holds whenever $(x,t) \in C^{\lambda,-}_{\hat r_j, r}(x_0,t_0)$. Finally, iteration over $j = 1, 2, \ldots, 2^k$ yields

$$\sup_{C^{\lambda,-}_r(x_0,t_0)} u \le c^{-2^k} \sup_{C^{\lambda,-}_{Kr,r}(x_0,t_0)} u.$$

By the choice of k we also have that $2^{k+1}{c_\Delta}^{k+1}\widehat{K} > K$ so $2^k > K^\nu/c$ with $\nu = 1/(1+\log_2(c_\Delta))$ and $c = \widehat{K}^\nu/2c_\Delta$. Thus

$$\sup_{C^{\lambda,-}_r(x_0,t_0)} u \le c^{-K^\nu/c} \sup_{C^{\lambda,-}_{Kr,r}(x_0,t_0)} u$$

which completes the proof. □

Theorem 5.5. *Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega \subset \mathbb{R}^n$ be a bounded λ-NTA-domain with parameters M, r_0 and let $\Omega_T = \Omega \times (0,T)$ for some $T > 0$. Let u be a non-negative solution of $Hu = 0$ in Ω_T vanishing continuously on S_T. Let δ, $0 < \delta < r_0/2$, be a fixed constant, let $(x_0,t_0) \in S_T$, $\delta^2 \le t_0$, and assume that $r < \delta/2$. Then there exists $c = c(H, M, \mathrm{diam}_\lambda(\Omega), T, \delta)$, $1 \le c < \infty$, such that*

$$u(x,t) \le c u\big(A_r(x_0,t_0)\big)$$

whenever $(x,t) \in \Omega_T \cap C^\lambda_r(x_0,t_0)$.

Proof. To begin the proof let $\delta < r_0$ be a fixed constant, let $(x_0,t_0) \in S_T$, $\delta^2 \le t_0$, and assume that $r < \delta/2$. Let u be a solution to $Hu = 0$ on Ω_T vanishing continuously on S_T. Extend u to a solution on $\Omega \times (0,\infty)$ by defining u to vanish continuously on $\partial\Omega \times (0,\infty)$. In the following, let ρ be the largest number $r \le \rho \le \delta/2$ satisfying the inequality

$$\sup_{\Omega_T \cap C^{\lambda,-}_r(x_0,t_0)} u(x,t) \le (r/\rho)^{\gamma_2} \sup_{\Omega_T \cap C^{\lambda,-}_\rho(x_0,t_0)} u(x,t), \tag{5.14}$$

where γ_2 is the exponent appearing in Lemma 5.1. Using Lemma 5.1, and the definition of the point $A^-_r(x_0,t_0)$, one sees that

$$u\big(A^-_\rho(x_0,t_0)\big) \le c(\rho/r)^{\gamma_2} u\big(A^-_r(x_0,t_0)\big). \tag{5.15}$$

Assuming that

$$\sup_{\Omega_T \cap C^{\lambda,-}_\rho(x_0,t_0)} u(x,t) \le c u\big(A^-_\rho(x_0,t_0)\big), \tag{5.16}$$

it follows from (5.14), (5.16) and (5.15) that

$$\sup_{\Omega_T \cap C^{\lambda,-}_r(x_0,t_0)} u(x,t) \le c u\big(A^-_r(x_0,t_0)\big). \tag{5.17}$$

In particular, Theorem 5.5 then follows from (5.17) and Lemma 3.5. Hence it suffices to show the estimate (5.16). To this end, let $K > 1$ be a degree of freedom to be chosen, and divide the proof into two cases. First, assume that $\delta/(2K) < \rho$. In this case ρ is large and, combining Lemma 5.2 and Lemma 5.3, one sees that

$$\sup_{\Omega_T \cap C_\rho^{\lambda,-}(x_0,t_0)} u(x,t) \le cu\big(A_\rho^+(x_0,t_0)\big) \le cu\big(A_\rho^-(x_0,t_0)\big), \tag{5.18}$$

for some $c = c(H, M, \mathrm{diam}(\Omega), T, \delta, K)$, $1 \le c < \infty$. Hence the proof is complete in this case. Next, assume that $r \le \rho \le \delta/(2K)$ and note, by the definition of ρ, that

$$\sup_{\Omega_T \cap C_\rho^{\lambda,-}(x_0,t_0)} u \ge K^{-\gamma_2} \sup_{\Omega_T \cap C_{K\rho}^{\lambda,-}(x_0,t_0)} u. \tag{5.19}$$

Using (5.19) we intend to prove that show that there exists $K = K(H, M) \ge 1$, such that

$$\sup_{\Omega_T \cap \big(B_\lambda(x_0,K\rho)\times\{t_0-4\rho^2\}\big)} u \ge 2^{-1} \sup_{\Omega_T \cap C_\rho^{\lambda,-}(x_0,t_0)} u, \tag{5.20}$$

and from this, the estimate (5.16) follows from by applications of Lemma 5.2 and the Harnack inequality. Hence it only remains to prove the estimate in (5.20). To do this we argue by contradiction and we assume that

$$\sup_{\big(\Omega \cap B_\lambda(x_0,K\rho)\big)\times\{t_0-4\rho^2\}} u \le 2^{-1} \sup_{\Omega_T \cap C_\rho^{\lambda,-}(x_0,t_0)} u \tag{5.21}$$

for all $K > 1$. Note that one may also assume that

$$\sup_{\Omega_T \cap C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)} u > \sup_{\Omega_T \cap C_\rho^{\lambda,-}(x_0,t_0)} u, \tag{5.22}$$

since otherwise (5.20) is trivially true. Let $\phi \in C_0^\infty\big(B_\lambda(x_0, K\rho)\big)$ be a function such that $0 \le \phi \le 1$ and $\phi = 1$ on $B_\lambda\big(x_0, (K - 2\rho)\big)$. Furthermore, let $f \in C\big(\partial_p C_{K\rho,2\rho}^\lambda(x_0,t_0)\big)$ be defined as follows

$$f(x,t) = \begin{cases} 0, & (x,t) \in \partial_p C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0) \setminus \Omega_T, \\ \big(1-\phi(x)\big)u(x,t), & (x,t) \in B_\lambda(x_0,K\rho) \times \{t_0 - 4\rho^2\} \cap \Omega_T, \\ u(x,t), & (x,t) \in \partial B_\lambda(x_0,K\rho) \times (t_0 - 4\rho^2, t_0) \cap \Omega_T. \end{cases}$$

