A note on extinction times for the general birth, death and catastrophe process

Phil Pollett* Hanjun Zhang† Benjamin J. Cairns‡

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Abstract

We consider a birth, death and catastrophe process where the transition rates are allowed to depend on the population size. We obtain an explicit expression for the expected time to extinction, which is valid in all cases where extinction occurs with probability 1.

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The model under consideration is a continuous-time Markov chain $(X(t),\, t \ge 0)$ taking values in $S = \{0, 1, \dots\}$, where $X(t)$ represents the number in a population at time t. When there are i individuals present the population size changes at rate f_i (> 0), and when a change occurs it is a birth with probability a (> 0) or catastrophe of size k (the removal of k individuals) with probability d_k $(k \ge 1)$. (Simple death events are catastrophes of size 1.) We assume that $d_k > 0$ for at least one $k \ge 1$ and $a + \sum_{k\ge1} d_k = 1$. Thus, the process has transition rates Q given by

$$q_{ij} = \begin{cases} f_i \sum_{k\ge i} d_k & j = 0,\ i \ge 1 \\ f_i d_{i-j} & j = 1, 2, \dots i-1,\ i \ge 2 \\ -f_i & j = i,\ i \ge 1 \\ f_i a & j = i+1,\ i \ge 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the sole absorbing state 0, corresponding to population extinction, is accessible from the irreducible class $\{1, 2, \dots\}$. The special case $f_i = \rho i$, where ρ (> 0) is a per-capita transition rate, was studied by Brockwell [2], Pakes [5], Pakes and Pollett [6] and Pollett [7]. Cairns and Pollett [1] studied the case of general f_i, providing an explicit expression for the

*Department of Mathematics, University of Queensland, Qld 4072, Australia. pkp@maths.uq.edu.au
†Department of Mathematics, University of Queensland, Qld 4072, Australia. hjz@maths.uq.edu.au
‡School of Biological Sciences, University of Bristol, Bristol BS8 1UG, UK. ben.cairns@bristol.ac.uk

expected time to extinction in the subcritical case. They noted that the supercritcal case could be handled by way of a standard transformation involving the extinction probabilities, and that the critical case could be handled if the variance of the catastrophe size distribution were finite. The purpose of this note is to point out that this latter condition is not needed.

Let $d(s) = a + \sum_{i=1}^{\infty} d_i s^{i+1}$, $|s| < 1$, being the probability generating function (pgf) of the jump-size distribution. Then the drift $D = 1 - d'(1-) = a - \sum_{i=1}^{\infty} i d_i$ satisfies $-\infty \leq D \leq 1$, and the process is said to be *subcritical*, *critical* or *supercritical* according as $D < 0$, $D = 0$ or $D > 0$. Now let $b(s) = d(s) - s$ and $e(s) = 1/b(s)$. It is well known (Section V.12 of Harris [3]) that $e(s)$ has a power series expansion with strictly positive coefficients (e_i, $i \geq 0$) and with radius of convergence σ, where σ is the smallest zero of $b(s)$ on $(0, 1]$. (Indeed this is true for any pgf $d(s)$ with $d(0) > 0$.) Furthermore, $\sigma = 1$ or $\sigma < 1$ according as $D \geq 0$ or $D < 0$, and $b(s) > 0$ for all $s \in [0, \sigma)$. If as we shall assume here $D \leq 0$, then the process is non-explosive and absorption occurs with probability 1 (Pakes [5]). We will prove the following result, which is Theorem 2 of [1] extended to include the $D = 0$ case.

Theorem 1 *For the subcritical or critical birth, death and catastrophe process, the expected extinction time τ_i, starting in state i, is finite if and only if $\kappa := \sum_{i=1}^{\infty} \sigma^i / f_i < \infty$, in which case $\tau_0 = 0$, $\tau_1 = \kappa/a$ and $\tau_i = \kappa e_{i-1} - \sum_{j=1}^{i-1} e_{i-1-j}/f_j$ $(i \geq 2)$.*

Cairns and Pollett's proof [1] for the subcritical case rested on $e_i/e_{i+1} \leq \sigma$ and $e_i/e_{i+1} \to \sigma$, facts which they deduced from results of Yang [8] concerning the invariant measure of a Markov branching process. However, they could be applied to the birth, death and catastrophe process only in the case $D < 0$. The next lemma shows that the two conditions hold whatever the sign of D. Thus, Theorem 1 is established.

Lemma 1 *Let $p(s)$ be a pgf with $p(0) > 0$ and let σ be the smallest zero of $p(s)$ on $(0, 1]$. Then, the coefficients (e_i, $i \geq 0$) of the power series $e(s) = 1/(p(s) - s)$ satisfy (i) $\sigma^{-i} \leq e_i \leq \sigma e_{i+1}$ and (ii) $\lim_{i \to \infty} e_i/e_{i+1} = \sigma$.*

Proof. Let $m(s) = \sum_{i=0}^{\infty} m_i s^i$ be a pgf with $m(0) > 0$ and $m'(1-) \leq 1$, so that 1 is the smallest root of $m(s) = s$ on $[0, 1]$. As already remarked, $h(s) = 1/(m(s) - s)$ is a power series with non-negative coefficients (h_i, $i \geq 0$). We will prove that $1 \leq h_i \leq h_{i+1}$ $(i \geq 0)$ and that $h_i/h_{i+1} \to 1$. The lemma will then follow on taking $m_1 = p_1\sigma + 1 - \sigma$ and $m_i = p_i \sigma^i$ $(i \neq 1)$.

