Polar Codes and Polar Lattices for Independent Fading Channels

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***Abstract*—In this paper, we design polar codes and polar lattices for i.i.d. fading channels when the channel state information is only available to receiver. For the binary input case, we show that one single polar code is sufficient to achieve the ergodic capacity. For the non-binary input case, polar codes are further extended to polar lattices to achieve the egodic Poltyrev capacity, i.e., the capacity without power limit. When the power constraint is taken into consideration, we show that polar lattices with lattice Gaussian shaping are able to achieve the egodic capacity of fading channels. The coding and shaping are both explicit, and our scheme works under any signal noise ratio. The overall complexity of encoding and decoding is $O(N \log^2 N)$.**

I. Introduction

Real-world wireless channels are generally modeled as time-varying fading channels due to multiple signal paths and user mobility. Compared with time-invariant channel models, the wireless fading channel models allow the channel gain to change randomly over time. In practice, we usually consider block fading channel models, where channel gain varies at a longer time scale than symbol transmission time. In this paper we study the fast block fading channel with stationary ergodic channel gains. By assuming a perfect interleaving/de-interleaving on symbols, the channel can be considered to be memoryless, which offers much convenience for coding design. We further assume that channel state information (CSI) is available to receiver, and the transmitter only has the channel distribution information (CDI).

Polar codes, introduced by Arıkan [1], are capacity achieving for binary-input memoryless symmetric channels (BMSCs). Besides channel coding, polar codes were then extended to source coding and their asymptotic performance was proved to be optimal [2]. As a combination of the application of polar codes for channel coding and lossless source coding, polar codes were further studied to achieve the capacity of binary-input memoryless asymmetric channels (BMACs) in [3]. For fading channels, there has been some previous work. A hierarchic polar coding scheme, approximating fading channels as a mixture of binary symmetric channels (BSCs) with different crossover probabilities, was proposed in [4]. The ergodic capacity is achieved through two phases of polarization. The first phase is to get each BSC polarized into a set of extremal subchannels, which is treated as a set of binary erasure channels (BECs). The second phase of polarization to get each BEC polarized. As a result, much longer block length than the standard polar codes is needed. Quasi-static fading channel with two states was discussed in [5]. Construction of polar codes for block Rayleigh fading channels when CSI or CDI is available for both transmitter and receiver is considered in [6]. In this work, however, we consider the case in which CSI is available to receiver, and transmitter only knows CDI. We show that the same channel capacity can be achieved as CSI is available to both, and the construction of polar codes is simplified.

As the counterpart of linear codes in the Euclidean space, lattice codes provide more freedom over signal constellation for communication systems. The existence of lattice codes achieving the additive white Gaussian noise (AWGN) channel capacity was established using the random coding argument [7]. Besides point-to-point communications, lattice codes are also useful in a wide range of applications in multiterminal communications (see [8] for an overview). Following the work on multilevel coset codes [9], polar lattices were constructed from polar codes according to "Construction D" [10] and proved to be AWGN-good [11]. With lattice Gaussian shaping [12], polar lattices were then shown to be capable of achieving the AWGN capacity [13]. More recently, lattice codes were investigated in ergodic fading channels [14] and proved to be capacity-achieving under the ambiguity decoding. However, the construction of such lattice codes for ergodic fading channels is still implicit. In this work, we will resolve this problem using polar lattices for the i.i.d. fading case.

Algebraic tools [15] play an important role in explicit lattice coding design for fading channles. It was shown that lattice codes constructed from algebraic number field can achieve full diversity over fading channels, which results in better error probability performance. A more recent work showed that number field lattices are able to achieve Gaussian and Rayleigh channel capacity within a constant gap [16]. It is still an open question whether this gap can be removed.

The paper is organized as follows: Section II presents the background of polar codes and polar lattices. The construction of polar codes for binary-input ergodic fading channels is investigated in Section III. In Section IV, we design polar lattices for fading channels without power constraint and prove that ergodic Poltyrev capacity can be achieved. Lattice Gaussian shaping is then implemented to obtain the optimum shaping gain. Finally, the paper is concluded in Section VI.