Now let u_1 be the solution to $Hu_1 = 0$ in $C_{K,2\rho}^{\lambda,-}(x_0,t_0)$ with $u_1 = f$ on $\partial_p C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)$. By the definition of u_1 we see, using (5.22) and the maximum principle, that

$$\sup_{C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)} u_1 \le \sup_{\Omega_T \cap C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)} u. \tag{5.23}$$

Extend u to $C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)$ by setting $u(x,t) = 0$ for $(x,t) \in \Omega_T \setminus C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)$. Since the function u is continuous on $C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)$ and a solution in $\Omega_T \cap C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)$, it is easily seen to be a weak subsolution in $C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0)$. It then follows from the maximum principle and (5.21) that

$$u(x,t) - \frac{1}{2} \sup_{\Omega_T \cap C_\rho^{\lambda,-}(x_0,t_0)} u \le u_1(x,t), \text{ whenever } (x,t) \in C_{K\rho,2\rho}^{\lambda,-}(x_0,t_0). \tag{5.24}$$

It now follows by (5.24), Lemma 5.4, (5.23), the maximum principle and (5.19) that

$$\begin{aligned}\frac{1}{2}\sup_{\Omega_T\cap C_\rho^{\lambda,-}(x_0,t_0)} u &\leq \sup_{C_\rho^{\lambda,-}(x_0,t_0)} u_1 \leq ce^{-(K-2)^\nu/c}\sup_{C^{\lambda,-}_{(K-2)\rho,\rho}(x_0,t_0)} u_1\\ &\leq ce^{-K^\nu/c}\sup_{\Omega_T\cap C^{\lambda,-}_{K\rho,2\rho}(x_0,t_0)} u \leq ce^{-K^\nu/c}K^{\gamma_2}\sup_{\Omega_T\cap C_\rho^{\lambda,-}(x_0,t_0)} u \end{aligned} \tag{5.25}$$

for every $K \geq \widehat{K}+2$, where $\widehat{K}=\widehat{K}(H,M)$ and ν are as in Lemma 5.4. For any $K\geq \widehat{K}+2$ such that

$$ce^{-K^\nu/c}K^{\gamma_2} < \frac{1}{2} \tag{5.26}$$

the estimate (5.25) is contradicted, and hence the proof of (5.20) is complete. This completes the proof of Theorem 5.5. □

Proof of Theorem 1.1. By Lemma 3.4 every NTA-domain is a λ-NTA-domain and thus Theorem 1.1 follows directly from Theorem 5.5 and the Harnack inequality. □

6 Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. The proof uses techniques available for uniformly parabolic operators, specifically the existence and properties of the Green function. However, using Lemma 4.1, the degenerate operators of interest here can be approximated by uniformly parabolic operators. Let H be as in (1.3) and assume that (1.4) and (1.5). Let Ω be a λ-NTA-domain with parameters M and r_0 and $T>0$. The adjoint operator of H is given by

$$H^* := \partial_t + \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}),\ (x,t)\in\mathbb{R}^n\times\mathbb{R}. \tag{6.1}$$

Note that all the results stated in the previous sections concerning solutions to $Hu=0$ remain, with appropriate reformulations, valid also for solutions to $H^*u=0$. In particular, there exists a unique probability measure $\omega^*(x,t,y,s)$ with support on $\partial_p^*\Omega_T = S_T\cup(\bar{\Omega}\times\{t=T\})$ such that the solution to the continuous Dirichlet problem $H^*u=0$ in Ω_T, $u=f$ on $\partial_p^*\Omega_T$, is given by

$$u(x,t)=\int_{\partial_p^*\Omega_T} f(y,s)d\omega^*(x,t,y,s)$$

for each $f\in C(\partial_p^*\Omega_T)$. In what follows adjoint versions of some of the lemmas established in the previous sections will be used. It should be clear from context how the lemmas are modified to hold for the adjoint operator. In the following we assume, in addition to (1.4) and (1.5), that there exist constants $c_1,c_2>0$ such that

$$c_1|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \leq c_2|\xi|^2. \tag{6.2}$$

Then, using [1] we can conclude that there exists a fundamental solution Γ to the operator H. A Green function for Ω_T, with pole at $(\hat{x},\hat{t}) \in \Omega_T$, can be defined as

$$G(x,t,\hat{x},\hat{t}) = \Gamma(x,t,\hat{x},\hat{t}) - \int_{\partial_p\Omega_T} \Gamma(y,s,\hat{x},\hat{t})d\omega(x,t,y,s) \tag{6.3}$$

where ω is the parabolic measure on $\partial_p\Omega_T$. By construction $G(x,t,\hat{x},\hat{t}) = 0$ whenever $t \leq \hat{t}$, or $(x,t) \in \partial_p\Omega_T$, and

$$H_{x,t}G(x,t,\hat{x},\hat{t}) = \delta_{(\hat{x},\hat{t})}(x,t), \tag{6.4}$$

where $\delta_{(\hat{x},\hat{t})}$ is the Dirac delta at $(\hat{x},\hat{t})$, in the sense of distributions. Furthermore,

$$G(x,t,\hat{x},\hat{t}) \leq \Gamma(x,t,\hat{x},\hat{t}) \text{ whenever } (x,t),(\hat{x},\hat{t}) \in \Omega_T, (x,t) \neq (\hat{x},\hat{t}). \tag{6.5}$$

Let G^* denote the Green function for the adjoint operator H^*. Then $G(x,t,\hat{x},\hat{t}) = G^*(\hat{x},\hat{t},x,t)$, that is

$$G(x,t,\hat{x},\hat{t}) = \Gamma(x,t,\hat{x},\hat{t}) - \int_{\partial_p^*\Omega_T} \Gamma(x,t,y,s)d\omega^*(\hat{x},\hat{t},y,s) \tag{6.6}$$

where ω^* is the adjoint parabolic measure. In particular, note that $G(x,t,\hat{x},\hat{t}) = 0$ whenever $\hat{t} \geq t$, or $(\hat{x},\hat{t}) \in \partial_p^*\Omega_T$, and

$$H^*_{\hat{x},\hat{t}}G(x,t,\hat{x},\hat{t}) = \delta_{(x,t)}(\hat{x},\hat{t}). \tag{6.7}$$

