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Robust Control of Automotive Active Seat-Suspension System Subject to Actuator Saturation

This paper deals with the problem of robust sampled-data control for an automotive seat-suspension system subject to control input saturation. By using the nature of the sector nonlinearity, a sampled-data based control input saturation in the control design is studied. A passenger dynamic behavior is considered in the modeling of seat-suspension system, which makes the model more precisely and brings about uncertainties as well in the developed model. Robust output feedback control strategy is adopted since some state variables, such as, body acceleration and body deflection, are unavailable. The desired controller can be achieved by solving the corresponding linear matrix inequalities (LMIs). Finally, a design example has been given to demonstrate the effectiveness and advantages of the proposed controller design approach. [DOI: 10.1115/1.4026833]

1 Introduction

In recent years, much attention has been drawn to the problem of vehicle suspension systems since it plays an important role in improving the ride comfort, vehicle safety, and minimizing the road damage, etc. [1–4]. The main purpose of this problem is to constrain some technical indicators within an acceptable level through isolating from road noise, bumps, and vibrations, etc. However, A trade-off should be made among these performance requirements, for example, enhancing ride comfort may lead to large suspension stroke and smaller damping in the wheel-hop mode [5]. In recent years, a considerable amount of research has been carried out to improve the performance of automotive suspension systems, see for example, Refs. [6–12] and the references therein.

Saturation of the control input appearing in mechanical systems is quite common, which may degrade the performance of the system. It is an important issue in designing the suspension control system to prevent saturation happening by constraining the control input within a certain level [9,13,14]. However, from the published results, it appears that general results pertaining to control input of seat-suspension systems subject to vector-nonlinear saturation are few, which motivates us to do further researches in this study.

To characterize the reality more precisely, the human body dynamic behavior is considered in the modeling process of the active suspension control system [14,15]. The state feedback control strategy cannot be used any more due to the unavailability of some state variables in the new dynamic, such as the deflection and velocity of passengers. The observer-based control strategy is adopted in Ref. [16]. It seems unreasonable in design process since the uncertainty item A_θ is modeled in the observer. To design the active suspension control system by using available measurements is a meaningful work, which is another motivation of our present study.

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In this study, we address the problem of robust H_∞ control for automotive active seat-suspension system. This problem aims at designing a robust controller such that the system with the saturation of the control input and the uncertainty of the passengers can achieve good active suspension performances related to ride comfort, suspension deflection limitation and road holding ability. Finally, a practical example is employed to illustrate the effectiveness of the proposed method.

The remainder of the paper is organized as follows: The problem formulation is given in Sec. 2. The controller design according to the technical indices of suspension control performances is presented in Sec. 3. Section 4 provides the design results and simulations. Finally, the study's findings are summarized in Sec. 5.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of real $n\times m$ matrices; I is the identity matrix of appropriate dimensions; $\|\cdot\|$ stands for the Euclidean vector norm or spectral norm as appropriate; The notation $X>0$ (respectively, $X<0$), for $X\in\mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, negative definite); The asterisk $*$ in a matrix is used to denote term that is induced by symmetry.

2 Problem Formulation

In this study, a 3-DOF quarter-car vertical suspension model, first established by Wei and Griffin [17], is considered, in which the human body dynamic behavior is considered. From the architecture of the seat-suspension system, shown in Fig. 1, one can see the human body is separated by two parts: the buttocks and legs part and the upper part. Those two parts are interconnected by a spring and a damper according to biodynamic responses.

The kinetic equations of the active seat suspension with consideration of human bodies are described by

$$\begin{aligned} m_u\ddot{z}_u &= k_s(z_s-z_u)+c_s(\dot{z}_s-\dot{z}_u)-k_t(z_u-z_r)-c_t(z_u-z_r)+u \\ m_s\ddot{z}_s &= k_h(z_h-z_s)+c_h(\dot{z}_h-\dot{z}_s)-k_s(z_s-z_u)-c_s(\dot{z}_s-\dot{z}_u)-u \\ m_{h_1}\ddot{z}_h &= -k_h(z_h-z_s)-c_h(\dot{z}_h-\dot{z}_s) \end{aligned} \quad (1)$$

where $m_s = m_{h2}+m'_s$.

For convenience of analysis, we define the state variable $x(t)=[x_1(t)x_2(t)x_3(t)x_4(t)x_5(t)x_6(t)]^T$ and