All random variables (RVs) are denoted by capital letters. Let P_X denote the probability distribution of a RV X taking values x in a set $\mathcal{X}$. The i-th realization of X is denoted by x^i. We also use the notation $x^{i:j}$ as a shorthand for a vector $(x^i, ..., x^j)$, which is a realization of RVs $X^{i:j} = (X^i, ..., X^j)$. For a set $\mathcal{I}$, $\mathcal{I}^c$ denotes its complement. We denote N independent uses of channel W by W^N. By channel combining and splitting, we get the combined channel W_N and the i-th subchannel $W_N^{(i)}$. The binary logarithm and natural logarithm are accordingly denoted by $\log$ and $\ln$, and information is measured in bits.

II. Preliminaries of Polar Codes and Polar Lattices

A. Polar Codes

Let $\tilde{W}$ be a BMSC with input alphabet $X \in \{0, 1\}$ and output alphabet Y. Given the capacity $C(\tilde{W})$ of $\tilde{W}$ and the rate $R < C(\tilde{W})$, the information bits of a polar code with block length $N = 2^m$ are indexed by a set of $\lfloor RN \rfloor$ rows of the generator matrix $G_N = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]^{\otimes m}$, where $\otimes$ denotes the Kronecker product. The matrix G_N combines N identical copies of $\tilde{W}$ to $\tilde{W}_N$. Then this combination can be successively split into N binary memoryless symmetric subchannels, denoted by $\tilde{W}_N^{(i)}$ with $1 \le i \le N$. By channel polarization, the fraction of good (roughly error-free) subchannels is about $C(\tilde{W})$ as $m \to \infty$. Therefore, to achieve the capacity, information bits are sent over those good subchannels and the rest are fed with frozen bits which are known before transmission. The indices of good subchannels can be identified according to their associated Bhattacharyya Parameters.

Definition 1. *Given a BMSC $\tilde{W}$ with transition probability $P_{Y|X}$, the Bhattacharyya parameter of $\tilde{W}$ is defined as*

$$\tilde{Z}(\tilde{W}) \quad \triangleq \sum_y \sqrt{P_{Y|X}(y|0)P_{Y|X}(y|1)}. \tag{1}$$

Based on the Bhattacharyya parameter, the information set $\tilde{\mathcal{I}}$ is defined as $\{i : \tilde{Z}(\tilde{W}_N^{(i)}) \le 2^{-N^\beta}\}$ for some $0 < \beta < \frac{1}{2}$, and the frozen set $\tilde{\mathcal{F}} = \tilde{\mathcal{I}}^c$. Let P_B denote the block error probability of a polar code under successive cancellation (SC) decoding. It can be upper-bounded as $P_B \le \Sigma_{i \in \tilde{\mathcal{I}}} \tilde{Z}(\tilde{W}_N^{(i)})$. An efficient algorithm to evaluate the Bhattacharyya parameter of subchannels for general BMSCs was presented in [17].

B. Lattice Codes and Polar Lattices

An n-dimensional lattice is a discrete subgroup of $\mathbb{R}^n$ which can be described by

$$\Lambda = \{\lambda = \mathbf{B}z : z \in \mathbb{Z}^n\}, \tag{2}$$

where the columns of the generator matrix $\mathbf{B} = [\mathrm{b}_1, \cdots, \mathrm{b}_n]$ are assumed to be linearly independent.

For a vector $x \in \mathbb{R}^n$, the nearest-neighbor quantizer associated with Λ is $Q_\Lambda(x) = \arg\min_{\lambda \in \Lambda} \|\lambda - x\|$. We define the modulo lattice operation by $x \bmod \Lambda \triangleq x - Q_\Lambda(x)$. The Voronoi region of Λ, defined by $\mathcal{V}(\Lambda) = \{x : Q_\Lambda(x) = 0\}$, specifies the nearest-neighbor decoding region. The Voronoi cell is one example of fundamental region of the lattice. A measurable set $\mathcal{R}(\Lambda) \subset \mathbb{R}^n$ is a fundamental region of the lattice Λ if $\cup_{\lambda \in \Lambda}(\mathcal{R}(\Lambda) + \lambda) = \mathbb{R}^n$ and if $(\mathcal{R}(\Lambda) + \lambda) \cap (\mathcal{R}(\Lambda) + \lambda')$ has measure 0 for any $\lambda \neq \lambda'$ in Λ. The volume of a fundamental region is equal to that of the Voronoi region $\mathcal{V}(\Lambda)$, which is given by $V(\Lambda) = |\det(B)|$.