Finally, note that if $\theta \in C_0^\infty(\mathbb{R}^{N+1})$, then

$$\begin{aligned}\theta(x,t) = &\int_{\partial_p\Omega_T} \theta(y,s)d\omega(x,t,y,s)\\ &- \int_{\Omega_T} \Big(\sum_{i,j=1}^n a_{ij}\partial_{y_i}G(x,t,y,s)\partial_{y_j}\theta - G(x,t,y,s)\partial_s\theta\Big)dyds\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}\theta(x,t) = &\int_{\partial_p^*\Omega_T} \theta(y,s)d\omega^*(x,t,y,s)\\ &- \int_{\Omega_T} \Big(\sum_{i,j=1}^n a_{ij}\partial_{y_i}G(y,s,x,t)\partial_{y_j}\theta + G(y,s,x,t)\partial_s\theta\Big)dyds,\end{aligned} \tag{6.9}$$

whenever $(x,t) \in \Omega_T$.

Note that in the following the assumption that H is uniformly parabolic, that is the assumption (6.2), is only used for the existence and properties of the Green function. Recall the notation $\Delta_r^\lambda(x_0,t_0) = C_r^\lambda(x_0,t_0) \cap S_T$.

Lemma 6.1. *Let H be as in* (1.3) *and assume* (1.4), (1.5) *and* (6.2). *Let Ω be a λ-NTA-domain with parameters M and r_0 and $T > 0$. Let $(x_0, t_0) \in S_T$ and let $r \leq \min\{r_0, \sqrt{(T - t_0)/2}\}$. Then there exists $c = c(H, M)$, $c \geq 1$, such that*

$$c^{-1}|B_\lambda(x_0, r)|G(x, t, A_r^+(x_0, t_0)) \leq \omega(x, t, \Delta_r^\lambda(x_0, t_0))$$

for every $(x, t) \in \Omega_T \cap \{(x, t) : \ t \geq t_0 + 4r^2\}$.

Proof. Introduce the sets

$$\begin{aligned} S_1 &= \{(x, t) \in \Omega_T : \ t = t_0 + 2r^2\} \setminus C^\lambda_{M^{-1}r}(A_r^+(x_0, t_0)), \\ S_2 &= \{(x, t) \in \Omega_T : \ t > t_0 + 2r^2\} \cap \partial C^\lambda_{M^{-1}r}(A_r^+(x_0, t_0)). \end{aligned} \tag{6.10}$$

Note that $G(x, t, A_r^+(x_0, t_0)) = 0$ if $(x, t) \in S_1$. By construction

$$G(x, t, A_r^+(x_0, t_0)) \leq \Gamma(x, t, A_r^+(x_0, t_0)) \text{ if } (x, t) \in S_2. \tag{6.11}$$

The upper bound on the fundamental solution derived in [27] implies that

$$\Gamma(x, t, A_r^+(x_0, t_0)) \leq c \left(\frac{1}{|B_\lambda(x, \sqrt{2}r)|} + \frac{1}{|B_\lambda(A_r(x_0), \sqrt{2}r)|} \right) e^{-c\left(\frac{d_\lambda(x, A_r(x_0))^2}{2r^2}\right)^{1/(1+n)}}, \tag{6.12}$$

for some $c = c(n, \Lambda) \geq 1$, whenever $(x, t) \in S_2$. In particular, using Lemma 2.1 and Lemma 2.2, it follows that

$$\Gamma(x, t, A_r^+(x_0, t_0)) \leq \frac{c}{|B_\lambda(x_0, r)|}, \tag{6.13}$$

whenever $(x, t) \in S_2$. Combining (6.11) and (6.13) we see that

$$|B_\lambda(x_0, r)|G(x, t, A_r^+(x_0, t_0)) \leq c, \tag{6.14}$$

whenever $(x, t) \in S_2$. Next, using Lemma 4.2 and the Harnack inequality it follows that

$$\omega(x, t, \Delta_r^\lambda(x_0, t_0)) \geq c^{-1}, \tag{6.15}$$

whenever $(x, t) \in S_2$. Combining (6.14) and (6.15), Lemma 6.1 now follows by the maximum principle. □

Lemma 6.2. *Let H be as in* (1.3) *and assume* (1.4), (1.5) *and* (6.2). *Let Ω be a λ-NTA-domain with parameters M and r_0 and $T > 0$. Let $(x_0, t_0) \in S_T$ and let $r \leq \max\{r_0, \sqrt{t_0/8}\}$. Then there exists $c = c(H, M)$, $1 \leq c < \infty$, such that*

$$\omega(x, t, \Delta_r^\lambda(x_0, t_0)) \leq c|B_\lambda(x_0, r)|G(x, t, A_{2r}^-(x_0, t_0)),$$

for every $(x, t) \in \Omega_T \setminus C_{2c_\Delta{}^2r, 2r}(x_0, t_0)$.

Proof. To simplify notation let $R = R_{x_0}(r)$. Let $\theta \in C^\infty(\mathbb{R}^{n+1})$ be such that $\theta \equiv 1$ on $C^\lambda_r(x_0,t_0)$ and $\theta \equiv 0$ on the complement of $C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)$. Recalling Remark 2.2, we may also assume that θ is such that $|\partial_t\theta| \leq cr^{-2}$ and such that $|\nabla_x\theta| \leq cR^{-1}$. Note that by definition

$$\omega(x,t,\Delta^\lambda_r(x_0,t_0)) \leq \int_{\partial_p\Omega_T} \theta(y,s)d\omega(x,t,y,s). \tag{6.16}$$

Furthermore, by the representation formula in (6.8), we see that

$$\begin{aligned}\theta(x,t) =& \int_{\partial_p\Omega_T} \theta(y,s)d\omega(x,t,y,s)\\ &- \int_{\Omega_T} (\sum_{i,j=1}^n a_{ij}\partial_{y_i}G(x,t,y,s)\partial_{y_j}\theta - G(x,t,y,s)\partial_s\theta\Big)dyds. \end{aligned}\tag{6.17}$$