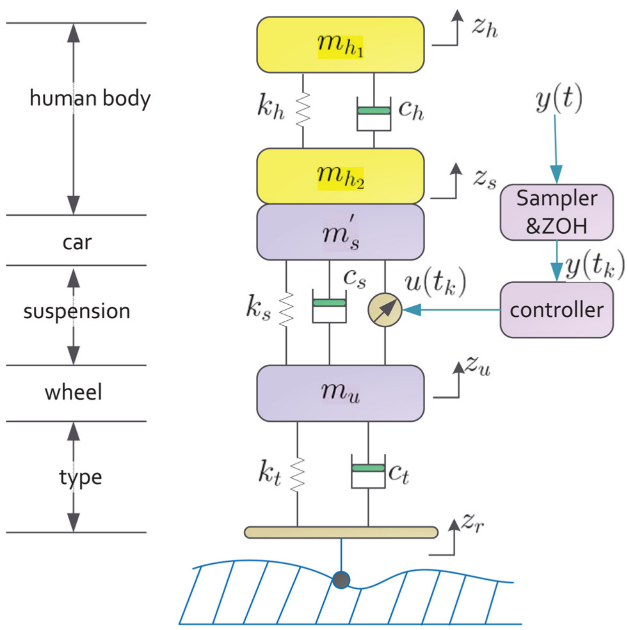

Fig. 1 Vibration model of seat-suspension system

$$
\begin{aligned}
x_1(t) &= z_h(t) - z_s(t), && \text{human body deflection}\\
x_2(t) &= z_s(t) - z_u(t), && \text{suspension deflection}\\
x_3(t) &= z_u(t) - z_r(t), && \text{tyre deflection}\\
x_4(t) &= \dot{z}_h(t), && \text{human body velocity}\\
x_5(t) &= \dot{z}_s(t), && \text{sprung mass velocity}\\
x_6(t) &= \dot{z}_u(t), && \text{unsprung mass velocity}
\end{aligned}
$$

Taking saturation nonlinearity of the control input into account, the state-space equation of the active seat-suspension model can be expressed by

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)) + B_\omega \omega(t) \tag{2}$$

where $\omega(t) = \dot{z}_r(t)$ denotes the road disturbance, $\sigma(u(t_k)) = \text{sign}(u(t))\min\{u_{\max}, |u(t)|\}$ represents the control input with a sector nonlinear saturation. From the kinetic equations of the seat-suspension system in Eq. (1) and the definition of the state variables, the system matrices in Eq. (2) can be obtained by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_h}{m_{h_1}} & 0 & 0 & -\frac{c_h}{m_{h_1}} & \frac{c_h}{m_{h_1}} & 0 \\ \frac{k_h}{m_s} & -\frac{k_s}{m_s} & 0 & \frac{c_h}{m_s} & -\frac{c_s + c_h}{m_s} & \frac{c_s}{m_s} \\ 0 & \frac{k_s}{m_u} & -\frac{k_t}{m_u} & 0 & \frac{c_s}{m_u} & -\frac{c_s + c_t}{m_u} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{m_s} & \frac{1}{m_u} \end{bmatrix}^T$$

$$B_\omega = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -\frac{c_t}{m_u} \end{bmatrix}^T$$

3 Robust Controller Design

In this section, we aim to design a controller such that the active seat-suspension system meets the following performances:

(1) Body acceleration $\ddot{z}_h(t)$ satisfies a prescribed level of H_∞ performance under the road disturbance, which is one of the most important performance indices of the active seat-suspension system and wildly used for ride comfort. The performance can be expressed by

$$\|\ddot{z}_h(t)\|_2 < \gamma^2 \|\omega\|_2 \tag{3}$$

(2) The maximum allowable stroke of the suspension should be taken into account since it is related to the safety of the vehicles structure. Therefore, the following limitation is required

$$|z_s(t) - z_u(t)| \le z_{1\max} \tag{4}$$

(3) In order to ensure a firm uninterrupted contact of wheels to road, the dynamic tyre load should not exceed the static one [18], i.e.,

$$|(z_u(t) - z_r(t))| < (m_h + m_u + m_s')g/k_t = z_{2\max} \tag{5}$$

where g denotes gravitational acceleration and $m_h = m_{h1} + m_{h2}$.

In order to analyze the above performance conveniently, we define the following controlled output:

$$z_1(t) = C_1 x(t) \tag{6}$$

$$z_2(t) = C_2 x(t) \tag{7}$$

where $C_1 = [-\frac{k_h}{m_{h_1}}\ 0\ 0 - \frac{c_h}{m_{h_1}}\ \frac{c_h}{m_{h_1}}\ 0], C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

Since the body deflection $x_1(t)$, tyre deflection $x_3(t)$ and the body velocity $x_4(t)$, in practice, are unmeasurable, the state feedback for the seat-suspension system with consideration of body dynamics is not suitable any more. Here, we select some measurable states $x_2(t)$, $x_5(t)$, $x_6(t)$ of the seat-suspension system in Eq. (2) as the system's output. The output $y(t)$ is then expressed by

$$y(t) = Cx(t) \tag{8}$$

where $C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Remark 1. In Eq. (8), the suspension deflection can be measured by using suitable displacement transducer; the sprung and the wheel velocities in vertical direction can be obtained by integrating the corresponding acceleration signals, respectively. A self-zeroing integrator which introduces a high-pass filter is needed in this scenario.

So far, the controller of the seat-suspension system (2) can be designed by a way of output feedback control. It follows

$$u(t) = Ky(t) \tag{9}$$

where K is the controller gain to be designed.