For an n-dimensional lattice Λ, define the volume-to-noise ratio (VNR) by

$$\gamma_\Lambda(\sigma) \triangleq V(\Lambda)^{\frac{2}{n}} / \sigma^2. \tag{3}$$

For $\sigma > 0$ and $c \in \mathbb{R}^n$, we define the Gaussian distribution of variance σ^2 centered at c as

$$f_{\sigma,c}(x) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{\|x-c\|^2}{2\sigma^2}}, \; x \in \mathbb{R}^n. \tag{4}$$

Let $f_{\sigma,0}(x) = f_\sigma(x)$ for short. For an AWGN channel with noise variance σ^2 per dimension, the probability of error $P_e(\Lambda, \sigma^2)$ of a minimum-distance decoder for Λ is

$$P_e(\Lambda, \sigma^2) = 1 - \int_{\mathcal{V}(\Lambda)} f_\sigma(x) dx. \tag{5}$$

The Λ-periodic function is defined as

$$f_{\sigma,\Lambda}(x) = \sum_{\lambda \in \Lambda} f_{\sigma,\lambda}(x) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \sum_{\lambda \in \Lambda} e^{-\frac{\|x-\lambda\|^2}{2\sigma^2}}. \tag{6}$$

We note that $f_{\sigma,\Lambda}(x)$ is a probability density function (PDF) if x is restricted to the fundamental region $\mathcal{R}(\Lambda)$. This distribution is actually the PDF of the Λ-aliased Gaussian noise, i.e., the Gaussian noise after the mod-Λ operation [9].

A sublattice $\Lambda' \subset \Lambda$ induces a partition (denoted by Λ/Λ') of Λ into equivalence groups modulo Λ'. The order of the partition, denoted by $|\Lambda/\Lambda'|$, is equal to the number of the cosets. If $|\Lambda/\Lambda'| = 2$, we call this a binary partition. Let $\Lambda(\Lambda_0)/\Lambda_1/\cdots/\Lambda_{r-1}/\Lambda'(\Lambda_r)$ for $r > 1$ be an n-dimensional lattice partition chain. The construction is known as "Construction D" [10, p.232]. For each partition $\Lambda_{\ell-1}/\Lambda_\ell$ ($1 \le \ell \le r$) a code C_ℓ over $\Lambda_{\ell-1}/\Lambda_\ell$ selects a sequence of coset representatives a_ℓ in a set A_ℓ of representatives for the cosets of Λ_ℓ. This construction requires a set of nested linear binary codes C_ℓ with block length N and dimension of information bits k_ℓ. Let ψ be the natural embedding of $\mathbb{F}_2^N$ into $\mathbb{Z}^N$, where $\mathbb{F}_2$ is the binary field. Consider $\mathrm{g}_1, \mathrm{g}_2, \cdots, \mathrm{g}_N$ be a basis of $\mathbb{F}_2^N$ such that $\mathrm{g}_1, \cdots \mathrm{g}_{k_\ell}$ span C_ℓ. When $n = 1$, the binary lattice L consists of all vectors of the form

$$\sum_{\ell=1}^{r} 2^{\ell-1} \sum_{j=1}^{k_\ell} \alpha_j^{(\ell)} \psi(\mathrm{g}_j) + 2^r z, \tag{7}$$

where $\alpha_j^{(\ell)} \in \{0, 1\}$ and $z \in \mathbb{Z}^N$. When $\{C_1, ..., C_r\}$ is a series of nested polar codes, we obtain a polar lattice.

III. Polar Codes for Binary-input fading channels

Consider the binary-input i.i.d. ergodic fading channel

$$Y = HX + Z, \tag{8}$$

where $X \in \{-1, +1\}$ is the binary signal after BPSK modulation, Y is the channel output, Z is a zero mean independent Gaussian noise with variance σ^2, and H is the channel gain. For convenience, we assume that H follows Rayleigh distribution with probability density function (PDF)

$$P_H(h) = \frac{h}{\sigma_h^2} e^{-\frac{h^2}{2\sigma_h^2}}, \tag{9}$$

where $\sigma_h = \sqrt{\frac{2}{\pi}} \cdot E[H]$. Note that our work can be easily generalized to other regular fading distributions [18].

Since we assume that H is available to the receiver, the fading channel can be modeled as a channel with input X and outputs (Y, H), as shown in Fig. 1.

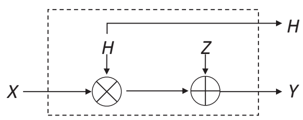

Fig. 1. Binary-input ergodic fading channel with CSI available to receiver.