By construction $\theta(x,t) = 0$ whenever $(x,t) \in \Omega_T \setminus C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)$, hence combining (6.16) and (6.17) gives

$$\omega(x,t,\Delta^\lambda_r(x_0,t_0)) \leq \int_{\Omega_T} (\sum_{i,j=1}^n a_{ij}\partial_{y_i}G(x,t,y,s)\partial_{y_j}\theta - G(x,t,y,s)\partial_s\theta\Big)dyds,$$

for all $(x,t) \in \Omega_T \setminus C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)$. Using the structure condition (1.4), the Hölder inequality and the construction of θ, it follows that

$$\begin{aligned}\omega(x,t,\Delta^\lambda_r(x_0,t_0)) \leq& cR^{-1}r(\lambda(B_\lambda(x_0,r))^{1/2}\bigg(\int_{C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)} |\partial_{y_i}G(x,t,y,s)|^2\lambda dyds\bigg)^{1/2}\\ &+ cr^{-2}\bigg(\int_{C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)} |G(x,t,y,s)|dyds\bigg), \end{aligned}\tag{6.18}$$

whenever $(x,t) \in \Omega_T \setminus C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)$. Furthermore, using the adjoint version of Lemma 2.4, it follows that

$$\begin{aligned}\int_{C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)} |\partial_{y_i}G(x,t,y,s)|^2\lambda dyds \leq& cR^{-2}\int_{C^\lambda_{2{c_\Delta}^2 r,2r}(x_0,t_0)} |G(x,t,y,s)|^2\lambda dyds\\ &+ cr^{-2}\int_{C^\lambda_{2{c_\Delta}^2 r,2r}(x_0,t_0)} |G(x,t,y,s)|^2 dyds, \end{aligned}\tag{6.19}$$

whenever $(x,t) \in \Omega_T \setminus C^\lambda_{2{c_\Delta}^2 r,2r}(x_0,t_0)$. In particular, using the adjoint version of Lemma 5.2, it is seen that

$$\begin{aligned}&\int_{C^\lambda_{\sqrt{2}c_\Delta r,\sqrt{2}r}(x_0,t_0)} |\partial_{y_i}G(x,t,y,s)|^2\lambda dyds\\ &\leq c\bigg(R^{-2}r^2\lambda(B_\lambda(x_0,2{c_\Delta}^2 r)) + |B_\lambda(x_0,2{c_\Delta}^2 r)|\bigg)\big(G(x,t,A^-_{2r}(x_0,t_0))\big)^2, \end{aligned}\tag{6.20}$$

for all $(x,t)\in\Omega_T\setminus C^\lambda_{2c_\Delta{}^2r,2r}(x_0,t_0)$. Combining the above estimates we can conclude that

$$
\begin{aligned}
&\frac{\omega(x,t,\Delta_r^\lambda(x_0,t_0))}{G(x,t,A^-_{2r}(x_0,t_0))}\\
&\leq cR^{-1}r(\lambda(B_\lambda(x_0,r)))^{1/2}\Big(R^{-2}r^2\lambda(B_\lambda(x_0,2c_\Delta{}^2r))+|B_\lambda(x_0,2c_\Delta r)|\Big)^{1/2}+c|B_\lambda(x_0,r)|\\
&\leq c|B_\lambda(x_0,r)| \qquad (6.21)
\end{aligned}
$$

where Lemma 2.1 and the fact that $\lambda(B_\lambda(x_0,r))\leq cR^2r^{-2}|B_\lambda(x_0,r)|$ have been used for the last inequality. This completes the proof of the lemma. □

Theorem 6.3. *Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega\subset\mathbb{R}^n$ be a bounded λ-NTA-domain with parameters M, r_0 and let $\Omega_T=\Omega\times(0,T)$ for some $T>0$. There then is an $\bar r_0=\bar r_0(H,M,r_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega))$, $0<\bar r_0<r_0$ such that the following is true. Let $0<\delta<\bar r_0$ be a fixed constant. Let $(x_0,t_0)\in S_T$ be such that $16\delta^2\leq t_0$ and $\delta^2\leq T-t_0$, and suppose that $r<\delta/2$. Then there exists a constant $c=c(H,M,\operatorname{diam}(\Omega),T,\delta)$, $1\leq c<\infty$, such that*

$$\omega(x,t,\Delta^\lambda_{2r}(x_0,t_0))\leq c\omega(x,t,\Delta^\lambda_r(x_0,t_0)),$$

whenever $(x,t)\in\Omega_T$ is such that $t\geq t_0+16r^2$.

Proof. Let H be as in (1.3) and assume that (1.4), (1.5). For each integer $\ell\geq 1$ let $\tilde a^\ell_{ij}$, λ^ℓ, $\tilde\lambda_1$ and $\tilde\lambda_2$ be as in Lemma 4.1. Let H_ℓ be the operator corresponding to $\tilde a^\ell_{ij}$. Let Ω be a λ-NTA-domain with parameters M and r_0 and let $T>0$. Applying Lemma 4.1 we have that $\Lambda_{\lambda^\ell}=\Lambda_{\lambda^\ell}(n,\Lambda)$, and using Lemma 3.4 we can conclude that there exist $\bar M^\ell=\bar M^\ell(H,M)$ and $\bar r^\ell_0=\bar r^\ell_0(H,M,r_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega),\operatorname{diam}_{\lambda^\ell}(\Omega))$ such that Ω is a λ^ℓ-NTA-domain with parameters $\bar M^\ell$ and $\bar r^\ell_0$. Furthermore, using Lemma 4.1 we have that for all ℓ large enough we can take $\bar M^\ell=\bar M$ and $\bar r^\ell_0=\bar r_0$ for some $\bar M^\ell=\bar M(H,M)$ and $\bar r_0=\bar r_0(H,M,r_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega))$. Note that $\bar M\geq M$ and $\bar r_0\leq r_0$. Let $\delta\leq\bar r_0$ be fixed and let $(x_0,t_0)\in S_T$ be such that $16\delta^2\leq t_0$ and $\delta^2\leq T-t_0$. Let also $r<\delta/2$. Let $\mu^\ell(x)=(\lambda^\ell(x))^{-n/2}$ whenever $x\in\mathbb{R}^n$ and let