From Eq. (2), one can know that it will incur a bad control performance due to the mismatch between the model and the practical suspension system if one uses the control strategy without introducing the passengers into the framework. However, the passengers' dynamic is uncertainty with the different passengers. Then, the dynamic (2) with relation to m_{h_1} and m_s can be further expressed by

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}\sigma(u(t)) + B_\omega \omega(t) \tag{10}$$

where $\bar{A} = A + \Delta A, \bar{B} = B + \Delta B, \Delta A$, and ΔB are unknown matrices representing parameter uncertainties. In this paper, the admissible parameter uncertainties are assumed to be of the following form:

$$[\Delta A \quad \Delta B] = DF(t)[E_1 \quad E_2] \tag{11}$$

where D, E_1, and E_2 are the known real constant matrices and $F(t)$ is an unknown matrix function with Lebesgue-measurable elements satisfying $F(t)^T F(t) \le I$.

It is assumed that the measurable state variables of the seat-suspension system are sampled and hold at instant t_k. Then, the seat-suspension system can be rewritten as

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}\sigma(u(t_k)) + B_\omega \omega(t), \qquad t_k \le t \le t_{k+1} \tag{12}$$

Defining $d(t) = t - t_k$, we have

$$t_k = t - (t - t_k) = t - d(t), \qquad t_k \le t \le t_{k+1} \tag{13}$$

It yields $0 \le d(t) \le t_{k+1} - t_k = h$, where h is a positive constant which represents the maximal sampling interval. From Eq. (9), it leads to

$$u(t_k) = KCx(t - d(t)) \qquad t_k \le t \le t_{k+1} \tag{14}$$

The following definition will be used to in deriving results.

Definition 1. [19] *A nonlinearity $\psi : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to satisfy a sector condition if*

$$(\psi(v) - L_1 v)^T (\psi(v) - L_2 v) \le 0, \quad \forall v \in \mathbb{R}^r \tag{15}$$

for some real matrices $L_1, L_2 \in \mathbb{R}^{r\times r}$, where $L = L_2 - L_1$ is a positive-definite symmetric matrix. In this case, we say that belongs to the sector $[L_1, L_2]$.

If we assume that there exist diagonal matrices H_1 and H_2 such that $0 \le H_1 < I \le H_2$, then the saturation function $\sigma(u(t_k))$ in Eq. (12) can be written as

$$\sigma(u(t_k)) = H_1 u(t_k) + \psi(u(t_k)) \tag{16}$$

where $\psi(u(t_k))$ is a nonlinear vector-valued function which satisfies a sector condition with $L_1 = 0$ and $L_2 = H$, in which $H = H_2 - H_1$, i.e., $\psi(u(t_k))$ satisfies

$$\psi^T(u(t_k))(\psi(u(t_k) - Hu(t_k))) \le 0 \tag{17}$$

Then, the closed-loop system can be expressed as

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}H_1 KCx(t - d(t)) + \bar{B}\psi(u(t_k)) + B_\omega \omega(t) \tag{18}$$

Before providing the solution to the problem of the active seat-suspension control, we recall the following useful lemmas.

Lemma 1. [20] *For any constant matrix $R \in \mathbb{R}^n, R > 0$, scalar $0 \le d(t) \le h$, and vector function $\dot{x} : [-h\ 0] \to \mathbb{R}^n$ such that the following integration is well defined, then it holds that*

$$-h\int_{t-h}^{0} \dot{x}^T(t) R\dot{x}(t) \le \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -R & * & * \\ R & -2R & * \\ 0 & R & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h) \end{bmatrix} \tag{19}$$

Lemma 2. *[19,21] Let $Y_0(\xi), Y_1(\xi(t)), \cdots, Y_p(\xi(t))$ be quadratic functions of $\xi(t) \in \mathbb{R}^n$*

$$Y_i(\xi(t)) = \xi(t)^T T_i \xi(t), \quad i = 0, 1, \cdots, p \tag{20}$$

with $T_i = T_i^T$. Then, the implication

$$Y_0(\xi(t)) \le 0, \cdots, Y_p(\xi(t)) \le 0 \Rightarrow Y_0(\xi(t)) \le 0 \tag{21}$$

holds if there exist $\kappa_1, \cdots, \kappa_p > 0$ such that

$$T_0 - \sum_{i=1}^{p} \kappa_i^{-1} T_i \le 0 \tag{22}$$

Theorem 1. *The sampled-data active seat-suspension system (12) subject to the actuator saturation is said to be asymptotically stable and meets the suspension performance constrains (3)–(5), if there exist $P > 0$, $Q > 0$, and $R > 0$ such that*

$$\bar{\Phi} = \begin{bmatrix} \bar{\Phi}_1 & * & * \\ hP\mathcal{A} & -PR^{-1}P & * \\ \mathcal{C}_1 & 0 & -I \end{bmatrix} < 0 \tag{23}$$

$$\begin{bmatrix} -\frac{z_{i\max}^2}{\vartheta} P & * \\ \{C_2\}_i & -I \end{bmatrix} < 0 \quad i = 1, 2 \tag{24}$$

where

$$\bar{\Phi}_1 = \begin{bmatrix} \Phi_{11} & * & * & * & * \\ \Phi_{21} & -2R & * & * & * \\ 0 & R & -Q-R & * & * \\ \bar{B}^T P & \kappa HKC & 0 & -\kappa I & * \\ B_\omega P & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$\Phi_{11} = P\bar{A} + \bar{A}^T P + Q - R, \quad \Phi_{21} = C^T K^T H_1^T \bar{B}^T P + R$$

$$\mathcal{A} = \begin{bmatrix} \bar{A} & \bar{B}H_1 KC & 0 & \bar{B} & B_\omega \end{bmatrix}, \quad \mathcal{C}_1 = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof. Choose a Lyapunov functional candidate for the system (18) as

$$V(t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Qx(s)ds + h\int_{-h}^{0}\int_{t-h}^{t} \dot{x}^T(v)R\dot{x}(v)dvds \tag{25}$$