The channel transition PDF of $\tilde{W}$ is given by

$$\begin{aligned} P_{Y,H|X}(y,h|x) &= P_H(h) P_{Y|X,H}(y|x,h) \\ &= P_H(h) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-xh)^2}{2\sigma^2}}. \end{aligned} \tag{10}$$

We define a permutation ϕ over the outputs (y, h) such that $\phi(y, h) = (-y, h)$. Check that $P_{Y,H|X}(y, h| + 1) = P_{Y,H|X}(\phi(y, h)| - 1)$ and hence $\tilde{W}$ is symmetric. It is well-known that uniform input distribution achieves the capacity of symmetric channels. Therefore, let X be uniform, the capacity of $\tilde{W}$ is given by

$$\begin{aligned} C(\tilde{W}) &= I(X;Y,H) = I(X;Y|H) \\ &= 1 - \frac{1}{\sqrt{2\pi}\sigma\sigma_h^2} \int_0^\infty h e^{-\frac{h^2}{2\sigma_h^2}} dh \cdot \\ &\qquad \int_{-\infty}^{\infty} \left(1 - \log(1 + e^{-\frac{2yh}{\sigma^2}})\right) dy, \end{aligned} \tag{11}$$

which is the same as the capacity when CSI is avaliable to both transmitter and receiver [6].

To achieve $C(\tilde{W})$, we combine N independent copies of $\tilde{W}$ to $\tilde{W}_N$ and split it to obtain subchannel $\tilde{W}_N^{(i)}$ for $1 \leq i \leq N$. Let $U^{1:N} = X^{1:N} G_N$, $\tilde{W}_N^{(i)}$ has input U^i and outputs $(U^{1:i-1}, Y^{1:N}, H^{1:N})$. Since $\tilde{W}$ is symmetric, $\tilde{W}_N^{(i)}$ is symmetric as well [1]. We can identify the information set $\mathcal{I}$ according to the Bhattacharyya parameter $\tilde{Z}(\tilde{W}_N^{(i)})$. Treating (Y, H) as the outputs, by Definition 1,

$$\tilde{Z}(\tilde{W}) = \sum_{y,h} \sqrt{P_{Y,H|X}(y,h|+1) P_{Y,H|X}(y,h|-1)}. \tag{12}$$

For general BMSCs, we can apply the degrading and upgrading merging algorithms [17], [19] to estimate $\tilde{Z}(\tilde{W}_N^{(i)})$ within acceptable accuracy.

In practice, the two approximations caused by the degrading and upgrading processes are typically close. Therefore, we focus on the degrading transform for brevity.

Define the likelihood ratio (LR) of (y, h) as

$$LR(y,h) \triangleq \frac{P_{Y,H|X}(y,h|+1)}{P_{Y,H|X}(y,h|-1)}. \tag{13}$$

By (10), we have $LR(y, h) = e^{\frac{2yh}{\sigma^2}}$ for the fading case. Clearly, $LR(y, h) \geq 1$ for any $y \geq 0$. Each $LR(y, h)$ corresponds to a BSC with crossover probability $\frac{1}{LR(y,h)+1}$ and its capacity is

$$C[LR(y,h)] = 1 - \mathfrak{h}_2\left(\frac{1}{LR(y,h)+1}\right), \tag{14}$$

where $\mathfrak{h}_2(\cdot)$ is the binary entropy function.

The fading channel $\tilde{W}$ is then quantized according to $C[LR(y, h)]$. Let $\mu = 2Q$ be the size of degraded channel output alphabet. The region $\{y \geq 0, h \geq 0\}$ is divided into Q sets

$$A_i = \left\{ y \geq 0, h \geq 0 : \frac{i-1}{Q} \leq C[LR(y,h)] < \frac{i}{Q} \right\}, \tag{15}$$

for $1 \leq i \leq L$. The outputs in A_i are mapped to one symbol, and $\tilde{W}$ is quantized to a mixture of Q BSCs with the crossover probability

$$p_i = \frac{\int_{A_i} P_{Y,H|X}(y,h|-1) dy dh}{\sum_{x=\{-1,+1\}} \int_{A_i} P_{Y,H|X}(y,h|x) dy dh}. \tag{16}$$

Note that p_i can be numerically evaluated. See [20] for more details. Let $\tilde{W}_Q$ denote the quantized channel from $\tilde{W}$ after the degrading transform. By [17, Lemma 13], the difference between the two channel capacities is upper-bounded by $\frac{1}{Q}$. When Q is sufficiently large, we can use $\tilde{W}_Q$ to approximate $\tilde{W}$ to construct polar codes. The size of the subchannel output alphabet after degrading merging is no more than $2Q$.