$$r^\ell_x(R)=\left(\int_{B(x,R)}\mu dx\right)^{1/n}\quad\text{for } R>0.$$

Using Lemma 4.1 we have that $\lambda^\ell\to\lambda$ almost everywhere in $\mathbb{R}^n$ as $\ell\to\infty$. Also by Lemma 4.1, $\tilde\lambda_1\leq\lambda^\ell\leq\tilde\lambda_2$ for all $\ell\geq1$. Thus $\mu^\ell\to\mu$ almost everywhere in $\mathbb{R}^n$, and by the Lebesgue theorem on dominated convergence $r^\ell_x(R)\to r_x(R)$, for every $x\in\mathbb{R}^n$ and $R>0$. Using this we see, in particular, that there exists $L\geq1$ such that $|r^\ell-r|<r-\delta/2$ for all $\ell\geq L$. To simplify notation let $r^\ell_1=r^\ell_{x_0}(R_{x_0}(\delta/2))$ and let $r^\ell_2=r^\ell_{x_0}(R_{x_0}(r+\delta/2))$. Let now $L\geq1$ be so large that also $r^\ell_1<r^\ell_2<\delta$ for all $\ell\geq L$. Let ω_ℓ denote the parabolic measure associated to the operator H_ℓ and Ω_T. Since, by construction, $\Delta^\lambda_{2r}(x_0,t_0)\subset\subset\Delta^\lambda_{r+\delta/2}(x_0,t_0)=\Delta^{\lambda^\ell}_{r^\ell_2}(x_0,t_0)$ for all $\ell\geq L$, it follows from Lemma 4.4 that

$$\omega\left(x,t,\Delta^\lambda_{2r}(x_0,t_0)\right)\leq\lim_{\ell\to\infty}\omega_\ell\left(x,t,\Delta^\lambda_{r+\delta/2}(x_0,t_0)\right)\leq\lim_{\ell\to\infty}\omega_\ell\big(x,t,\Delta^{\lambda^\ell}_{r^\ell_2}(x_0,t_0)\big). \qquad (6.22)$$

Similarly, since $\Delta^{\lambda^\ell}_{r_1^\ell}(x_0,t_0)=\Delta^\lambda_{\delta/2}(x_0,t_0)\subset\subset\Delta^\lambda_r(x_0,t_0)$, we see that

$$\lim_{\ell\to\infty}\omega_\ell\big(\Delta^{\lambda^\ell}_{r_1^\ell}(x_0,t_0)\big)\leq\omega\big(\Delta^\lambda_r(x_0,t_0)\big). \tag{6.23}$$

Note that the operator H_ℓ, for $\ell\geq L$ fixed, satisfies (6.2), and hence we can apply Lemma 6.1 and Lemma 6.2 to the corresponding parabolic measure ω_ℓ and the corresponding Green function G_ℓ. In particular, using Lemma 6.2 we see that

$$\omega_\ell(x,t,\Delta^{\lambda^\ell}_{r_2^\ell}(x_0,t_0))\leq c|B_{\lambda^\ell}(x_0,r_2^\ell)|G_\ell\big(x,t,A^{\lambda^\ell,-}_{2r_2^\ell}(x_0,t_0)\big). \tag{6.24}$$

Furthermore, using Lemma 2.1, Lemma 2.2, the adjoint versions of Theorem 5.5 and the Harnack inequality, we have that

$$|B_{\lambda^\ell}(x_0,r_2^\ell)|G_\ell\big(x,t,A^{\lambda^\ell,-}_{2r_2^\ell}(x_0,t_0)\big)\leq c|B_{\lambda^\ell}(x_0,r_1^\ell)|G_\ell\big(x,t,A^{\lambda^\ell,+}_{r_1^\ell}(x_0,t_0)\big), \tag{6.25}$$

and, using Lemma 6.1,

$$|B_\lambda(x_0,r_1^\ell)|G_\ell\big(x,t,A^+_{r_1^\ell}(x_0,t_0)\big)\leq c\omega_\ell\big(x,t,\Delta^{\lambda^\ell}_{r_1^\ell}(x_0,t_0)\big). \tag{6.26}$$

Theorem 6.3 now follows by combining (6.22), (6.24), (6.25), (6.26) and (6.23). □

Proof of Theorem 1.2. By Lemma 3.4 every NTA-domain is a λ-NTA-domain. Theorem 1.2 now follows by iterating Theorem 6.3 and applying Lemma 2.1. □

7 Proof of Theorem 1.3

The purpose of this section is to prove Theorem 1.3. Throughout this section, let H be as in (1.3) and assume (1.4) and (1.5). Let also $\Omega\subset\mathbb{R}^n$ be a bounded λ-NTA-domain with parameters M, r_0 and let $\Omega_T=\Omega\times(0,T)$ for some $T>0$.

Lemma 7.1. *Let H be as in* (1.3) *and assume that* (1.4), (1.5)*. Then there is some $\bar r_0=\bar r_0(H,M,r_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega))$ and $K=K(H,M)$ such that the following is true. Suppose that $(x_0,t_0)\in S_T$ and*

$$r<\min\{\bar r_0/4,\sqrt{(T-t_0)/16)},\sqrt{t_0/16}\}.$$

Suppose that u and v are two non-negative solutions to $Hu=0$ in $\Omega_T\cap C^\lambda_{4r}(x_0,t_0)$ and assume that u and v vanish continuously on $S_T\cap C^\lambda_{2r}(x_0,t_0)$. Then there exists a constant $c=c(H,M)$, $c\geq1$ and $K=K(H,M)$, $K\geq1$, such that

$$\sup_{\Omega_T\cap C^\lambda_{r/K}(x_0,t_0)}\frac{u}{v}\leq c\frac{u\big(A^+_{2r}(x_0,t_0)\big)}{v\big(A^-_{2r}(x_0,t_0)\big)}.$$