Then, we have

$$\dot{V}(t) = 2x^T P\mathcal{A}\xi(t) + x^T(t)Qx(t) - x^T(t-h)Qx(t-h) + h^2\dot{x}^T(t)R\dot{x}(t) - h\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds$$

where $\xi(t) = [x^T(t)x^T(t-d(t))x^T(t-h)\psi^T(u(t_k))\omega^T(t)]^T$.

From Lemma 1, It yields

$$z_1^T(t)z_1(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(t) \le \xi^T(t)[\Phi_0 + h^2\mathcal{A}^T R\mathcal{A}]\xi(t) \tag{26}$$

where

$$\Phi_0 = \begin{bmatrix} \Phi_{11} + \mathcal{C}^T\mathcal{C} & * & * & * & * \\ \Phi_{21} & -2R & * & * & * \\ 0 & R & -Q-R & * & * \\ \bar{B}^T P & 0 & 0 & 0 & * \\ B_\omega P & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

Combining Eqs. (14) and (17), we can obtain

$$\psi^T(u(t_k))(\psi(u(t_k) - HKCx(t - d(t)))) \le 0 \tag{27}$$

which can be rewritten as

$$\xi^T(t)\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{bmatrix}\begin{bmatrix} 0 & -HKC & 0 & I & 0 \end{bmatrix}\xi(t) \le 0$$

Applying Schur complement and Lemma 2, one can see that Eq. (23) is a sufficient condition to guarantee Eq. (27) and

$$z_1^T(t)z_1(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(t) < 0 \tag{28}$$

Under zero initial conditions, integrating both side of Eq. (28) yields

$$0 < V(t) < \int_0^t [\gamma^2\omega^T(t)\omega(t) - z_1^T(t)z_1(t)]dt \tag{29}$$

Then, we can conclude that $\|z_1(t)\|_2 < \gamma\|\omega(t)\|_2$ for all nonzero $\omega(t) \in L_2[0,\infty)$. H_∞ performance is then established.

Consider the second and third performance requirements of the active seat-suspension system listed in Sec. 2, i.e., $|\{z_2(t)\}_i| < z_{i\max}, i = 1,2$, which can be written by

$$x^T(t)\{C_2\}_i^T\{C_2\}_i x(t) < z_{i\max}^2 \qquad i = 1,2 \tag{30}$$

Note that

$$\begin{aligned} x^T(t)\{C_2\}_i^T\{C_2\}_i x(t) &= x^T(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}\{C_2\}_i^T\{C_2\}_i P^{-\frac{1}{2}}P^{\frac{1}{2}}x(t) \\ &\le \lambda_{\mathrm{m}}\left(P^{-\frac{1}{2}}\{C_2\}_i^T\{C_2\}_i P^{-\frac{1}{2}}\right)x(t)^T P x(t) \end{aligned} \tag{31}$$

where $\lambda_{\mathrm{m}}(\cdot)$ represents maximal eigenvalue.

Define $\vartheta = \inf_{t>0}\{(V(0) + \int_0^t [\gamma^2\omega^T(t)\omega(t)]dt\}$. It is true that

$$x^T(t)Px(t) \le V(t) \le \vartheta \tag{32}$$

from Eqs. (29) and (25). Then, it follows that

$$z_{i\max}^2 > \lambda\left(P^{-\frac{1}{2}}\{C_2\}_i^T\{C_2\}_i P^{-\frac{1}{2}}\right) \qquad i = 1,2 \tag{33}$$

Then, the constraint in Eq. (30) holds if

$$P^{-\frac{1}{2}}\{C_2\}_i^T\{C_2\}_i P^{-\frac{1}{2}} < \frac{z_{i\max}^2}{\vartheta} \tag{34}$$

which, by Schur complement, is equivalent to Eq. (24). The proof is completed.

By a commonly used analysis method, the following result can be obtained for the stability of the active seat-suspension system with consideration of the modeling uncertainty based on Theorem 1.

THEOREM 2. *The sampled-data active seat-suspension system (12) subject to the actuator saturation is said to be asymptotically stable and meets the suspension performance constrains (3)–(5), if there exist $P > 0$, $Q > 0$, and $R > 0$ such that Eq. (24) and*

$$\Pi = \begin{bmatrix} \Phi & * & * \\ \Xi_1 & -\varepsilon I & * \\ \Xi_2 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{35}$$

hold, where Φ is obtained from $\bar{\Phi}$ in Eq. (23) by replacing $\bar{A}$ and $\bar{B}$ with A and B, respectively, and

$$\begin{aligned} \Xi_1 &= \begin{bmatrix} \varepsilon D^T P & 0 & 0 & 0 & 0 & \varepsilon h D^T P & 0 \end{bmatrix} \\ \Xi_2 &= \begin{bmatrix} E_1 & E_2 H_1 K C & 0 & E_2 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Theorem 2 gives the conditions to guarantee the robust stability of the active seat-suspension system with consideration of the design requirements. However, the controller gain cannot be obtained directly due to some nonlinear items existing in Eqs. (35) and (24). Next we will give a method to search the controller gain by a set of tractable LMIs.