The proof of the following theorem is standard and fully given in [17]. We omit it for brevity.

Theorem 1. *Let $\tilde{W} : X \to (Y, H)$ be a binary-input ergodic fading channel. Let N denote the block length and $\mu = 2Q$ denote the limit of the size of output alphabet. A polar code constructed by the degrading merging algorithm achieves the capacity $C(\tilde{W})$ when N and μ are both sufficiently large. The block error probability under SC decoding is unpper-bounded by $N 2^{-N^\beta}$ for $0 < \beta < \frac{1}{2}$.*

Simulation results of polar codes with different block length for binary Rayleigh fading channels in shown in Fig. 2, where $\sigma_h = 1.2575$, $\sigma = 1$ ($\sigma_h^2/\sigma^2 = 5$ dB), and $C(\tilde{W}) = 0.671$.

IV. Polar Lattices for Gaussian Fading Channels

In this part, we extend polar codes to polar lattices for ergodic fading channels. The reason for this extension is that the input of fading channels is not necessarily limited to be binary. In general, input X is subject to a power constraint P, i.e., $E[\|X\|^2] \leq P$. In this case, lattice codes offer more choices of input constellation. Our work follows a similar idea of [13]. We firstly construct polar lattices which achieve the Poltyrev capacity of egodic fading channels and then perform lattice Gaussian shaping to achieve the ergodic capacity.

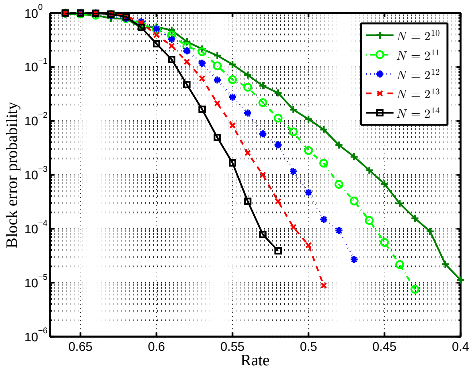

Fig. 2. Performance of polar codes for Rayleigh fading channel when $N = 2^{10}, 2^{11}, ..., 2^{14}$.

A. The Mod Λ channel and Λ/Λ' channel [9]

A mod-Λ Gaussian channel is a Gaussian channel with an input in $\mathcal{V}(\Lambda)$ and with a mod-$\mathcal{V}(\Lambda)$ operator at the receiver front end. The capacity of the mod-Λ channel for noise variance σ^2 is

$$C(\Lambda, \sigma^2) = \log V(\Lambda) - \mathfrak{h}(\Lambda, \sigma^2), \tag{17}$$

where $\mathfrak{h}(\Lambda, \sigma^2) = -\int_{\mathcal{V}(\Lambda)} f_{\sigma,\Lambda}(x) \log f_{\sigma,\Lambda}(x)dx$ is the differential entropy of the Λ-aliased noise over $\mathcal{V}(\Lambda)$.

For a lattice partition Λ/Λ', the Λ/Λ' channel is a mod-Λ' channel whose input is restricted to discrete lattice points in $(\Lambda + a) \cap \mathcal{R}(\Lambda')$ for some translate a. The capacity of the Λ/Λ' channel is given by [9]

$$\begin{aligned} C(\Lambda/\Lambda', \sigma^2) &= C(\Lambda', \sigma^2) - C(\Lambda, \sigma^2) \\ &= \mathfrak{h}(\Lambda, \sigma^2) - \mathfrak{h}(\Lambda', \sigma^2) + \log\left(V(\Lambda')/V(\Lambda)\right). \end{aligned} \tag{18}$$

As we mentioned, we use the "Construction D" method to construct polar lattices. Let $\Lambda/\Lambda_1/\cdots/\Lambda_{r-1}/\Lambda'$ be an n-dimensional self-similar lattice partition chain. For each partition $\Lambda_{\ell-1}/\Lambda_\ell$ a code over $\Lambda_{\ell-1}/\Lambda_\ell$ selects a sequence of representatives a_ℓ for the cosets of Λ_ℓ. If each partition is a binary partition, the codes $\mathcal{C}_\ell$ are binary codes. Moreover, based on this partition chain, the capacity $C(\Lambda/\Lambda', \sigma^2)$ can also be expanded as

$$C(\Lambda/\Lambda', \sigma^2) = C(\Lambda/\Lambda_1, \sigma^2) + \cdots + C(\Lambda_{r-1}/\Lambda', \sigma^2). \tag{19}$$