Proof. It is enough to assume that H satisfies (6.2) since the general result follows by the same type of approximation argument as used in the proof of Theorem 6.3. Let K be a constant to be chosen. If K is large enough we can conclude, using Lemma 2.1 and Lemma 2.2, that there exists $\{(x_i,t_i)\}_{i=1}^N$, $(x_i,t_i)\in S_T\cap\partial_pC^\lambda_{r/c_\Delta}(x_0,t_0)$, $N=N(K,n,\Lambda)$, such that

$C^\lambda_{r/K}(x_i,t_i)\subset C^\lambda_{2r}(x_0,t_0)\setminus C_{r/K}(x_0,t_0)$ for every $1\le i\le N$ and such that $S_T\cap\partial_p C^\lambda_{r/c_\Delta}(x_0,t_0)\subset\bigcup_{i=1}^N C^\lambda_{r/K}(x_i,t_i)$. Let c_1 be the constant appearing in the statement of Lemma 4.2. We will base our argument on the auxiliary function

$$\Psi(x,t)\ =\ \sum_{i=1}^N\omega\big(x,t,\Delta^\lambda_{c_1r/K}(x_i,t_i)\big)+|B_\lambda(x_0,r)|G\big(x,t,A^-_{4r}(x_0,t_0)\big).\qquad(7.1)$$

The first step is to prove that there exists $c(H,M)\ge 1$ such that

$$u(x,t)\le cu\big(A^+_{2r}(x_0,t_0)\big)\Psi(x,t),\qquad(7.2)$$

whenever $(x,t)\in\Omega_T\cap C^\lambda_{r/K}(x_0,t_0)$. First, taking $K\ge c_1$ it follows from Lemma 4.2 that $\omega\big(x,t,\Delta^\lambda_{c_1r/K}(x_i,t_i)\big)\ge c^{-1}$ whenever $(x,t)\in\Omega_T\cap C^\lambda_{r/K}(x_i,t_i)$ and thus $\Psi(x,t)\ge c^{-1}$ whenever

$$(x,t)\in\Omega_T\cap\partial_p C^\lambda_{r/c_\Delta}(x_0,t_0)\cap\bigcup_{i=1}^N C^\lambda_{r/K}(x_i,t_i).$$

Next, consider

$$(x,t)\in\big(\Omega_T\cap\partial_p C^\lambda_{r/c_\Delta}(x_0,t_0)\big)\setminus\bigcup_{i=1}^N C^\lambda_{r/K}(x_i,t_i).$$

In this case it follows from Lemma 6.2, the Harnack inequality and Lemma 4.2 that

$$|B_\lambda(x_0,r)|G\big(x,t,A^-_{4r}(x_0,t_0)\big)\ge c^{-1}\qquad(7.3)$$

for some $c=c(H,M)$, $1\le c<\infty$. Thus we may conclude that $\Psi(x,t)\ge c^{-1}$ for all $(x,t)\in\Omega_T\cap\partial_p C^\lambda_{r/c_\Delta}(x_0,t_0)$. Furthermore, using Lemma 5.2 and the Harnack inequality we have that

$$\sup_{C^\lambda_{r/K}(x_i,t_i)} u\le cu(A^+_{r/K}(x_i,t_i))\le cu(A^+_{2r}(x_0,t_0))$$

for every i, if K is large enough. Hence (7.2) holds whenever $(x,t)\in\partial_p C^\lambda_{r/c_\Delta}(x_0,t_0)$. Since Ψ is a solution to $Hu=0$ in $\Omega_T\cap C^\lambda_{r/c_\Delta}(x_0,t_0)$ and since u vanishes on $\Delta^\lambda_{2r}(x_0,t_0)$ it follows, by the maximum principle, that (7.2) holds, in particular, for all $(x,t)\in\Omega_T\cap C^\lambda_{r/K}(x_0,t_0)$.
The second step is to prove that

$$v(x,t)\ge c^{-1}v\big(A^-_{2r}(x_0,t_0)\big)\Psi(x,t)\qquad(7.4)$$

whenever $(x,t)\in\Omega_T\cap\partial_p C^\lambda_{r/K}(x_0,t_0)$. However, arguing as in Lemma 6.1, using appropriate sets S_1 and S_2, estimates for the Green function, Harnack inequality and the maximum principle, we immediately see that

$$v(x,t)\ge c^{-1}|B_\lambda(x,r)|G\big(x,t,A^-_{4r}(x_0,t_0)\big)v(A^-_{2r}(x_0,t_0)),\qquad(7.5)$$

whenever $(x,t)\in\Omega_T\cap C^\lambda_{r/K}(x_0,t_0)$. By the maximum principle, it suffices to prove that

$$c|B_\lambda(x,r)|G\big(x,t,A^-_{4r}(x_0,t_0)\big)\ge\Psi(x,t)\qquad(7.6)$$

whenever $(x,t) \in \Omega_T \cap \partial_p C^\lambda_{r/K}(x_0,t_0)$. Since $|B_\lambda(x,r)| \approx |B_\lambda(x_0,r)|$ whenever $(x,t) \in \Omega_T \cap C^\lambda_{2r}(x_0,t_0)$ it follows from the construction of Ψ that we only have to prove that

$$\omega\big(x,t,\Delta^\lambda_{c_1 r/K}(x_i,t_i)\big) \leq c|B_\lambda(x,r)|G\big(x,t,A^-_{4r}(x_0,t_0)\big). \tag{7.7}$$

whenever $(x,t) \in \Omega_T \cap \partial_p C^\lambda_{r/K}(x_0,t_0)$. However, for $K = K(H,M)$ large, this follows from Lemma 6.2, Lemma 2.1, Lemma 2.2 and the Harnack inequality for the adjoint equation. This completes the proof of the lemma. □