THEOREM 3. *For given positive scalars ρ, γ, ϑ and h, the sampled-data closed-loop seat-suspension system (12) with consideration of the control input saturation is asymptotically stable and meets the active suspension performance (3) and (4), if there exist $X > 0, W > 0, \tilde{Q} > 0, \tilde{R} > 0$, and $\delta > 0$, $\varepsilon > 0$ such that*

$$\begin{bmatrix} \tilde{\Phi} & * & * \\ \tilde{\Xi}_1 & -\varepsilon I & * \\ \tilde{\Xi}_2 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} -\frac{z_{i\max}^2}{\vartheta}X & * \\ \{C_2\}_i X & -I \end{bmatrix} < 0 \quad i = 1,2 \tag{37}$$

$$CX = WC \tag{38}$$

Moreover, the controller parameter is given by $K = YW^{-1}$. where

$$\tilde{\Phi} = \begin{bmatrix} \hat{\Phi}_1 & * & * \\ h\widehat{\mathcal{A}} & -2\rho X + \rho^2\tilde{R} & * \\ \widehat{\mathcal{C}} & 0 & -I \end{bmatrix}$$

$$\hat{\Phi}_1 = \begin{bmatrix} \tilde{\Phi}_{11} & * & * & * & * \\ \tilde{\Phi}_{21} & -2\tilde{R} & * & * & * \\ 0 & \tilde{R} & -\tilde{Q} - \tilde{R} & * & * \\ \delta\bar{B}^T & HYC & 0 & -\delta I & * \\ B_\omega & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$\tilde{\Phi}_{11} = AX + XA^T + \tilde{Q} - \tilde{R}, \quad \tilde{\Phi}_{21} = Y^T H_1^T \bar{B}^T + \tilde{R}$$

$$\widehat{\mathcal{A}} = [AX \quad BH_1Y \quad 0 \quad B \quad B_\omega], \quad \widehat{\mathcal{C}} = [C_1X \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\tilde{\Xi}_1 = \begin{bmatrix} \varepsilon D^T & 0 & 0 & 0 & 0 & \varepsilon h D^T & 0 \end{bmatrix}$$

$$\tilde{\Xi}_2 = \begin{bmatrix} E_1X & E_2H_1Y & 0 & E_2X & 0 & 0 & 0 \end{bmatrix}$$

Proof. Note that

$$(\rho R - P)R^{-1}(\rho R - P) \ge 0 \tag{39}$$

where ρ is a positive scalar. Then, it is true that

$$-PR^{-1}P \le -2\rho P + \rho^2 R \tag{40}$$

It follows that $\Pi \le \tilde{\Pi}$, where $\tilde{\Pi}$ is a matrix by replacing the item $-PR^{-1}P$ in Eq. (35) with $-2\rho^2 P + \rho R$.

Defining $P^{-1} = X, \delta = \kappa^{-1}, XQX = \tilde{Q}, XRX = \tilde{R}, KW = Y, J_1 = \mathrm{diag}\{X, X, X, \delta, I, X, I, I, I\}$ and $J_2 = \mathrm{diag}\{X, I\}$, pre- and postmultiplying Eqs. (35) and (24) and their transposes, respectively, together with Eq. (38), we can know Eqs. (36) and (37) hold. This completes the proof.

One can see that it is difficult to find a feasible solution by Theorem 3 due to Eq. (38). Now we introduce the following algorithm to address this problem.

It is noted that Eq. (38) is equivalent to trace $\left[(CX - WC)^T(CX - WC)\right] = 0$, which can be converted to the following optimization problem by using Schur complement

$$\begin{cases} \begin{bmatrix} -\sigma I & * \\ WC - CX & -I \end{bmatrix} < 0 \\ \sigma \to 0 \end{cases} \tag{41}$$

where the scalar σ is a small enough positive scalar. Then, the controller gain can be resolved by Eqs. (36), (37), and (41).

Table 1 Nominal value of the quarter-car model

Symbol	$m_{h1}+m_{h2}$ (kg)	c_h (Ns/m)	k_h (N/m)	m_s (kg)	k_s (N/m)	c_s (Ns/m)	m_u (kg)	k_t (N/m)	c_t (Ns/m)
Values	43.4 + 7.8	1485	44,130	972.2	42719.6	1095	113.6	101,115	14.6

Remark 2. The output matrix C is assumed to be invertible in Refs. [22,23]. The method cannot be used in this study since C is not a square matrix. Although some intelligent optimization algorithm can be used to find feasible solutions on continuous-time system with SOF control design, for the sake of technical simplicity, we take the above algorithm to tackle this problem.