In order to approach the Poltyrev capacity of AWGN channels, for a polar lattice L, we would like to have $\gamma_L(\sigma) \to 2\pi e$ while $P_e(L, \sigma^2) \to 0$. According to the analysis in [9], we need a negligible capacity $C(\Lambda, \sigma^2)$, a small error probability $P_e(\Lambda', \sigma^2)$, and a capacity-approaching polar code for each $\Lambda_{\ell-1}/\Lambda_\ell$ partition channel.

B. Polar Lattices for Fading Channels without Power Limit

For the ergodic fading channels, the channel gain varies. The above analysis for time-unvarying AWGN channels need to be generalized. Since receiver knows the CSI, the fading effect can be removed by multiplying Y with $\frac{1}{H}$ at the receiver's end. We define the fading mod-Λ channel as follows.

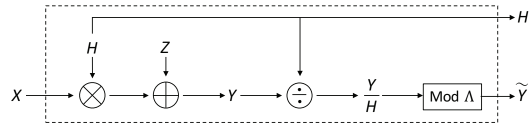

Fig. 3. A block diagram of the fading mod-Λ channel.

Definition 2. *A fading mod-Λ channel is a Gaussian fading channel with an input in $\mathcal{V}(\Lambda)$, and an output being scaled by $\frac{1}{H}$ before the mod-$\mathcal{V}(\Lambda)$ operation. A block diagram of this model is shown in Fig 3.*

The fading mod-Λ channel is closely related to a mod-Λ channel with noise variance $\frac{\sigma^2}{h^2}$. The channel transition PDF of the fading mod-Λ channel is given by

$$\begin{aligned} P_{\tilde{Y},H|X}(\tilde{y}, h|x) &= P_{Y,H|X}(y = \tilde{y}h + h\cdot\Lambda, h|x)\frac{dy}{d\tilde{y}} \\ &= h \cdot P_H(h) \sum_{\lambda\in\Lambda} P_{Y|X,H}(y = \tilde{y}h + \lambda h|x, h) \\ &= P_H(h) \sum_{\lambda\in\Lambda} \frac{1}{\sqrt{2\pi}\frac{\sigma}{h}} e^{-\frac{(\tilde{y}+\lambda-x)^2}{2(\frac{\sigma}{h})^2}}, \end{aligned} \tag{20}$$

where the second term in the last equation is the channel transition PDF of a mod-Λ channel with noise variance $\frac{\sigma^2}{h^2}$. Therefore, the fading mod-Λ channel can be viewed as an independent combination of a Rayleigh distributed variable H and a mod-Λ channel with noise variance $\frac{\sigma^2}{H^2}$. The capacity of the mod-Λ channel is

$$C_H(\Lambda, \sigma^2) = E_h\left[C\left(\Lambda, \frac{\sigma^2}{h^2}\right)\right]. \tag{21}$$

Similarly, A fading Λ/Λ' channel is a fading mod-Λ' channel whose input is restricted to discrete lattice points in $(\Lambda + a) \cap \mathcal{R}(\Lambda')$ for some translate a. It can be viewed as an independent combination of a Rayleigh distributed variable H and a Λ/Λ' channel with noise variance $\frac{\sigma^2}{H^2}$. The capacity of the fading Λ/Λ' channel is given by

$$C_H(\Lambda/\Lambda', \sigma^2) = E_h\left[C\left(\Lambda', \frac{\sigma^2}{h^2}\right)\right] - E_h\left[C\left(\Lambda, \frac{\sigma^2}{h^2}\right)\right]. \tag{22}$$

Since the Λ/Λ' channel is symmetric [9], it is easy to check that the Λ/Λ' fading channel is symmetric as well. Polar lattices can be constructed to achieve the (ergodic) Poltyrev capacity of the ergodic fading channel, as we did for the AWGN channel. Recall that the Poltyrev capacity C_∞ of a general additive-noise channel is defined as the capacity per unit volume in [8, Theorem 6.3.1]. For the independent AWGN channels, it is given by $-\mathfrak{h}(\sigma^2)$, where $\mathfrak{h}(\sigma^2)$ denotes the differential entropy of a Gaussian variable with variance σ^2.