Following Harris (Section V.12 of [3]) write

$$\frac{1}{m(s) - s} = \frac{1}{(1 - s)(1 - \Psi(s))}, \tag{1}$$

where $\Psi(s) = (1 - m(s))/(1 - s)$, being analytic at $s = 0$ and having a power series expansion with non-negative coefficients: $\Psi(s) = 1 - m_0 + (1 - m_0 - m_1)s + (1 - m_0 - m_1 - m_2)s^2 + \cdots$. Setting

$$\sum_{i=0}^{\infty} g_i s^i = \frac{1}{1 - \Psi(s)} = 1 + \Psi(s) + \Psi(s)^2 + \Psi(s)^3 + \cdots, \tag{2}$$

we see that $g_0 \geq 1$ and $g_i \geq 0$ $(i \geq 1)$. From (1) and (2) we have that

$$\sum_{i=0}^{\infty} h_i s^i = \frac{1}{m(s) - s} = (1 + s + s^2 + s^3 + \cdots) \sum_{i=0}^{\infty} g_i s^i.$$

Hence $h_i = \sum_{j=0}^{i} g_j$, and so $h_{i+1} \geq h_i \geq 1$.

Next, since

$$\sum_{i=0}^{\infty} g_i s^i = \frac{1}{1 - \Psi(s)} = \frac{\Psi(s)}{1 - \Psi(s)} + 1,$$

we have $g_0 = \psi_0 g_0 + 1$ and $g_i = \sum_{j=0}^{i} \psi_j g_{i-j}$ $(i \geq 1)$, where $\psi_i = 1 - m_0 - m_1 - \cdots - m_i$. It follows that $g_0 = 1/(1 - \psi_0)$ and $g_i = g_0 \sum_{j=1}^{i} \psi_j g_{i-j}$ $(i \geq 1)$. We will prove that $g_i \leq g_0$ $(i \geq 0)$ using mathematical induction. The statement is clearly true for $i = 0$. So, suppose that, for some fixed $i \geq 1$, $g_j \leq g_0$ for all $j \leq i - 1$. Then,

$$g_i = \frac{1}{1 - \psi_0} \sum_{j=1}^{i} \psi_j g_{i-j} \leq \frac{g_0}{1 - \psi_0} \sum_{j=1}^{i} \psi_j \leq g_0,$$

where in the last inequality we have used that fact that $\sum_{j=0}^{\infty} \psi_j = m'(1-) \leq 1$.

Since $h_i = \sum_{j=0}^{i} g_j$, we have $1 \leq h_{i+1}/h_i = 1 + g_{i+1}/h_i$. Also, since (h_i) is an increasing sequence, the limit $L = \lim_{i \to \infty} h_i$ exists, but might be infinite. If $L < \infty$, then $g_i \to 0$, and so $g_{i+1}/h_i \to 0$. Otherwise, if $L = \infty$, then $g_{i+1}/h_i \to 0$, because $g_i \leq g_0$. In either case we have $h_{i+1}/h_i \to 1$.

On setting $m_1 = p_1 \sigma + 1 - \sigma$ and $m_i = p_i \sigma^i$ $(i \neq 1)$, it is clear that $m_0 = p_0 > 0$, $\sum_{i=0}^{\infty} m_i = 1$ and $m(s) = \sum_{i=0}^{\infty} p_i \sigma^i s^i + (1-\sigma)s$. Hence $m(s) - s = p(\sigma s) - \sigma s$, and so $h_i = e_i \sigma^i$ and 1 is the smallest root of $m(s) = s$ on $[0, 1]$ because σ is the smallest root of $p(s) = s$ on $[0, 1]$. The first claim, that $\sigma^{-i} \leq e_i \leq \sigma e_{i+1}$ $(i \geq 0)$, now follows because $1 \leq h_i \leq h_{i+1}$. To prove the second claim observe that $\lim_{i \to \infty} e_i/e_{i+1} = \sigma \lim_{i \to \infty} h_i/h_{i+1} = \sigma$. The proof is complete.

Example. We will suppose that catastrophe sizes follow the zeta distribution (also known as the Zipf or 'discrete Pareto' distribution), an example of a power law that has been used to model the effect of a variety of hazards, such as the area burnt in forest fires [4]. We set $d_k = (1 - a)k^{-r}/\zeta(r)$, where $\zeta(\cdot)$ is the Riemann zeta function and r is a parameter that determines the weight ascribed to the tail of the distribution. Then, $b(s) = a - s + (1 - a)s \operatorname{Li}_r(s)/\zeta(r)$, where $\operatorname{Li}_r(\cdot)$ is the (real-valued) polylogarithm function of order r. The distribution $(d_k,\, k \geq 1)$ is only defined for $r > 1$ and its mean is finite only if $r > 2$; for $1 < r \leq 2$, $D = -\infty$, while for $r > 2$, D is finite and its value depends on a. The variance of the distribution is finite if and only if $r > 3$, and so when $2 < r \leq 3$ the mean is finite but the variance is infinite. For example, when $r = 3$, it can be shown that $D < 0$, $D = 0$ or $D \geq 0$, according as a is less than, equal to, or greater than $\pi^2/[6\zeta(3) + \pi^2]$. The expected extinction times $(\tau_i, i \geq 0)$ may then be obtained from Theorem 1. If in particular $D = 0$, then $\sigma = 1$ and thus the τ_i are finite if and only if $\kappa = \sum_{i=1}^{\infty} 1/f_i$ is finite (that is, if changes in the population size become 'fast enough' as the population size increases). For the linear case $f_i = \rho i$, the birth, death and catastrophe process with zeta catastrophes and $r = 3$ has finite or infinite expected extinction times according as $D < 0$ or $D \geq 0$. On the other hand, if $f_i = \rho i(i+1)$ then the expected extinction times are finite when $D \leq 0$; when $D > 0$ the process is explosive (Theorem 1 of [1]).

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