4 An Application Example

The proposed approach will be applied to design the controller of the automotive active seat-suspension system in this section. The nominal values of the quarter-car model [9,24] are listed in Table 1. The norm-bounded parameters of uncertainties in Eq. (12) can be expressed by $D = I$ and

$$E_1 = \alpha_1 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad E_2 = \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad (42)$$

where $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, and $F(t) = \sin 0.01t$.

The following three different road profiles are considered to illustrate the effectiveness of our proposed method:

(a) Shock (see Fig. 2(*a*)). Shock (single bump) are discrete events of relatively short duration and high intensity, for example, an isolated bump or pothole in an otherwise smooth road surface. Such a disturbance can be described as [9]

$$z_r(t) = \begin{cases} \frac{A}{2}\left(1 - \cos\left(\frac{2\pi V_0}{l} t\right)\right), & 0 \le t \le \frac{l}{V_0} \\ 0, & t > \frac{l}{V_0} \end{cases}$$

where A is the height of the bump and l is the length of the bump. Here we choose $A = 0.06$ m, $l = 5$ m, $V_0 = 4.5$ km/h.

(b) Zero-mean white noise as a disturbance $\omega(t)$ (see Fig. 2(*b*)). It represents a case of rough road profile.

(c) Superposition of multinonlinear functions with different frequency(see Fig. 2(*c*)). It can be described as

$$z_r(t) = 0.02\sin 2\pi t + 0.001\sin 10\pi t + 0.001\sin 16\pi t$$

by which one can analysis the influence on the automotive seat-suspension system with various frequency.

From the technical requirements of the active seat-suspension system stated in Sec. 2, the following technical parameters are taken: $\gamma = 12$, $H = 1$, $H_1 = 0.5$, $u_{max} = 1600$ N, $z_{1max} = 0.03$ m, $z_{2max} = 0.015$ m, and $h = 10$ ms. From Theorem 2 together with its corresponding algorithm, we can obtain the controller gain $K = 10^4 \times [5.67861.3079 - 0.2743]$.

Next we will evaluate the control quantities from the following aspects: (1) Body acceleration $z_1(t)$, (2) suspension deflection and $\{z_2(t)\}_1$, and (3) tyre deflection $\{z_2(t)\}_2$. The following response curves are based on the case of $m_{h1} = 46$ kg, $m_{h2} = 8$ kg. The responses of the active suspension control system under isolated bump are plotted in Figs. 3 and 4, from which it can be seen that a