For the independent ergodic fading channels, C_∞ is generalized as [18]

$$\begin{aligned} C_\infty &= -E_h\left[\mathfrak{h}\left(\frac{\sigma^2}{h^2}\right)\right] = E_h\left[\frac{1}{2}\log\left(\frac{h^2}{2\pi\sigma^2}\right)\right] \\ &= -\frac{1}{2}\log\left(2\pi e\sigma^2 \cdot \frac{e^\zeta}{\sigma_h^2}\right), \end{aligned} \tag{23}$$

where $\zeta = -\int_0^\infty e^{-x} \ln x dx$ is the Euler-Mascheroni constant.

To approach the Poltyrev capacity C_∞, we construct polar lattices according to the following three design criteria:

(a) Λ gives negligible capacity $E_h\left[C\left(\Lambda, \frac{\sigma^2}{h^2}\right)\right]$.

(b) Λ' has a small error probability $E_h\left[P_e\left(\Lambda', \frac{\sigma^2}{h^2}\right)\right]$.

(c) Each component polar code $\mathcal{C}_\ell$ is a capacity-approaching code for the $\Lambda_{\ell-1}/\Lambda_\ell$ fading channel.

For criterion (a), we pick a top lattice Λ for a large channel gain h_r such that $\mathfrak{h}\left(\Lambda, \frac{\sigma^2}{h_r^2}\right) \approx \log V(\Lambda)$. For criterion (b), we pick a bottom lattice Λ' for a small channel gain h_l such that $P_e\left(\Lambda', \frac{\sigma^2}{h_l^2}\right) \to 0$. For criterion (c), we choose a binary partition chain and construct binary polar codes to achieve the capacity of the $\Lambda_{\ell-1}/\Lambda_\ell$ fading channel for $1 \leq \ell \leq r$. Since the $\Lambda_{\ell-1}/\Lambda_\ell$ fading channel is a BMSC, treating $(\tilde{Y}, H)$ as the outputs, the construction method proposed in Sect. III can be used. The constructed polar codes can be proved to be sequentially nested. See [20] for more details.

Theorem 2. *For an independent ergodic Rayleigh fading channel with given σ_h^2 and σ^2, select an n-dimensional binary lattice partition chain $\Lambda/\Lambda_1/\cdots/\Lambda_{r-1}/\Lambda'$ such that both the criterion (a) and (b) are satisfied. Construct a polar lattice L from this partition chain and r nested polar codes with block length N. For a fixed dimension n and some constant $\delta \geq 1$, L can achieve the Poltyrev capacity of the ergodic fading channel, i.e., $\gamma_L(\sigma) \to 2\pi e \cdot \frac{e^\zeta}{\sigma_h^2}$ and $P_e(L, \sigma^2) = O(\frac{1}{N^{2\delta-1}}) \to 0$, as $r = O(n\delta \log N)$ and $N \to \infty$.*

Proof: See the proof of [20, Theorem 2]. ∎

C. Polar Lattices With Gaussian Shaping

When the CSI is only known to the receiver, for a given power constraint $E[\|X\|^2] \leq P$, the optimal input distribution for the ergodic fading channel is the continuous Gaussian distribution with variance P [21], which is the same as that for AWGN channels. Therefore, the lattice Gaussian shaping technique proposed for the AWGN-good polar lattices in [13] can be applied to the fading case with minor modification.

Let the input X be Gaussian distributed, the ergodic channel capacity is given by [21]

$$I(X;Y,H) = E_h\left[\frac{1}{2}\log\left(1 + \frac{Ph^2}{\sigma^2}\right)\right], \tag{24}$$

where $\frac{1}{2}\log(1 + \frac{Ph^2}{\sigma^2})$ is the capacity of an AWGN channel with noise variance $\frac{\sigma^2}{h^2}$ and power constraint P. Our strategy is to pick a lattice Gaussian distribution which is able to achieve the AWGN capacity for almost all possible h. We can choose a large h_r such that the lattice Gaussian distribution defined over Λ achieves the capacity $\frac{1}{2}\log(1 + \frac{Ph_r^2}{\sigma^2})$, and finally the ergodic capacity can be approached by this distribution. The details of implementing the Gaussian shaping on polar lattices are given in [20] and omitted here due to the limited space.