Theorem 7.2. *Let H be as in* (1.3)*, assume* (1.4) *and* (1.5)*. Let $\Omega \subset \mathbb{R}^n$ be a bounded λ-NTA-domain with parameters M, r_0 and let $\Omega_T = \Omega \times (0,T)$ for some $T > 0$. There then is an $\bar r_0 = \bar r_0(H,M,r_0,\operatorname{diam}(\Omega),\operatorname{diam}_\lambda(\Omega))$, $0 < \bar r_0 < r_0$ such that the following is true. Let u,v be non-negative solutions of $Hu = 0$ in Ω_T vanishing continuously on S_T. Let δ, $0 < \delta < \bar r_0$, be a fixed constant. Then u/v is Hölder continuous on the closure of $\Omega \times (\delta^2,T]$. Furthermore, let $(x_0,t_0) \in S_T$, $\delta^2 \leq t_0$, and assume that $r < \delta/2$. Then there exist $c = c(H,M,\operatorname{diam}(\Omega),T,\delta)$, $1 \leq c < \infty$, and $\alpha = \alpha(H,M,\operatorname{diam}(\Omega),T,\delta)$, $\alpha \in (0,1)$, such that*

$$\left|\frac{u(x,t)}{v(x,t)} - \frac{u(y,s)}{v(y,s)}\right| \leq c\left(\frac{d_{p,\lambda}\big((x,t),(y,s)\big)}{r}\right)^\alpha \frac{u\big(A_r(x_0,t_0)\big)}{v\big(A_r(x_0,t_0)\big)}$$

whenever $(x,t),(y,s) \in \Omega_T \cap C^\lambda_{r/4}(x_0,t_0)$.

Proof. Let u,v be non-negative solutions of $Hu = 0$ in Ω_T vanishing continuously on S_T. Let $\bar r_0$ be as in Lemma 7.1 and let δ, $0 < \delta < \bar r_0$, be a fixed constant. Let $(x_0,t_0) \in S_T$, $\delta^2 \leq t_0$, and assume that $r < \delta/2$. To avoid trivialities in the following argument, assume in the following that u and v are defined in $\Omega \times (0,\infty)$. This is easily achieved by continuing u and v beyond $t = T$ in the natural way. Given (x,t) in the closure of $\Omega \times (\delta^2,T]$ and $\rho > 0$, define

$$W^{u,v}(x,t,\rho) = \sup_{\Omega\times(\delta^2,T]\cap C^\lambda_\rho(x,t)} \frac{u}{v} - \inf_{\Omega\times(\delta^2,T]\cap C^\lambda_\rho(x,t)} \frac{u}{v}. \tag{7.8}$$

To start with, note that it follows from Lemma 7.1 and Theorem 5.5 that

$$W^{u,v}(x_0,t_0,2r) < \infty \text{ whenever } (x_0,t_0) \in \partial\Omega \times (\delta^2,T] \text{ and } r \text{ as above.} \tag{7.9}$$

In the following, let (x,t) be an arbitrary point in $\big(\Omega \times (\delta^2,T]\big) \cap C^\lambda_{r/4}(x_0,t_0)$ and let $d = d_\lambda(x,\partial\Omega) = d_{p,\lambda}(x,t,S_T)$. Given $0 < \rho \leq r/4$, consider two cases: $\rho \leq d$ (interior case) and $\rho > d$ (boundary case).

We first consider the case $\rho \leq d$. Let

$$\hat u(y,s) := \big(W^{u,v}(x,t,\rho)\big)^{-1}\left(u(y,s) - \left(\inf_{(\Omega\times(\delta^2,T])\cap C^\lambda_\rho(x,t)} u/v\right)v(y,s)\right),$$

and note that

$$\begin{aligned} &(i) \qquad 0 \leq \frac{\hat u(y,s)}{v(y,s)} \leq 1, \text{ whenever } (y,s) \in (\Omega \times (\delta^2,T]) \cap C^\lambda_\rho(x,t),\\ &(ii) \qquad W^{\hat u,v}(x,t,\rho) = 1. \end{aligned} \tag{7.10}$$

In addition, assume first that

$$\frac{\hat{u}(x,t-\rho^2/2)}{v(x,t-\rho^2/2)} \geq \frac{1}{2}. \tag{7.11}$$

Now, note that $H\hat{u} = 0$ in Ω_T. Therefore, it follows from the Harnack inequality that

$$\hat{u}(x,t-\rho^2/2) \leq c\hat{u}(y,s) \text{ whenever } (y,s) \in C^{\lambda}_{\rho/2}(x,t), \tag{7.12}$$

and that

$$v(y,s) \leq cv(x,t+\rho^2/2) \text{ whenever } (y,s) \in C^{\lambda}_{\rho/2}(x,t). \tag{7.13}$$

Moreover, it follows from Theorem 5.5 that

$$v(x,t+\rho^2/2) \leq cv(x,t-\rho^2/2). \tag{7.14}$$

Combining (7.10)-(7.14), one then deduces that

$$\frac{1}{2} \leq \frac{\hat{u}(x,t-\rho^2/2)}{v(x,t-\rho^2/2)} \leq c\frac{\hat{u}(y,s)}{v(y,s)} \leq c, \tag{7.15}$$

whenever $(y,s) \in C^{\lambda}_{\rho/2}(x,t)$. Hence

$$W^{\hat{u},v}(x,t,\rho/2) \leq \theta, \tag{7.16}$$

where $\theta = 1 - 1/(2c) \in (0,1)$. Recalling the definition of $\hat{u}$, and rearranging (7.16) one now sees that

$$W^{u,v}(x,t,\rho/2) \leq \theta W^{u,v}(x,t,\rho). \tag{7.17}$$

Assume now, on the contrary, that (7.11) does not hold and that instead

$$\frac{\hat{u}(x,t-\rho^2/2)}{v(x,t-\rho^2/2)} < \frac{1}{2}. \tag{7.18}$$

In this case let $\bar{u} = v - \hat{u}$. Then (7.10) and (7.11) hold with $\hat{u}$ replaced by $\bar{u}$. We can then first conclude that $W^{\bar{u},v}(x,t,\rho/2) \leq \theta$ and subsequently again that (7.17) holds. Next, iterating the estimate in (7.17) we deduce that

$$W^{u,v}(x,t,\rho) \leq \left(\frac{2\rho}{d}\right)^{\sigma_1} W(x,t,d) \tag{7.19}$$

for $\sigma_1 = -\log_2\theta$.