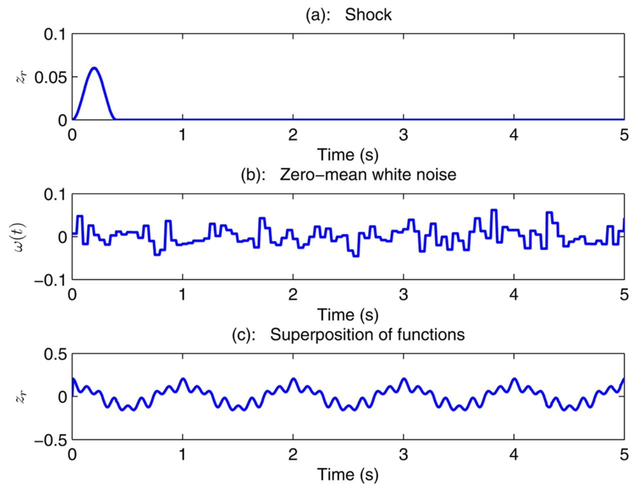

Fig. 2 Three tyres of road profile

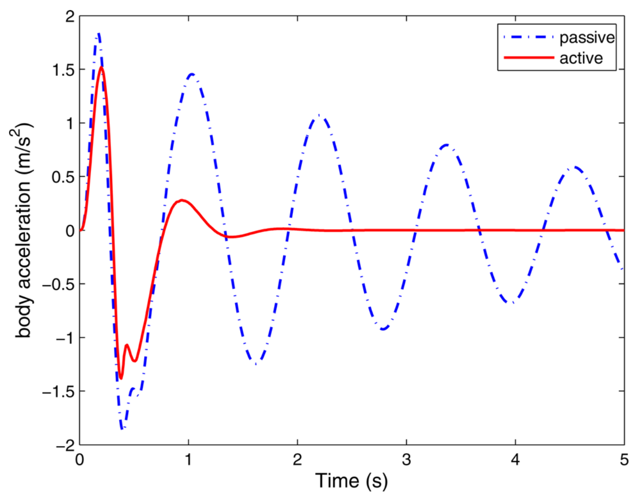

Fig. 3 Body acceleration under case a

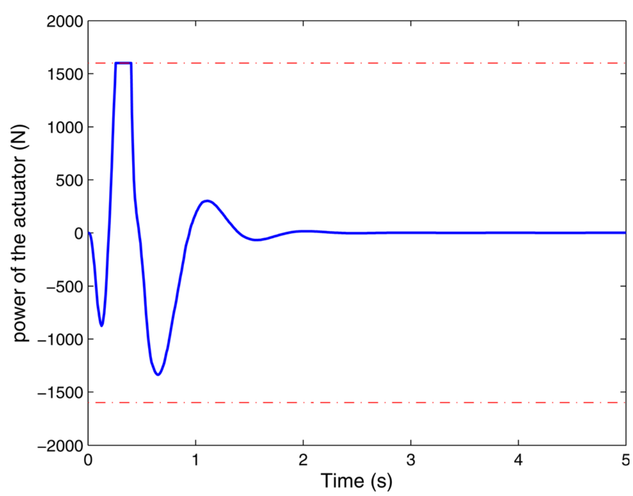

Fig. 4 Control force under case a

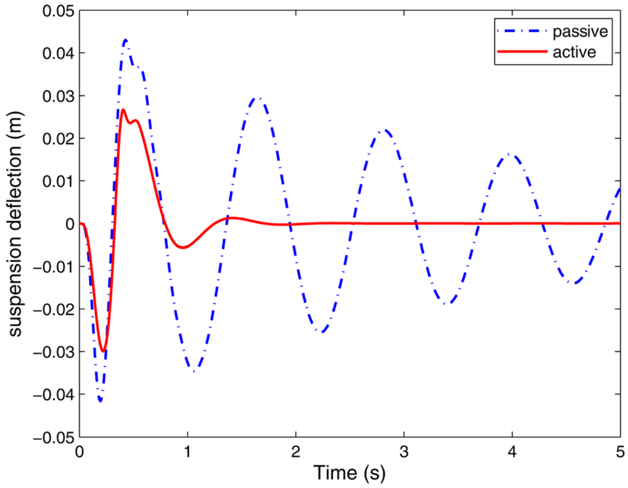

Fig. 5 Suspension deflection under case a

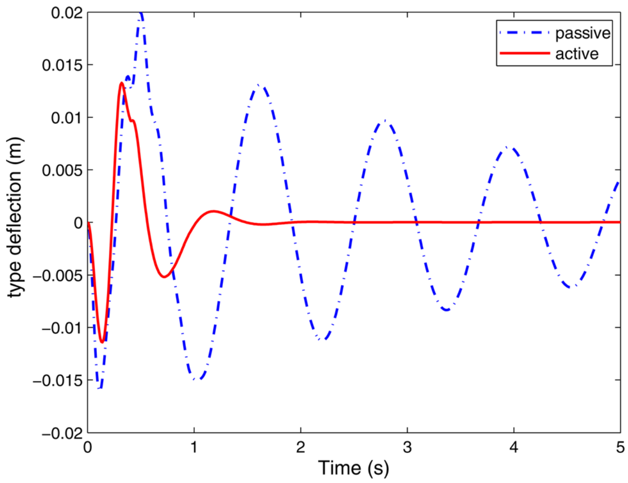

Fig. 6 Tyre deflection under case a

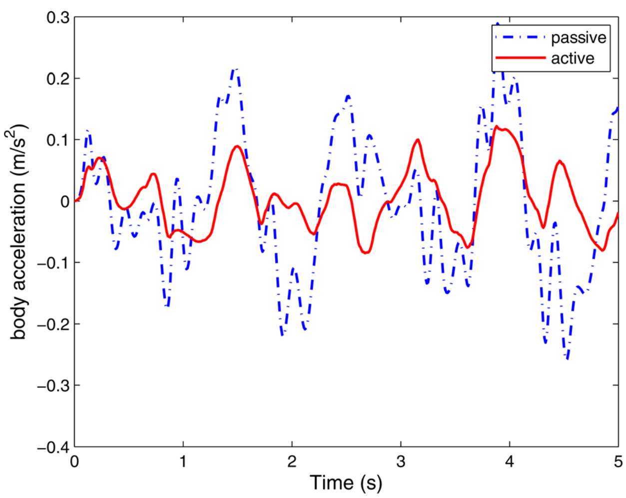

Fig. 7 Body acceleration under case b

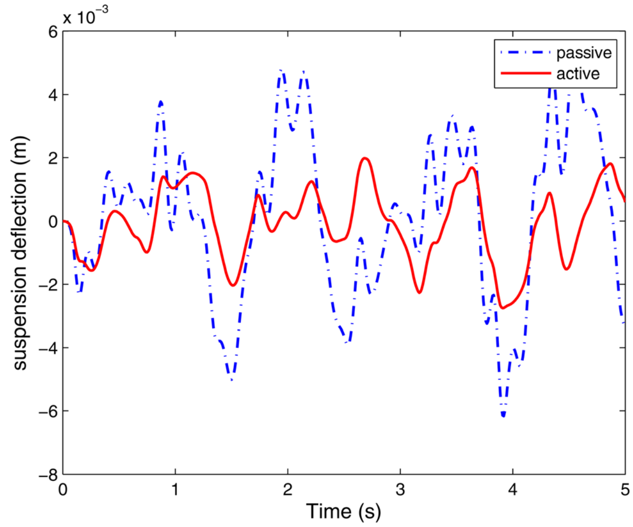

Fig. 8 Suspension deflection under case b

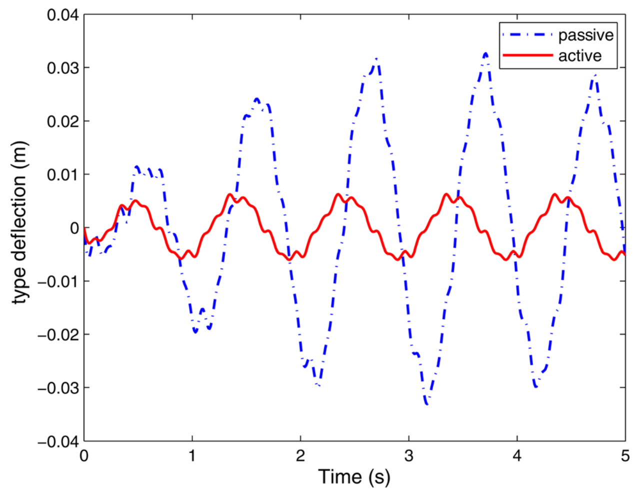

Fig. 9 Tyre deflection under case b

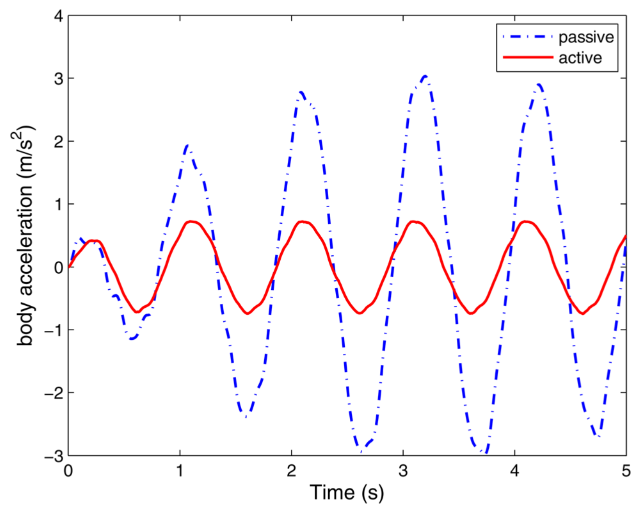

Fig. 10 Body acceleration under case c

better performance can be got in comparison with the passive mode by using the designed controller subjected to the nonlinear saturation shown in Fig. 4. From Fig. 3, one can see that the ride comfort is greatly improved, and the safety can be guaranteed from Figs. 5 and 6.

Figures 7–9 depict the responses of body acceleration, suspension deflection and tyre deflection, which demonstrate the effectiveness of the designed controller for the automotive seat-suspension system under white noise disturbance from ground.

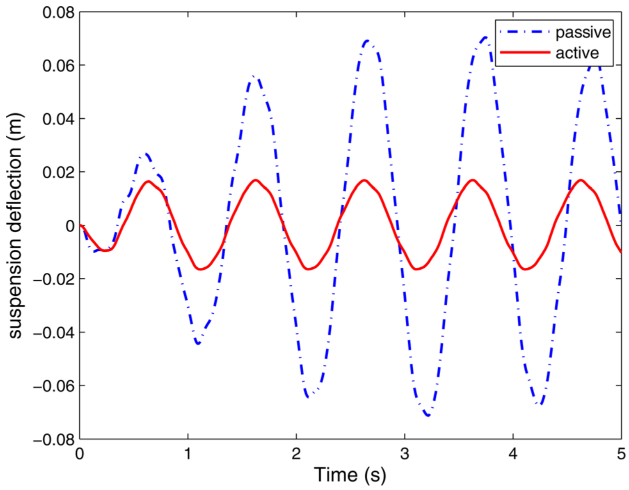

Fig. 11 Suspension deflection under case c

Figures 10 and 11 illustrate the proposed method can lead to good control performances under the road profiles with different frequencies as well, especially in 4–8 Hz frequency range which is regarded as more sensitive frequency range to human bodies in the vertical direction according to ISO-2631. From Figs. 7–11, one can be see that the control force constrained by nonlinear sector saturation can also meet the active suspension design requirements by using the proposed output feedback controller.

5 Conclusion

This paper has investigated the problem of H_∞ robust control for a class of automotive active seat-suspension system with sampling measurements. The active vehicle suspension system subject to control input saturation has been studied by using the nature of the sector nonlinearity. Due to some unavailable physical variables, the output feedback control strategy is adopted. The ride comfort, good handling, and bounded control force, etc., of the system are considered in the design process. A quarter-car model has been considered and the effectiveness of the proposed approach has been illustrated by a practical design example.

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Nomenclature

c_h = the damping coefficient of the human body
c_s = the suspension damping coefficient
c_t = the tyre damping coefficient
k_h = the stiffness coefficient of the components inside human body
k_s = the stiffness coefficient of the suspension
k_t = the tyre stiffness coefficient
m_{h1} = the mass of the upper part of a seated man
m_{h2} = the mass of the buttocks and legs together with the seat cushion
m'_s = the sprung mass
m_u = the unsprung mass
z_h = the displacements of the upper part of a seated man
z_r = the displacements of the road
z_s = the displacements of the sprung mass
z_u = the displacements of the unsprung mass

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