V. Conclusion

Explicit construction of polar codes and polar lattices for ergodic fading channels is proposed in this paper. By treating channel gain as a part of channel outputs, the work of polar codes and polar lattices for time-unvarying channels is generalized to the fading case. We prove that standard polar codes are able to achieve the capacity of binary-input ergodic fading channels, and polar lattices are able to achieve the capacity with certain input constraint.

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References

[1] E. Arıkan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3051–3073, July 2009.

[2] S. Korada and R. Urbanke, "Polar codes are optimal for lossy source coding," *IEEE Trans. Inf. Theory*, vol. 56, no. 4, pp. 1751–1768, April 2010.

[3] J. Honda and H. Yamamoto, "Polar coding without alphabet extension for asymmetric models," *IEEE Trans. Inf. Theory*, vol. 59, no. 12, pp. 7829–7838, Dec. 2013.

[4] H. Si, O. Koyluoglu, and S. Vishwanath, "Polar coding for fading channels: Binary and exponential channel cases," *IEEE Trans. Commun.*, vol. 62, no. 8, pp. 2638–2650, Aug. 2014.

[5] J. Boutros and E. Biglieri, "Polarization of quasi-static fading channels," in *Proc. 2013 IEEE Int. Symp. Inform. Theory*, July 2013, pp. 769–773.

[6] A. Bravo-Santos, "Polar codes for the Rayleigh fading channel," *IEEE Commun. Lett.*, vol. 17, no. 12, pp. 2352–2355, December 2013.

[7] U. Erez and R. Zamir, "Achieving 1/2 log (1+SNR) on the AWGN channel with lattice encoding and decoding," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2293–2314, Oct. 2004.

[8] R. Zamir, *Lattice Coding for Signals and Networks: A Structured Coding Approach to Quantization, Modulation, and Multiuser Information Theory*. Cambridge, UK: Cambridge University Press, 2014.

[9] G. D. Forney Jr., M. Trott, and S.-Y. Chung, "Sphere-bound-achieving coset codes and multilevel coset codes," *IEEE Trans. Inf. Theory*, vol. 46, no. 3, pp. 820–850, May 2000.

[10] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices, and Groups*. New York: Springer, 1993.

[11] Y. Yan, C. Ling, and X. Wu, "Polar lattices: Where Arıkan meets Forney," in *Proc. 2013 IEEE Int. Symp. Inform. Theory*, Istanbul, Turkey, July 2013, pp. 1292–1296.

[12] C. Ling and J. Belfiore, "Achieving AWGN channel capacity with lattice Gaussian coding," *IEEE Trans. Inf. Theory*, vol. 60, no. 10, pp. 5918–5929, Oct. 2014.

[13] Y. Yan, L. Liu, C. Ling, and X. Wu, "Construction of capacity-achieving lattice codes: Polar lattices," Nov. 2014. [Online]. Available: http://arxiv.org/abs/1411.0187

[14] A. Hindy and A. Nosratinia, "Achieving the ergodic capacity with lattice codes," in *Proc. 2015 IEEE Int. Symp. Inform. Theory*, June 2015, pp. 441–445.

[15] F. Oggier and E. Viterbo, *Algebraic number theory and code design for Rayleigh fading channels*. The Netherlands: Now publishers inc, 2004.

[16] R. Vehkalahti and L. Luzzi, "Number field lattices achieve Gaussian and Rayleigh channel capacity within a constant gap," in *Proc. 2015 IEEE Int. Symp. Inform. Theory*, June 2015, pp. 436–440.

[17] I. Tal and A. Vardy, "How to construct polar codes," *IEEE Trans. Inf. Theory*, vol. 59, no. 10, pp. 6562–6582, Oct. 2013.

[18] S. Vituri and M. Feder, "Dispersion of infinite constellations in fast fading channels," April 2014. [Online]. Available: http://arxiv.org/abs/1206.5401

[19] R. Pedarsani, S. Hassani, I. Tal, and I. Telatar, "On the construction of polar codes," in *Proc. 2011 IEEE Int. Symp. Inform. Theory*, July 2011, pp. 11–15.

[20] L. Liu and C. Ling, "Polar codes and polar lattices for independent fading channels," Jan. 2016. [Online]. Available: http://arxiv.org/abs/XXXX.XXXX

[21] A. El Gamal and Y. H. Kim, *Network information theory*. Cambridge university press, 2011.