We next consider the case $\rho > d$. Let $\tilde{x}_0 \in \partial\Omega$ be such that $d = d_\lambda(x,\tilde{x}_0)$. It then holds that $C^{\lambda}_{\rho}(x,t) \subset C^{\lambda}_{2c_\Delta\rho}(\tilde{x}_0,t)$, and hence that $W^{u,v}(x,t,\rho) \leq W^{u,v}(\tilde{x}_0,t,2c_\Delta\rho)$. Let K be as in the statement of Lemma 7.1. We first assume that $4Kc_\Delta\rho < r/2$. Let now $\hat{u}$ be defined by

$$\hat{u}(y,s) \;=\; \left(W^{u,v}(\tilde{x}_0,t,8Kc_\Delta\rho)\right)^{-1}\left(u(y,s) - \left(\inf_{(\Omega\times(\delta^2,T])\cap C^{\lambda}_{8Kc_\Delta\rho}(\tilde{x}_0,t)} u/v\right)v(y,s)\right).$$

As in the interior case, it then holds that

$$
\begin{aligned}
&(i) \qquad 0\leq \frac{\hat u(y,s)}{v(y,s)}\leq 1,\ \text{whenever } (y,s)\in(\Omega\times(\delta^2,T])\cap C^\lambda_{8Kc_\Delta\rho}(\tilde x_0,t),\\
&(ii) \qquad W^{\hat u,v}(\tilde x_0,t,8Kc_\Delta\rho)=1.
\end{aligned}
\tag{7.20}
$$

Assume first that

$$
\frac{\hat u\big(A^-_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}{v\big(A^-_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}\geq\frac12. \tag{7.21}
$$

Since $\hat u$ and v are solutions to $Hu=0$ on Ω_T, non-negative in $\Omega_T\cap C^\lambda_{8Kc_\Delta\rho}(\tilde x_0,t)$, and $\hat u$ and v vanish continuously on S_T, it follows from Lemma 7.1 that

$$
\frac{\hat u\big(A^-_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}{v\big(A^+_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}\leq c\frac{\hat u(y,s)}{v(y,s)}\leq c, \tag{7.22}
$$

whenever $(y,s)\in(\Omega\times(\delta^2,T])\cap C^\lambda_{2c_\Delta\rho}(\tilde x_0,t)$. By Theorem 5.5, it follows that

$$
\frac{\hat u\big(A^-_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}{v\big(A^-_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}\leq c\frac{\hat u\big(A^-_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}{v\big(A^+_{4Kc_\Delta\rho}(\tilde x_0,t)\big)}. \tag{7.23}
$$

Hence it now follows from (7.22), (7.23) and (7.21) that

$$
\frac12\leq\frac{\hat u(y,s)}{v(y,s)}\leq c^2,
$$

whenever $(y,s)\in(\Omega\times(\delta^2,T])\cap C^\lambda_{2c_\Delta\rho}(\tilde x_0,t)$ and therefore

$$
W^{\hat u,v}(\tilde x_0,t,2c_\Delta\rho)\leq\theta, \tag{7.24}
$$

where $\theta=1-1/(2c^2)\in(0,1)$. Rewriting this expression we see that

$$
W^{u,v}(x,t,\rho)\leq W^{u,v}(\tilde x_0,t,2c_\Delta\rho)\leq\theta W^{u,v}(\tilde x_0,t,8Kc_\Delta\rho). \tag{7.25}
$$

Assume now, on the contrary, that (7.21) does not hold and instead that

$$
\frac{\hat u(A^-_{4Kc_\Delta\rho}(\tilde x_0,t))}{v(A^-_{4Kc_\Delta\rho}(\tilde x_0,t))}<\frac12. \tag{7.26}
$$

In this case, let $\bar u=v-\hat u$. Then (7.20) and (7.21) holds with $\hat u$ replaced by $\bar u$. One can then first conclude that $W^{\bar u,v}(\tilde x_0,t,2c_\Delta\rho)\leq\theta$ and subsequently again that (7.25) holds. Iterating (7.25) gives that

$$
\begin{aligned}
W^{u,v}(x,t,\rho)&\leq\theta W^{u,v}(\tilde x_0,t,8Kc_\Delta\rho)\\
&\leq\left(\frac{8Kc_\Delta\rho}{r}\right)^{\sigma_2}W^{u,v}(x_0,t_0,r),
\end{aligned}
\tag{7.27}
$$

where $\sigma_2 = -\ln(\theta)/\ln(c^*K)$ and $c^* = c^*(H, M)$, $1 \leq c^* < \infty$. Obviously this also holds if $4Kc_\Delta\rho \geq r/2$.

From (7.19) and (7.27) it follows that if $\rho \leq d < r$, then

$$W^{u,v}(x,t,\rho) \leq \left(\frac{2\rho}{d}\right)^{\sigma_1} W^{u,v}(x,t,d) \leq \left(\frac{2\rho}{d}\right)^{\sigma_1} \left(\frac{8Kc_\Delta d}{r}\right)^{\sigma_2} W^{u,v}(x_0,t_0,r). \tag{7.28}$$

With $\alpha = \min\{\sigma_1, \sigma_2\}$, (7.27) and (7.17) imply that

$$W^{u,v}(x,t,\rho) \leq c\left(\frac{\rho}{r}\right)^{\alpha} W^{u,v}(x_0,t_0,r), \tag{7.29}$$

for all $\rho \leq r/4$. Now for $(y,s) \in \Omega_T \cap C^\lambda_{r/4}(x_0,t_0)$ let $\hat\rho = d_{p,\lambda}(x,t,y,s)$. It then follows from (7.29) and Lemma 7.1 together with Theorem 5.5 that

$$\left|\frac{u(x,t)}{v(x,t} - \frac{u(y,s)}{v(y,s)}\right| \leq W^{u,v}(x,t,\hat\rho) \leq c\left(\frac{\hat\rho}{r}\right)^{\alpha} W^{u,v}(x_0,t_0,r) \leq c\left(\frac{\hat\rho}{r}\right)^{\alpha} \frac{u\big(A_r(x,t)\big)}{v\big(A_r(x,t)\big)}.$$

This completes the proof of Theorem 7.2. □

Proof of Theorem 1.3. Using Lemma 3.4 and Lemma 2.1, Theorem 1.3 follows directly from Theorem 7.2. □

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